CONTROLLING CHAOS BY PREDICTIVE CONTROL

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Abstract: New approach to controlling chaos in discrete-time nonlinear autonomous systems is proposed. We stabilize the desired unstable periodic orbit via small control, based on the prediction of the trajectory. The knowledge of the periodic orbit is not required, just its existence. The method is validated for one-dimensional as well as for multidimensional maps. Numerical simulation for logistic, tent, Henon maps demonstrates the effectiveness of the approach. The method is simple, but the main limitation for its use is the lack of noises and disturbances in the system description.

Keywords: Chaos, control synthesis, nonlinear systems, unstable, stabilization, predictive control, periodic motion.

1. INTRODUCTION

Control of chaotic dynamical systems attracted much attention during recent years, see e.g. the pioneering paper (Ott, *et al.*, 1990) and the surveys (Arecchi, *et al.*, 1998; Fradkov and Pogromsky, 1998; Bocaletti, *et al.*, 2000; Chen and Yu, 2003; Andrievski and Fradkov, 2003-2004). As stated in (Arecchi, *et al.*, 1998) "controlling chaos consists in perturbing chaotic system in order to stabilize a given unstable periodic orbit embedded in the chaotic attractor". In the present paper we suggest a novel approach to the problem for discrete-time nonlinear systems, which in the absence of control have the form

$$x_{k+1} = f(x_k). \tag{1}$$

The idea (proposed by V.Maslov) is to predict a trajectory of the system and to use the additive control in the form

$$u(x) = \varepsilon(f_{m+s}(x) - f_m(x)), \tag{2}$$

where ε is a small step-size (the simple rule for its fit will be provided), m is a prediction horizon and s is the desired period. Here and elsewhere f_m denotes m^{th} iteration of the function f, i.e. $f_1(x) = f(x), f_m(x) = f(f_{m-1}(x))$. In contrast with the method of delayed feedback control (DFC, proposed originally by Pyragas for continuous-time systems and extended to discrete-time case in (Ushio, 1996)), which uses delayed iterations, (2) exploits predicted iterations of a point x_k. This allows to overcome many difficulties and limitations of DFC method and to validate stabilization effect of control (2) for ε small enough. In paper (Ushio and Yamamoto, 1999) a particular case of (2) with $m = 0$ has been proposed (further results can be found in (Hino, *et al.*, 2002)). However, in this case it is impossible to make ε small. The main feature of the function $u(x)$ is as follows: it can be done small enough for m large in uniform norm while its derivative becomes large. This changes the nature of the periodic orbit and affects it to be stable.

One of the possible applications of the proposed approach is not control itself, but checking conjectures on existence of periodic orbits for nonlinear iterations. It is very hard to find such periodic orbit, if it is unstable. By use of the algorithm we make the orbit stable and thus are able to detect it.

The paper is organized as follows. In Section 2 we analyze method (2) for one-dimensional case and report simulation results for such classical chaotic systems as logistic and tent ones. Section 3 is devoted to n-dimensional case; Henon map is considered as an example. Implementation issues are discussed in Section 4.

2. SCALAR CASE

In this section we consider one-dimensional nonlinear discrete-time open-loop system

$$x_{k+1} = f(x_k),\ x_k \in \mathbf{R^1},\ k = 1, \ldots. \qquad (3)$$

Let $x_1^*, x_2^*, \ldots, x_s^*$ be a *s-cycle (period s orbit)* of (3), that is $x_{i+1}^* = f_i(x_1^*), i = 1, \ldots, s-1, x_1^* = f_s(x_1^*)$. In particular $s = 1$ relates to a fixed point of f. In what follows we do not assume that the cycle is known, the only assumption is the existence of a cycle of period s. This information is often available in advance (see e.g. the famous Sharkovski's theorem on the ordering of cycles (Sharkovski, 1964)). The case of interest is an unstable cycle (*UPO – unstable periodic orbit*); our goal is to stabilize it by small control.

