IMPROVED DELAY-DEPENDENT ROBUST STABILITY CRITERIA FOR TIME DELAY SYSTEMS

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Abstract: This paper proposes novel robust stability and robust stabilizability conditions for time delay systems with time-varying delay and nonlinear time-varying perturbations. Using the traditional quadratic approach with Lyapunov-Krasovskii functional in which the Lyapunov-Krasovskii matrices are independent from the time as well as from the nonlinear perturbations, some delay-dependent robust stability and robust stabilizability criteria are derived. The main contribution of this research is to give quadratic-based conditions for stability analysis and controller synthesis of time delay systems without using any bound for cross-terms presented in the time derivative equation. This allows the proposed stability and stabilizability criteria to provide less conservative results than that of reported methods which often employ bounding of the cross-terms. Several examples are considered from the literature to illustrate the effectiveness of the proposed method.

Keywords:Time-delay, nonlinear perturbations, robust stability, robust stabilizability, linear matrix inequalities.

1. INTRODUCTION

Time delays which naturally exist in dynamical physical systems may cause instability and deteriorated performance. This situation has triggered a plenty of attention devoted to the study of various engineering, process control, biological and economical systems which are mathematically modeled with delay-differential systems as the speed of information processing is finite. The stability analysis of delay-differential systems have been widely investigated. For a consolidated work in this area, one can refer to (Hale and Lunel, 1993; Niculescu, 2001; Gu, et al., 2003). Moreover, some recent advances on time-delay systems are well documented in (Niculescu and Gu, 2004). There exist also some survey studies (Kolmanovskii, 1999) and (Richard, 2003) in which motivations to study time delay systems and some recent progresses on stability and robust stability of time delay systems are presented. In general, the stability analysis of systems with time-delay can be classified into two categories in accordance with the dependence of the stability result's upon the size of the delay time. In other words, the so-called delay-dependent stability criteria carry delay information while the so-called delay-independent stability criteria do not include any information on the size of the delay.

The stability of time delay systems with nonlinear perturbations has been widely studied in the literature, see for example (Su and Huang, 1991; Xu, 1994; Su, 1994) and the references therein. Based on induced norms and matrix measures, (Oucheriah, 1995) presented a delay-dependent sufficient condition that guarantees the robust stability of linear uncertain time-delay systems. A delay-dependent robust stability criterion is given by (Gu, et al., 1998) for a class of linear uncertain systems with time-varying delay. The robust stability of interval time-delay systems with delay-dependence is investigated by (Liu and Su, 1998). Employing the quadratic stability theory based on checking the Hamiltonian matrix and solving an algebraic Riccati equation, (Yan, 2001) derived an upper bound on the size of the delay with a delay-dependent stability criterion. (Kim, 2001) studied the robust stability of time-delayed linear systems with time-varying and norm-bounded uncertainties. Using Lyapunov method and quadratic stability theory, an upper delay bound is presented by (Cao and Wang, 2004) based on the Hamiltonian matrix and an algebraic Riccati equation.

In this paper, the problem of robust stability and robust stabilizability analysis is considered for a class of systems with time-varying delay and nonlinear perturbations. Lyapunov method and quadratic stability theory are employed to obtain some new delay-dependent robust stability and robust stabilizability criteria which are presented in the form of solvable linear matrix inequalities through interior-point algorithms (Boyd et al.,1994). The nonquadratic, so-called cross-terms are usually bounded in the quadratic stability analysis. For example, for some vectors $x \in \Re^{n\times 1}$, $y \in \Re^{n\times 1}$, a cross term like $\pm 2x^T y$ is bounded with $x^T x + y^T y$ and it is often replaced with $x^T x + y^T y$ in the stability analysis. Bounding of the cross terms as described with the former example, is in fact an application of a worst case scenario which may not always take place in practice. However, to a certain extent it will do induce some conservatism to the robustness bound on the time delay. From the point of this view, the proposed method is quite different from the existing approaches. No bounding of the cross terms is employed in the derivation of the robust stability and stabilizability results. Thus, they have the potential advantage of being less conservative than those of existing methods that make use of bounding for the cross-terms. Moreover, the proposed method is not based on the solution of any Lyapunov function and/or algebraic Riccati equation. As a result, there exist no need of parameter tuning for which an optimal procedure is often unavailable. Finally, four numerical examples are introduced to illustrate the effectiveness of our method.

2. PROBLEM STATEMENT

Let us consider a class of time delay systems with time-varying delay and nonlinear perturbations described with the following differential equation

$$\begin{aligned} \dot{x}(t) &= Ax(t) + A_d x(t-\tau(t)) + Bu(t) \\ &\quad + f(t, x(t)) + g(t, x(t-\tau(t))) \end{aligned} \tag{1}$$

$$x(t) = \Phi(t), \quad \forall t \in [-\bar{\tau}, 0], \quad \bar{\tau} > 0 \tag{2}$$

where $x(t) \in \Re^n$ is the system state, $u(t) \in \Re^m$ is the control input, A, $A_d \in \Re^{n\times n}$, $B \in \Re^{n\times m}$ are the system matrices which are constant and known, $\tau(t) > 0$ is a continuous time-varying function representing the time-delay and satisfying $0 \le \tau(t) \le \bar{\tau}$ and $\dot{\tau}(t) \le \delta \le 1$, $\bar{\tau}$ and δ are constant nonnegative scalars, the time-varying nonlinear continuous functions $f(t, x(t))$ and $g(t, x(t-\tau(t)))$ are unknown and represent the nonlinear parameter perturbations with respect to the current state $x(t)$ and the delayed state $x(t-\tau(t))$, respectively. These nonlinear parameter perturbations may be structured or unstructured and satisfy $f(t,0)=0$ and $g(t,0)=0$ for any $t \ge 0$. We assume that some bounds are available satisfying the following inequalities,

$$\|f(t, x(t))\| \le \alpha \|x(t)\|, \qquad \alpha > 0 \tag{3}$$

$$\|g(t, x(t-\tau(t)))\| \le \beta \|x(t-\tau(t))\|, \quad \beta > 0 \tag{4}$$

where positive scalars α and β are known. Moreover, the existence and uniqueness of the solution of system (1), (2) is assumed. The aim is to derive delay-dependent robust stability and robust stabilizability criteria such that the time delay system (1), (2) with nonlinear perturbations satisfying (3), (4) is quaranteed to be robustly asymptotically stable. We use Leibniz-Newton model transformation and Lyapunov-Krasovskii functional method combined with linear matrix inequalities approach. We do not bound any cross-term in the stability analysis.

3. MAIN RESULTS

In this section, we present delay-dependent robust stability analysis and delay-dependent robust controller synthesis for the time-delay system with nonlinear perturbations (1)-(2).

3. 1 Delay-dependent robust stability criteria

We first assume that system (1), (2) is unforced, i.e. $u(t) \equiv 0$. Since $x(t)$ is continuously differentiable for $t \ge 0$, we use a well-known model transformation (Richard, 2003), so-called Leibniz-Newton formula to get

$$\begin{aligned} x(t-\tau(t)) &= x(t) - \int_{t-\tau(t)}^{t} \dot{x}(s)ds = x(t) - \int_{t-\tau(t)}^{t} [Ax(s) \\ &\quad + A_d x(s-\tau(s)) + f(s, x(s)) + g(s, x(s-\tau(s)))]ds \end{aligned}$$

for $t \ge 0$. Then system (1), (2) can be reexpressed as

$$\begin{aligned} \dot{x}(t) &= (A + A_d)x(t) - A_d \int_{t-\tau(t)}^{t} [Ax(s) + A_d x(s-\tau(s)) \\ &\quad + f(s, x(s)) + g(s, x(s-\tau(s)))]ds \\ &\quad + f(t, x(t)) + g(t, x(t-\tau(t))) \end{aligned} \tag{5}$$

$$x(t) = \Psi(t), \quad \forall t \in [-2\bar{\tau}, 0], \quad \bar{\tau} > 0 \tag{6}$$

Remark 1: The stability of the original system (1), (2) is ensured by the global uniform asymptotic stability of the transformed system (5), (6), (Hale and Lunel, 1993).