We suppose that the function f maps some bounded interval $[a, b]$ into itself and is differentiable: $f : [a, b] \to [a, b], f \in C^1$. The number $\mu = f'(x_s^*) \cdot \ldots \cdot f'(x_1^*)$ is called a *multiplicator* of the cycle. A sufficient condition for stability of the cycle (the cycle is an *attractor*) is $|\mu| < 1$, while a sufficient condition for instability of the cycle (the cycle is a *repeller*) is $|\mu| > 1$. We suppose that the cycle under consideration is unstable and $|\mu| > 1$. To stabilize it, we replace f in the right hand side of (3) by its correction, including an additive control term. Thus the closed-loop system becomes

$$x_{k+1} = F(x_k), \qquad (4)$$

$$F(x) = f(x) - \varepsilon(f_{(p+1)s+1}(x) - f_{ps+1}(x)),$$
$$\frac{|\varepsilon - \varepsilon^*|}{\varepsilon^*} < \frac{1}{|\mu|^{1/s}},\quad \varepsilon^* = \frac{1}{\mu^p(\mu-1)}, \qquad (5)$$

where p is an integer. Note that ε^* becomes arbitrary small when p is large enough, thus the control term has the same property, because f_m are bounded for all m and ε decreases simultaneously with ε^*.

Theorem 1. Suppose (3) has an unstable s-cycle with the multiplicator $\mu, |\mu| > 1$. Then the same cycle is stable for system (4) for any $p \geq 1$ and any ε, satisfying (5).

Proof A cycle $x_1^*, x_2^*, \ldots, x_s^*$ of f remains the cycle for f_m with any m, thus $F(x_i^*) = f(x_i^*) - \varepsilon(f_{p(s+1)+1}(x_i^*) - f_{ps+1}(x_i^*)) = x_{i+1}^*$ and this is also the cycle for F. Now calculate its multiplicator for (4): $\nu = F'(x_s^*) \cdot \ldots \cdot F'(x_1^*)$. Having in mind that $f_s'(x_i^*) = \mu, f_{ps}'(x_i^*) = \mu^p, f_{ps+1}'(x_i^*) = \mu^p f'(x_{i+1}^*)$, we get $F'(x_i^*) = (1 - \varepsilon\mu^p(\mu-1))f'(x_i^*)$. Multiplying these equalities for $i = 1, \ldots, s$ we arrive to the formula for the multiplicator of F:

$$\nu = (1 - \varepsilon\mu^p(\mu - 1))^s \mu. \qquad (6)$$

To verify the stability of the cycle, it suffices to show that $|\nu| < 1$. But indeed $|\nu| = |(1 - \varepsilon\mu^p(\mu-1))|^s|\mu| < |(1 - (\varepsilon^*(1 \pm (1/|\mu|^{1/s}))\mu^p(\mu - 1))|^s|\mu| = 1$, because the function $|1 - c\varepsilon|^s$ attains its maximum for the extreme values of ε. $\diamond$

A challenging issue related to the proposed control is its global behavior. Theorem 1 ensures local convergence only. However, if we apply the algorithm to stabilize chaotic motion, which has mixing property, then we can expect that beyond a neighborhood of the cycle the trajectory of the controlled system has mixing properties as well (F is close to f), so after some number of iterations it will arrive to the attracting neighborhood of the cycle. Simulation results confirm this conclusion for all cases. Let us consider two most popular examples of one-dimensional chaotic systems.

Example 1 – logistic map. Let

$$f(x) = \lambda x(1 - x),\ 0 \leq \lambda \leq 4. \qquad (7)$$

Then $f : [0, 1] \to [0, 1]$. The behavior of iterations (3) for this map is very well studied, see e.g. (Gumovsky and Mira, 1980). For $\lambda < 1$ the fixed point $x^* = 0$ is stable, for $1 < \lambda < 3$ another fixed point $x^* = 1 - 1/\lambda$ becomes stable, then after the bifurcation a stable 2-cycle arises and so on. What is important for our purposes: there exist s-cycles with any s for $\lambda > 3.84$, but all of them are unstable. The behavior of trajectories in this case is completely chaotic. Thus it is of interest to stabilize periodic orbits for λ close to 4; we set $\lambda = 3.9$. Simulation was performed as follows. We take 100 initial points x_0 on the uniform grid for [0, 1] and run K iterations of method (4), (5) with various s, p, μ; the points x_K are plotted. In (5) we take $\varepsilon = \varepsilon^* = 1/\mu^p(1 - \mu)$, where μ is calculated for the desired s-cycle as described in Section 4. We report just typical results. For $s = 1$ (stabilization of fixed points) the value of $\mu = 2 - \lambda = -1.9$ for $x^* = 1 - 1/\lambda = 0.7436$ can be calculated explicitly, and the method indeed globally stabilizes the desired fixed point very fast, see Figure 1 for $K = 150, p = 10, \varepsilon \simeq -0.0005$; the median of the number of iterations to achieve stabilization is $N = 50$. If we want to get smaller ε we should take larger p, for instance if $p = 20$