Remark 2: The Leibniz-Newton model transformation transforms a system (1), (2) with discrete delay to a system with distributed delay (5), (6). It is well known (Gu and Niculescu, 2000) that the eigenvalues of the transformed system consists of those of the original system and additional eigenvalues. If the critical delay value that causes one of the additional eigenvalues cross the imaginary axis, is less than the stability delay limit of the original system, then any stability criteria obtained using such transformation will be conservative.

Here we assume that system (1) without time delay and nonlinear perturbations is asymptotically stable, i.e. $A + A_d$ is Hurwitz stable. The stability results are summarized in Theorem 1.

Theorem 1: Consider the time-delay system (1), (2), and the nonlinear time-varying perturbations satisfying (3), (4). Then given scalars $\bar{\tau} > 0$, $\delta \le 1$, this system is robustly stable for any time delay satisfying $0 \le \tau(t) \le \bar{\tau}$, $\dot{\tau}(t) \le \delta$ if there exist symmetric and positive definite matrices P, Q, R, S and Z such that the following LMI set is satisfied

$$\Omega = \begin{bmatrix} \Omega_{11} & 0 & P & P \\ * & -(1-\delta)S + \beta \mathrm{I} & 0 & 0 \\ * & * & -\mathrm{I} & 0 \\ * & * & * & -\mathrm{I} \end{bmatrix} < 0 \quad (7)$$

$$\Psi = \begin{bmatrix} Z & PA_d A & PA_d A_d & PA_d & PA_d \\ * & \Psi_{22} & 0 & 0 & 0 \\ * & * & \Psi_{33} & 0 & 0 \\ * & * & * & \mathrm{I} & 0 \\ * & * & * & * & \mathrm{I} \end{bmatrix} \ge 0 \quad (8)$$

where $\Omega_{11} = (A + A_d)^T P + P(A + A_d) + S + \alpha \mathrm{I}$ $+ \bar{\tau}(Q + R + Z)$, $\Psi_{22} = (1-\delta)Q - \delta R - \alpha \mathrm{I}$, $\Psi_{33} = (1-2\delta)R - \beta \mathrm{I}$, I is the identity matrix.

Proof: Let us choose a candidate Lyapunov-Krasovskii functional in the following form

$$V(t, x(t)) = \sum_{i=1}^{4} V_i(x(t)) \quad (9)$$

with

$$V_1(x(t)) = x^T(t)Px(t) \quad (10)$$

$$V_2(t, x(t)) = \int_{-\tau(t)}^{0} \int_{t+\theta}^{t} x^T(s)Qx(s)dsd\theta \quad (11)$$

$$V_3(t, x(t)) = \int_{-\tau(t)}^{0} \int_{t-\tau(t)+\theta}^{t} x^T(s)Rx(s)dsd\theta \quad (12)$$

$$V_4(t, x(t)) = \int_{t-\tau(t)}^{t} x^T(s)Sx(s)ds \quad (13)$$

where P, Q, R and S are symmetric and positive definite matrices with appropriate dimensions. Note that here we employ the traditional quadratic approach with Lyapunov-Krasovskii functional in which the Lyapunov-Krasovskii matrices are independent from the time as well as from the nonlinear perturbations Then we take the time derivative of (9) along the state trajectory of (5), (6)

$$\dot{V}(t, x(t)) = \sum_{i=1}^{4} \dot{V}_i(x(t)) \quad (14)$$

We can compute $\dot{V}_i(x(t))$ $(i = 1, \ldots 4)$ as follows

$$\dot{V}_1(x(t)) = \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t = x^T(t)\left[(A + A_d)^T P + P(A + A_d)\right]x(t) - 2x^T(t)PA_d \int_{t-\tau(t)}^{t} \left[Ax(s) + A_d x(s - \tau(s)) + f(s, x(s)) + g(s, x(s - \tau(s)))\right]ds + 2x^T(t)Pf(t, x(t)) + 2x^T(t)Pg(t, x(t - \tau(t))) \quad (15)$$