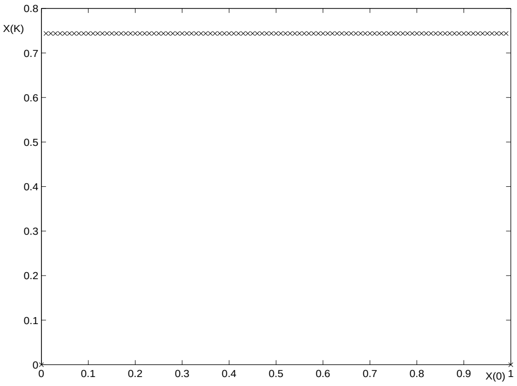

Fig. 1. Logistic map: stabilizing the fixed point

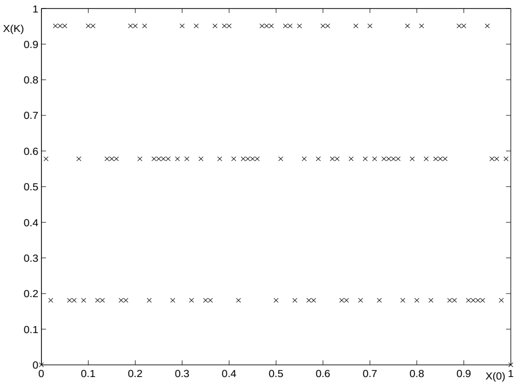

Fig. 2. Logistic map: stabilizing 3-cycle

then $\varepsilon \simeq 0.9 \cdot 10^{-6}$, however the number of iterations increases: $N = 320$. Stabilization of 2-cycle is also fast and simple, values of $p \simeq 15, \varepsilon \simeq 10^{-10}$ are possible. For $s = 3$ there are two 3-cycles, for the first $\mu = -5.17$ was estimated and $K = 1500, p = 5$ was taken to achieve global stabilization($N = 210, \varepsilon \simeq 4 \cdot 10^{-5}$), see Figure 2. For $s = 7$, two 7-cycles were stabilized (with $\mu = -90, \mu = 95$ respectively). However the large number of iterations was necessary to achieve global convergence: $K = 10000$. Larger periods were also detected, for instance eight 11-cycles were found.Here is one of them with $\mu = 871, p = 1, K = 15000, (x_1^*, \ldots, x_{11}^*)$=(0.8847, 0.3979, 0.9343, 0.2393, 0.7099, 0.8031, 0.6167, 0.9219, 0.2809, 0.7877, 0.6522). Probably, it is very hard to find this cycle analytically, so detecting such cycles is one of the useful applications of the stabilization algorithm. The record was $s = 31$, for $p = 0, \varepsilon \simeq 10^{-8}$ as many as 133 such cycles were detected.

Example 2 - tent map. Let

$$f(x) = \lambda(1 - |2x - 1|),\ 0 \le \lambda \le 1. \qquad (8)$$

Here also $f : [0, 1] \to [0, 1]$. Iterations of this map have much similarity with that of logistic map — it exhibits chaotic behavior for λ close to 1. However, there is an essential difference — all cycles of (8) are unstable for all $\lambda > 0.5$. Indeed, $|f'(x)| = 2\lambda > 1$ for any point $x \neq 0.5$, and $|\mu| = (2\lambda)^s > 1$ for any s-cycle if its points are not binary rational. Nevertheless it is possible to stabilize these cycles by control law (4), (5). Its application is very simple, because just values $\mu = \pm(2\lambda)^s$ should be considered. We choose $\lambda = 1$. In this case the values of n_s (the number of s-cycles) and corresponding values of multiplicators μ for them are known (Gumovsky and Mira, 1980), they are given below.

Table 1. Number of s-cycles and values of multiplicators

s	1	2	3	4	5	6
n_s	2	1	2	3	6	9
μ	± 2	-4	± 8	± 16	± 32	± 64

For $s = 1, \mu > 0$ the fixed point $x^* = 0$ is stabilized, for $s = 1, \mu < 0$ – the point $x^* = 2/3$. For the case $s = 2$ one 2-cycle is detected with $\mu < 0$, while for $s = 3$ and $s = 4$ – two cycles. It is possible to stabilize 5-cycles; all six of them were found (for each sign of μ three 5-cycles become stable simultaneously). The value of p was chosen to get $ps \sim 25$, then $\varepsilon \sim 10^{-8}$.