$$\dot{V}_2(t, x(t)) = \frac{d}{dt}\left\{ \int_{-\tau(t)}^{0} \left[F_1(t) - F_1(t+\theta)\right]d\theta \right\} = \frac{d}{dt}\left[\tau(t)F_1(t)\right] - \frac{d}{dt}\left[F_2(t) - F_2(t - \tau(t))\right] = \tau(t)\dot{F}_1(t) + \dot{\tau}(t)F_1(t) - \left\{F_1(t) - \left[1 - \dot{\tau}(t)\right]F_1(t - \tau(t))\right\} = \tau(t)F_0(t) - \left[1 - \dot{\tau}(t)\right] \times \left[F_1(t) - F_1(t - \tau(t))\right] = \tau(t)F_0(t) - \left[1 - \dot{\tau}(t)\right] \int_{t-\tau(t)}^{t} F_0(s)ds = \tau(t)x^T(t)Qx(t) - \left[1 - \dot{\tau}(t)\right] \int_{t-\tau(t)}^{t} x^T(s)Qx(s)ds \quad (16)$$

where $x^T(\cdot)Qx(\cdot) = F_0(\cdot) = \dot{F}_1(\cdot)$ and $F_1(\cdot) = \dot{F}_2(\cdot)$, and

$$\dot{V}_3(t, x(t)) = \frac{d}{dt}\left\{ \int_{-\tau(t)}^{0} \left[G_1(t) - G_1(t - \tau(t) + \theta)\right]d\theta \right\} = \frac{d}{dt}\left[\tau(t)G_1(t)\right] - \frac{d}{dt}\left[G_2(t - \tau(t)) - G_2(t - 2\tau(t))\right] = \tau(t)\dot{G}_1(t) + \dot{\tau}(t)G_1(t) - \left[1 - \dot{\tau}(t)\right]G_1(t - \tau(t)) + \left[1 - 2\dot{\tau}(t)\right]G_1(t - 2\tau(t)) = \tau(t)G_0(t) + \dot{\tau}(t)\left[G_1(t) - G_1(t - \tau(t))\right] - \left[1 - 2\dot{\tau}(t)\right]\left[G_1(t - \tau(t)) - G_1(t - 2\tau(t))\right] = \tau(t)G_0(t) + \dot{\tau}(t) \int_{t-\tau(t)}^{t} G_0(s)ds - \left[1 - 2\dot{\tau}(t)\right] \int_{t-\tau(t)}^{t} G_0(s - \tau(s))ds = \tau(t)\,x^T(t)Rx(t) + \dot{\tau}(t) \int_{t-\tau(t)}^{t} x^T(s)Rx(s)ds - \left[1 - 2\dot{\tau}(t)\right] \int_{t-\tau(t)}^{t} x^T(s - \tau(s))Rx(s - \tau(s))ds \quad (17)$$

where $x^T(\cdot)Rx(\cdot) = G_0(\cdot) = \dot{G}_1(\cdot)$ and $G_1(\cdot) = \dot{G}_2(\cdot)$,

$$\dot{V}_4(t, x(t)) = x^T(t)Sx(t) - \left[1 - \dot{\tau}(t)\right]x^T(t - \tau(t))Sx(t - \tau(t)) \quad (18)$$

Substituting (15)-(18) into (14) and adding and subtracting some quadratic terms gives

$$\begin{aligned} \dot{V}(t, x(t)) \le\; & x^T(t)\left[(A + A_d)^T P + P(A + A_d) + S + \bar{\tau}(Q + R + Z)\right]x(t) - (1-\delta)x^T(t - \tau(t))Sx(t - \tau(t)) \\ & - \int_{t-\tau(t)}^{t} \left[x^T(t)Zx(t) + 2x^T(t)PA_d Ax(s) + 2x^T(t)PA_d A_d x(s - \tau(s)) + 2x^T(t)PA_d f(s, x(s)) \right. \\ & + 2x^T(t)PA_d g(s, x(s - \tau(s))) + x^T(s)\left[(1-\delta)Q - \delta R\right] \times x(s) + (1-2\delta)x^T(s - \tau(s))Rx(s - \tau(s)) \\ & + f^T(s, x(s))f(s, x(s)) - f^T(s, x(s))f(s, x(s)) + g^T(s, x(s - \tau(s)))g(s, x(s - \tau(s))) \\ & \left. - g^T(s, x(s - \tau(s)))g(s, x(s - \tau(s)))\right]ds + 2x^T(t)Pf(t, x(t)) + 2x^T(t)Pg(t, x(t - \tau(t))) \\ & + f^T(t, x(t))f(t, x(t)) - f^T(t, x(t))f(t, x(t)) + g^T(t, x(t - \tau(t)))g(t, x(t - \tau(t))) \\ & - g^T(t, x(t - \tau(t)))g(t, x(t - \tau(t))) \end{aligned} \quad (19)$$

where Z is a symmetric and positive definite matrix. Using (3), (4) we rearrange $\dot{V}(t, x(t))$ in (19) to get

$$\dot{V}(t, x(t)) \le \chi^T(t)\Omega\chi(t) - \int_{t-\tau(t)}^{t} \zeta^T(t, s)\Psi\zeta(t, s)ds \quad (20)$$

where $\chi(t) = \left[x^T(t) \quad x^T(t - \tau(t)) \quad f^T(t, x(t)) \quad g^T(t, x(t - \tau(t)))\right]^T$,

and

$\zeta = \left[x^T(t) \quad x^T(s) \quad x^T(s - \tau(s)) \quad f^T(s, x(s)) \quad g^T(s, x(s - \tau(s)))\right]^T$,

Ω and Ψ are as defined in (7), (8). Therefore, if there exist symmetric and positive definite matrices P, Q, R, S and Z and positive scalar $\bar{\tau}$, δ satisfying (7), (8) then we have

$$\dot{V}(t,x(t)) \le -\lambda_{\min}(-\Omega)\|\chi(t)\|^2 \quad (21)$$

It follows from (21) that system (5), (6) is robustly globally asymptotically stable for any time delay satisfying $0 \le \tau(t) \le \bar{\tau}$ and $\dot{\tau}(t) \le \delta \le 1$. Hence the proof is completed.

Remark 3: Note that no bounding on the cross terms in the time-derivative equation (14) is used, thus reducing the conservatism of the approach.

If system (1), (2) does not involve any nonlinear perturbations, we have the following result.

Corollary 1: Consider the nominal system (1) and (2) with $f(t,x(t)) = 0$, and $g(t,x(t-\tau(t))) = 0$, then given positive scalars $\bar{\tau}$, δ, this system is globally asymptotically stable for any time-delay satisfying $0 \le \tau(t) \le \bar{\tau}$, $\dot{\tau}(t) \le \delta \le 1$ if there exist symmetric and positive definite matrices P, Q, R, S and Z solving the following LMI set

$$\begin{bmatrix} (A+A_d)^T P + P(A+A_d) + \bar{\tau}(Q+R+Z) + S & 0 \\ * & -(1-\delta)S \end{bmatrix} < 0 \quad (22)$$

$$\begin{bmatrix} Z & PA_d A & PA_d A_d \\ * & (1-\delta)Q - \delta R & 0 \\ * & * & (1-2\delta)R \end{bmatrix} \ge 0 \quad (23)$$

Proof: Similar to the proof of Theorem 1. Thus it can be omitted.

3. 2 Delay-dependent robust stabilizability criteria

In order to investigate the delay-dependent robust stabilizability of the time-delay system (1), (2), we consider a state-feedback controller chosen as

$$u(t) = Kx(t) \quad (24)$$

Substituting this control law into (1) yields

$$\dot{x}(t) = A_c x(t) + A_d x(t-\tau(t)) + f(t,x(t)) + g(t,x(t-\tau(t))) \quad (25)$$

where $A_c = A + BK$, K is a design matrix to be selected appropriately. The robust stabilizability criteria are summarized in Theorem 2.