3. VECTOR CASE

We consider n-dimensional counterpart of (3):

$$x_{k+1} = f(x_k),\ x_k \in \mathbf{R}^n,\ k = 1, \ldots. \qquad (9)$$

The definition of s-cycle and the multiplicator remains the same, but now it is $n \times n$ Jacobian matrix $M = f'(x_s^*) \cdot \ldots \cdot f'(x_1^*)$. Note that multiplicator depends on ordering of points, i.e. which point in the cycle is chosen as the first one. For instance, if x_i^* is taken as the starting one, we get $M_i = f'(x_{i-1}^*) \cdot \ldots \cdot f'(x_i^*)$, were indices of the arguments are taken in the cyclical decreasing order $i-1, i-2, \ldots, 1, s, s-1, \ldots, i$; thus $M = M_1$ and in general $M_i \neq M_1, i \neq 1$. However the eigenvalues of the matrices $M_1, \ldots, M_s$ coincide (arbitrary matrices AB, BA have common eigenvalues: if $ABe = \lambda e$, then multiplying by B we have $BABe = \lambda Be, BAf = \lambda f, f = Be$). We denote $\mu_i, i = 1, \ldots, n$ the eigenvalues of any M_j. The cycle is stable if $\rho = \max_i |\mu_i| < 1$ and unstable if $\rho > 1$. We can also write M_i in the form $M_i = A_i B_i, A_i = f'(x_{i-1}^*) \cdot \ldots \cdot f'(x_1^*), B_i = f'(x_s^*) \cdot \ldots \cdot f'(x_i^*),\ A_1 = I, B_1 = M, B_i A_i = M$. The same control law as in scalar case is exploited

$$x_{k+1} = F(x_k), (10)$$
$$F(x) = f(x) - \varepsilon(f_{(p+1)s+1}(x) - f_{ps+1}(x)),$$
$$\frac{|\varepsilon - \varepsilon^*|}{\varepsilon^*} < \frac{1}{|\mu|^{1/s}},\quad \varepsilon^* = \frac{1}{\mu^p(\mu - 1)},$$

but the choice of μ is specified below. The simplest stabilization result reads as follows.

Theorem 2. Suppose (9) has an unstable s-cycle with the multiplicator $M, \rho > 1$. Assume $\mu_n = \mu$ is real, $|\mu| = \rho$, while $|\mu_i| < 1, i = 1, \dots, n-1$. Then the same cycle is stable for system (10) provided p is large enough.

Proof We follow the lines of the proof of Theorem 1; the difference arises because matrices are not commutative. To calculate the matrix multiplicator of F for the cycle $x_1^*, x_2^*, \dots, x_s^*$: $N = F'(x_s^*) \cdot \dots \cdot F'(x_1^*)$ we should calculate each term of the product. By using the chain rule for differentiation $f'_m(x_i^*) = f'_{m-1}(x_{i+1}^*) f'(x_i^*)$ and definition of multiplicators M_i we get $f'_{ps}(x_i^*) = M_i^p, f'_{ps+1}(x_i^*) = M_{i+1}^p f'(x_i^*) = f'(x_i^*) M_i^p, M_i^p = A_i M^{p-1} B_i$ and hence $F'(x_i^*) = f'(x_i^*)(I - \varepsilon A_i (M^p - M^{p-1}) B_i)$. By induction we easily obtain $F'(x_{i-1}^*) \cdot \dots \cdot F'(x_1^*) = A_i (I - \varepsilon M^p (M - I))^{i-1}$ and finally $N = F'(x_s^*) \cdot \dots \cdot F'(x_1^*) = A_{s+1}(I - \varepsilon M^p (M - I))^s = M(I - \varepsilon M^p (M - I))^s$. The eigenvalues ν_i of N can be expressed via the eigenvalues μ_i of M as $\nu_i = \mu_i (1 - \varepsilon \mu_i^p (\mu_i - 1))^s$. Now, for $i = n$ one has $\mu_n = \mu$ and due to (10) we have $|\nu_n| < 1$ as in the proof of Theorem 1. For $i \neq n$ we have $|\nu_i| \leq |\mu_i| (1 + \frac{|\mu_i|^p}{|\mu|^p} c_i)^s$, where c_i does not depend on p. But $|\mu_i| < 1$ under the assumption of the theorem, and $|\mu_i|^p / |\mu|^p$ tends to 0 when p increases. Thus $|\nu_i| < 1$ for p large enough. We conclude that $r = \max_{1 \leq i \leq n} |\nu_i| < 1$ for such p, that is the cycle is stable for F.$\diamond$

The assumption on location of eigenvalues of M can be relaxed — there exists a modification of the algorithm, which stabilizes an arbitrary unstable periodic orbit. However this version includes a matrix gain instead of the scalar gain ε.