Theorem 2: Consider the time-delay system (1), (2) with the controller (24) and the nonlinear time-varying perturbations satisfying (3), (4). Then given scalars $\bar{\tau}$, δ, the system (25) is robustly stabilizable for any time delay satisfying $0 \le \tau(t) \le \bar{\tau}$, $\dot{\tau}(t) \le \delta \le 1$ if there exist symmetric and positive definite matrices X, $\bar{Q}$, $\bar{R}$, $\bar{S}$ and $\bar{Z}$ such that the following LMI set is satisfied

$$\Pi = \begin{bmatrix} \Pi_{11} & 0 & I & I & \sqrt{\alpha}X & 0 \\ * & \Pi_{22} & 0 & 0 & 0 & \sqrt{\beta}X \\ * & * & -I & 0 & 0 & 0 \\ * & * & * & -I & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \quad (26)$$

$$\Sigma = \begin{bmatrix} \bar{Z} & \Sigma_{12} & A_d A_d X & A_d & A_d & 0 & 0 \\ * & \Sigma_{22} & 0 & 0 & 0 & \sqrt{\alpha}X & 0 \\ * & * & \Sigma_{33} & 0 & 0 & 0 & \sqrt{\beta}X \\ * & * & * & I & 0 & 0 & 0 \\ * & * & * & * & I & 0 & 0 \\ * & * & * & * & * & I & 0 \\ * & * & * & * & * & * & I \end{bmatrix} \ge 0 \quad (27)$$

where
$\Pi_{11} = X(A+A_d)^T + (A+A_d)X + BY + Y^T B^T + \bar{S} + \bar{\tau}(\bar{Q}+\bar{R}+\bar{Z})$, $\Pi_{22} = -(1-\delta)\bar{S}$,
$\Sigma_{12} = A_d AX + A_d BY$, $\Sigma_{22} = (1-\delta)\bar{Q} - \delta\bar{R}$,
$\Sigma_{33} = (1-2\delta)\bar{R}$.

Proof: We choose a symmetric and positive-definite matrix X such that $X = P^{-1}$ where P is defined I,n Theorem 1. Then we pre- and post- multiply (7), (8) with $\mathrm{diag}\{X, X, I, I\}$, $\mathrm{diag}\{X, X, X, I, I\}$, respectively to get

$$\Gamma = \begin{bmatrix} \Gamma_{11} & 0 & I & I \\ * & \Gamma_{22} & 0 & 0 \\ * & * & -I & 0 \\ * & * & * & -I \end{bmatrix} < 0 \quad (28)$$

$$\Xi = \begin{bmatrix} \bar{Z} & A_d AX & A_d A_d X & A_d & A_d \\ * & \Xi_{22} & 0 & 0 & 0 \\ * & * & \Xi_{33} & 0 & 0 \\ * & * & * & I & 0 \\ * & * & * & * & I \end{bmatrix} \ge 0 \quad (29)$$

where $\Gamma_{11} = X(A+A_d)^T + (A+A_d)X + \bar{S} + \alpha XX + \bar{\tau}(\bar{Q}+\bar{R}+\bar{Z})$, $\Gamma_{22} = -(1-\delta)\bar{S} + \beta XX$,
$\Xi_{22} = (1-\delta)\bar{Q} - \delta\bar{R} - \alpha XX$, $\Xi_{33} = (1-2\delta)\bar{R} - \beta XX$.
We replace A in (28), (29) with A_c by choosing $K = YX^{-1}$ with Y being an arbitrary matrix to be selected. Finally applying Schur's complement yields the LMIs given in (26), (27). This completes the proof.

Remark 4: It follows from the stabilizability results given in Theorem 2 that the controller synthesis based on the stability analysis proposed in Theorem 1 is shown to be convex for the presented approach.

The case when there exist no nonlinear perturbations in system (25) is considered in Corollary 2.

Corollary 2: Consider the nominal system (25) with $f(t,x(t)) = 0$, and $g(t,x(t-\tau(t))) = 0$, then given positive scalars $\bar{\tau}$, δ, this system is globally asymptotically stabilizable for any time-delay satisfying $0 \le \tau(t) \le \bar{\tau}$, $\dot{\tau}(t) \le \delta \le 1$ if there exist symmetric and positive definite matrices P, Q, R, S and Z such that the following LMI set is satisfied

$$\begin{bmatrix} X(A+A_d)^T+(A+A_d)X \\ +BY+Y^TB^T+\bar{S} & 0 & \mathrm{I} & \mathrm{I} \\ +\bar{\tau}(\bar{Q}+\bar{R}+\bar{Z}) & & & \\ * & -(1-\delta)\bar{S} & 0 & 0 \\ * & * & -\mathrm{I} & 0 \\ * & * & * & -\mathrm{I} \end{bmatrix} < 0 \tag{30}$$

$$\begin{bmatrix} \bar{Z} & A_dAX+A_dBY & A_dA_dX & A_d & A_d \\ * & (1-\delta)\bar{Q}-\delta\bar{R} & 0 & 0 & 0 \\ * & * & (1-2\delta)\bar{R} & 0 & 0 \\ * & * & * & \mathrm{I} & 0 \\ * & * & * & * & \mathrm{I} \end{bmatrix} \geq 0 \tag{31}$$

Proof: As the proof follows from the proof of Theorem 2, it is omitted.

4. NUMERICAL EXAMPLES

In this section, we consider four numerical examples which are chosen from the existing work in the literature.

Example 1: Let us consider the following time delay system with constant time-delay and nonlinear time-varying perturbations

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) + f(t,x(t)) + g(t,x(t-\tau)) \tag{32}$$

with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

$$f(t,x(t)) = \begin{bmatrix} \delta_1 \cos(t)|x_1(t)| & \delta_2 \sin(t)|x_2(t)| \end{bmatrix}^T$$

$$g(t,x(t-\tau)) = \begin{bmatrix} \gamma_1 \cos(t)|x_1(t-\tau)| & \gamma_2 \sin(t)|x_2(t-\tau)| \end{bmatrix}^T$$

where $|\delta_i| \leq \alpha = 0.05$ and $|\gamma_i| \leq \beta = 0.1$ $(i=1,2)$. It is immediately noticed that due to the absolute value function involved in $f(t,x(t))$ and $g(t,x(t-\tau))$, the nonlinear time-varying parameter perturbations are not linear with respect to state $x(t)$ and delayed state $x(t-\tau)$, i.e. neither $f(t,x(t))$ nor $g(t,x(t-\tau))$ can be represented in the form of $\Delta A(t)x(t)$ and $\Delta B(t)x(t-\tau)$, respectively. As the method proposed in (Su, 1994; Gu et al., 1998); Liu and Su, 1998; Kim, 2001) consider the uncertainties to be defined linearly with respect to state $x(t)$ and delayed state $x(t-\tau)$, they are not applicable for this particular example. Applying the condition in Theorem 1 given by (Yan, 2001), it is easy to construct a Hamiltonian matrix H with $\gamma = 0.4987$. Then it is stated that H has no eigenvalues on the imaginary axis and the system is robustly stable when $\tau < 0.3102$. Finally the stability robustness bound which is obtained by (Cao and Wang, 2004) is 0.4332. Solving the LMI's (7)-(8) in Theorem 1, we conclude that system (32) is robustly globally asymptotically stable for any time delay satisfying $0 \leq \tau \leq 0.4368$. This result is less conservative than that of (Cao and Wang, 2004).

Example 2: Let us consider the special case of the time delay system (1), (2), that is a nominal time-delay system with constant time-delay, i.e. $f(t,x(t)) = 0$ and $g(t,x(t-\tau)) = 0$,

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) \tag{33}$$

with

$$A = \begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix}, \quad A_d = \begin{bmatrix} -0.5 & 0.1 \\ 0.3 & 0 \end{bmatrix}$$

It is shown in (Cao and Wang, 2004) that since some of the preconditions of which are related to the system dynamics, i.e.