Example 3 – Henon map. This is the classical 2-D example, originated at (Henon, 1976), see also (Mira, 1987):

$$y_{k+1} = 1 - 1.4 y_k^2 + z_k, \quad z_{k+1} = 0.3 y_k. \quad (11)$$

It is well known that the map has "strange attractor". Fig. 4 depicts an individual trajectory for some x_0, its complicated movement along points of the strange attractor is typical. It is known, that there exists the unstable fixed point $x^* = (0.6314, 0.1894)$, for this point the eigenvalues of matrix M are $(-1.92, 0.15)$, thus assumptions of Theorem 2 are satisfied with $\mu = -1.92$. There is one 2-cycle $x_1^* = (-0.4758, 0.2927), x_2^* = (0.9758, -0.1427)$, it is also unstable.

Figure 5 plots y-component of a typical trajectory when we stabilize the fixed point, $\mu = -1.92$. The fixed point possesses global stability.

Similar results are obtained for stabilization of 2-cycle ($s = 2, \mu = -3.01$), see Fig.6.

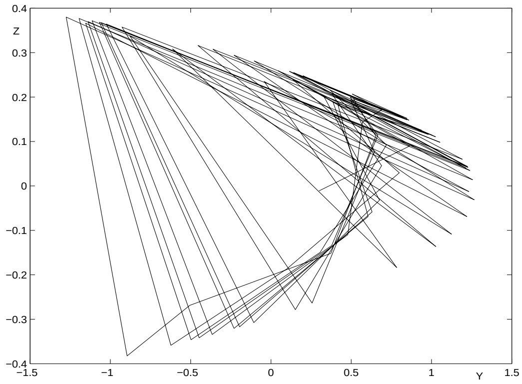

Fig. 3. Henon map: an individual trajectory

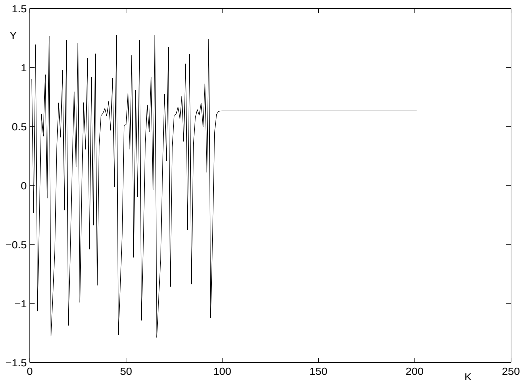

Fig. 4. Henon map: stabilizing the fixed point

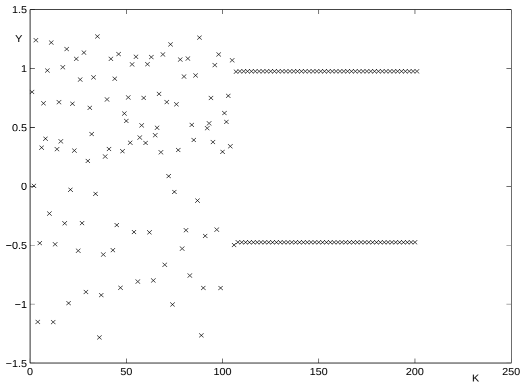

Fig. 5. Henon map: stabilizing 2-cycle

For $s = 4$ corresponding cycles and values of μ are not known. By trials it was found that $\mu = -9$ stabilizes 4-cycle. The results are presented at Fig. 6 (for a typical trajectory its last 20 iterations are shown on x plane, all iterations coincide with 4-cycle).

Note that in all these experiments the typical values were $\varepsilon \sim 10^{-4} - 10^{-5}$.

4. IMPLEMENTATION ISSUES

Choice of μ. In some examples above the value of the multiplicator of the cycle to be stabilized was known apriori or required minor calculations (fixed points or 2-cycles for the logistic map; any

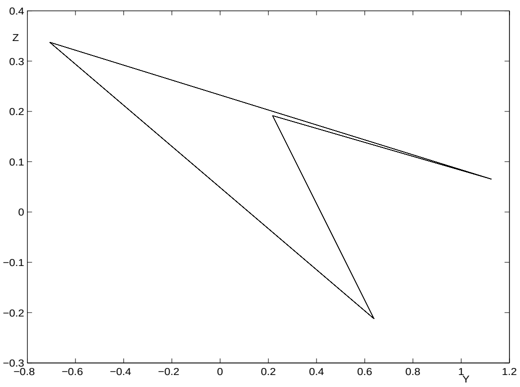

Fig. 6. Henon map: stabilizing 4-cycle, x-plane

cycles for the tent map). However sometimes the value of μ is not available (large s; the case of f with no analytic expression but given with some code etc.) Then it can be estimated. The estimates are especially simple for scalar case ($n = 1$). We introduce the function $g(x) = f_s(x) - x$ and calculate its values on a uniform grid $a = x_0 < x_1 < \ldots < x_N = b, x_{i+1} - x_i = d$ (it is assumed that the interval $S = [a, b], f : S \to S$ is known). Then we detect points of interchange of sign: $g(x_i)g(x_{i+1}) < 0$, they are candidates for zeros of g, i.e. for s-cycles of f. The points, which are t-cycles ($t < s$ being a divisor of s) are also zeros of g; they should be neglected. The quantities $(g(x_{i+1}) - g(x_i))/d$ are good estimates for μ provided that d is small enough.

For n-dimensional case this method can be revised. We can minimize the function $||g(x)||$ either on a grid or by any minimization procedure, say `fmin` in `Matlab`. Suppose that x_0 is one of local minima with $||g(x_0)|| \approx 0$. Then we perform m iterations $x_1 = f_s(x_0), \ldots, x_m = f_s(x_{m-1})$ and calculate $a = (x_m - x_{m-1}, x_{m-1} - x_{m-2}), r_1 = ||x_m - x_{m-1}||, r_2 = ||x_{m-1} - x_{m-2}||, q = a/(r_1 r_2)$. Then, if $|q|$ is close to 1, value a/r_2^2 is an estimate for μ.

Choice of p. Formulas (5), (10) ensure that larger is p smaller is ε. However there are some limitations for the growth of p, they are due to round off errors of computer arithmetic. Thus functions $f_m(x)$ can not be calculated precisely for large m. Let us illustrate this for some examples. For $f(x) = 4x(1-x)$ we have $f_m(0) = 0$ for any m, while $f_m(\epsilon) \approx 4^m\epsilon$ for small ϵ and m not too large. Thus an error in x, equal to the standard machine accuracy `eps`$=\epsilon = 2^{-52}$, causes an error in calculation of $f_m(x)$, equal to 2^{2m-52}, hence a reasonable value of m, which does not lead to dramatic consequences, is $m < 20$. In other situations the limitation is not so severe; if points $x_i, x_{i+1} = f(x_i), i = 1, \ldots, m$ are approximately uniformly distributed on [0,1], then $E|f'(x)| = 2$, and $E|f'_m(x)| = 2^m$, and reasonable limitation is $m < 40$. The same result holds for the tent map $f(x) = (1 - |2x-1|), |f'_m(x)| = 2^m$ for any x, m. We conclude that it is more or less safe to choose $sp \sim 25$, for these examples, and this rule was verified for all simulations.

Choice of K. As we mentioned above, Theorems 1, 2 validate just local stability of periodic orbits. Usually bigger is s and p, smaller is the basin of attraction of the controlled cycle. Due to chaotic nature of the trajectories they get this basin, but after larger number of iterations K. This effect explains why larger K are required to achieve stabilization when s, p increase. For instance, $K = 10000$ iterations were needed in Example 1 for global stabilization of the 7-cycle, while $K = 150$ was sufficient for stabilization of the fixed point.

5. CONCLUSIONS

We provided a simple and effective method for stabilization of unstable s-cycles of nonlinear discrete-time systems by use of small control. It is based on prediction of a current point on m and $m + s$ iterations forward, where m is of the form $ps+1, p$ being large enough. The main assumption is a possibility to perform this prediction precisely enough, that is the function $f(x)$ should be known (or given by an algorithm) and no disturbances are allowed. The method can be also used for detecting of unknown periodic orbits of nonlinear maps.

The extension of the approach for differential equations and for synchronization of chaotic oscillators will be reported later.

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