$$\mu_1(A+B) = 1.9 > 0, \quad \mu_2(A+B) = 0.0255 > 0$$

$$\mu_\infty(A+B) = 1.3 > 0$$

are not satisfied, a bound on τ can not be found to guarantee the stability of the system by using the stability criteria proposed in (Oucheriah, 1995), (Liu and Su, 1998). Thus these methods are not applicable for this particular example. However, (Xu, 1994) computed a robust delay bound of 0.0667 while the allowable time delay is increased up to 0.1298 by (Su, 1994). Moreover, the robust stability bound obtained in (Yan, 2001) is 0.4991. Using the stability condition given by Kim in Corollary 1 of (Kim, 2001), the robust delay bound is computed as 1.7828. Finally, (Cao and Wang, 2004) showed that the robustness bound on the time delay is obtained as 0.7062 by their method. Applying Corollary 1 of this paper and solving the LMI's (22), (23) with LMI Control Toolbox shows that the global asymptotical stability of system (33) is ensured for any time-delay with the upper bound of 1.7953. This result is less conservative than that of the stability bound given in (Kim, 2001) and (Cao and Wang, 2004).

Example 3: Let us consider another constant time-delayed system with nonlinear perturbations given in the form of

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau) + \Delta f(t,x(t),x(t-\tau)) \tag{34}$$

with

$$A = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}$$

and

$$\Delta f(t,x(t),x(t-\tau)) = \Delta A(t)x(t) + \Delta B(t)x(t-\tau)$$

where

$$\Delta A(t) = \begin{bmatrix} 0.3\cos(t) & 0 \\ 0 & 0.2\sin(t) \end{bmatrix}$$

$$\Delta B(t) = \begin{bmatrix} 0.2\cos(t) & 0 \\ 0 & 0.3\sin(t) \end{bmatrix}$$

thus it is clear that we have $\alpha = \beta = 0.3$. (Xu, 1994; Gu, et al., 1998) have obtained an allowable delay bound of 0.1575 while (Su and Huang, 1991) computed the robust stability bound for the time delay as 0.1614. It is shown by (Oucheriah, 1995) that robustness bound can be raised up to 0.1892. Finally, (Liu and Su, 1998) achieved a robust stability bound of 0.2103. Employing the robust stability criteria (7)-(8) given in Theorem 1, it is easily concluded that the allowable robustness bound on the delay time is calculated as 0.2442. This shows an increment of %16.1198 when compared with the result of (Liu and Su, 1998).

Example 4: Let us consider the time-delay system

$$\dot{x}(t) = Ax(t) + A_d x(t-\tau(t)) + Bu(t) + f(t, x(t)) + g(t, x(t-\tau(t))) \quad (35)$$

where $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A_d = \begin{bmatrix} -1 & -1 \\ 0 & -0.9 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

and $f(t, x(t))$, $g(t, x(t-\tau(t)))$ are similarly defined as in Example 1 with time-varying time-delay. Note that this system with $\tau(t) = 0$, $f(t, x(t)) = 0$, $g(t, x(t-\tau(t))) = 0$ is not asymptotically stable and the pair (A, B) is not stabilizable. However, the pair $(A + A_d, B)$ is stabilizable, thus the stabilizability of system (35) is unavoidably delay-dependent. The methods of (Li and De Souza, 1997a, b) cannot be applied to this particular system because of the same reasoning about $f(t, x(t))$ and $g(t, x(t-\tau(t)))$ discussed in Example 1. When the time-delay is time-invariant, i.e. $\delta = 0$, applying Theorem 2 and solving (26), (27) gives a delay bound of 0.2894. If the bound on the derivative of the time-varying time-delay is $\delta = 0.1$, the robustness bound on the time delay is obtained as 0.1689.

5. CONCLUSIONS

Some novel delay-dependent robust stability and stabilizability criteria are introduced for time-delay systems with time-varying delay and nonlinear perturbations by employing the Lyapunov method and quadratic stability theory. The nonquadratic or so-called cross terms are not replaced with their quadratic bounds in the stability and stabilizability analysis. Thus the robust stability and stabilizability criteria are potentially less conservative when compared with some existing methods which usually bound the cross-terms. Moreover, unlike some Lyapunov or Riccati equation approaches, there is also no need for the tuning of any parameters in the proposed method. Based on the solution of linear matrix inequalities approach with interior-point algorithms, the upper robustness bound on the size of the time delay are easily computed for four numerical examples. The achieved results demonstrate that the proposed method is capable of giving less conservative robustness bounds on the time delay than that of some reported methods in the literature.

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