ON THE ROBUST OUTPUT CONTROL VIA REFLECTION VECTORS

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Abstract: A robust version of the output controller design for discrete-time systems is introduced. Instead of a single stable point a stable polytope (or simplex) is defined in the closed loop characteristic polynomials coefficients space. A constructive procedure for generating simplexis inside the "nicely stable" region is given starting from the unit hypercube of reflection coefficients of monic polynomials. This procedure is quite straightforward because for a special family of polynomials the linear cover of so-called reflection vectors is stable. The roots placement of reflection vectors is studied. If the stable simplex is preselected then the robust output controller design task is solved by quadratic programming approach. If the stable simplex (or polytope) of reflection vectors is not given then a simple search procedure is needed.

Keywords: robust control, pole placement, stability, discrete-time systems.

1. INTRODUCTION

The modal control or pole placement method is a common approach for designing closed-loop controllers in order to meet desired control specifications. The objective of assigning closed loop poles is often replaced by assigning a characteristic polynomial because 1) this polynomial plays a central role in the stability analysis of linear control systems and 2) polynomial coefficients are simply (affinely) related to controller and plant parameters .

Another practical issue is that of model uncertainty. If the model uncertainty is relatively small, then it is possible to use sensitivity-based methods. If the model uncertainty is large some robust formulation of the problem is needed, such as multimodel (Ackermann, 1993; Magni, 2002) or polytopic plant model approach (Jetto, 99; Rotstein *et al.*, 1991) .

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The main hindrance of the parametric methods is the well-known fact that the stability domain in the space of polynomial coefficients is non-convex in general. That is why several convex approximations of the stability region such as ellipsoids (Galafiore and El Ghaoui, 2004; Henrion *et al.*, 2003), hyperrectangles (Kharitonov, 1978; Chapellat and Bhattacharyya, 1989; Katbab and Jury, 1990) , polytopes (Jetto, 99; Rotstein *et al.*, 1991) and simplexes are well known and widely used in in robust control .

In this paper we consider a polytopic plant model, i.e. the set of possible plant parameters are defined as a convex polytope. This kind of modeling allows the determination of properties that are common to all elements in the set from the analysis of its vertices only. Thus the complexity of computations is determined by the number N of vertices of the polytope.

We are looking for Schur stable simplexes inside the "nicely stable" region with vertices defined by so-called reflection vectors of the closed-loop characteristic polynomial. If the stable simplex is preselected then the robust output controller design task can be solved by quadratic programming approach. If the stable simplex (or polytope) of reflection vectors is not given then a simple search procedure is needed.

The paper is organized as follows. In section 2 we recall the stability condition via reflection coefficients and introduce reflection vectors of a monic Schur polynomial. The roots placement of reflection vectors is studied. In section 3 the problem of stable simplex building around a given stable point is considered. At last, in section 4, the robust controller design problem will be solved by quadratic programming approach.

2. REFLECTION COEFFICIENTS AND REFLECTION VECTORS OF SCHUR POLYNOMIALS

A polynomial $a(z)$ of degree n with real coefficients $a_i \in \mathcal{R}$, $i = 0, ..., n$

$$a(z) = a_n z^n + ... + a_1 z + a_0$$

is said to be Schur polynomial if all its roots are placed inside the unit circle. A linear discrete-time dynamical system is stable if its characteristic polynomial is Schur, i.e. if all its poles lie inside the unit circle.

Besides the unit circle criterion some other criterias are known for checking the stability of a linear system. It is interesting to mention that the well-known Jury's stability test leads precisely to the stability hypercube of reflection coefficients . In the following we use the stability criterion via reflection coefficients.

Let us recall the recursive definition of reflection coefficients $k_i \in \mathcal{R}$ of a polynomial $a(z)$ (Oppenheim and Schaffer, 1989)

$$k_i = -a_i^{(i)}, \tag{1}$$

$$a_i^{(n)} = \frac{a_{n-i}}{a_n}, \qquad i = 1, ..., n; \tag{2}$$

$$a_j^{(i-1)} = \frac{a_j^{(i)} + k_i a_{i-j}^{(i)}}{1 - k_i^2},\ j = 1, ..., i-1. \tag{3}$$

Reflection coefficients are well-known in signal processing and digital filters. They are called also partial correlation coefficients and k-parameters (Oppenheim and Schaffer, 1989). The stability criterion via reflection coefficient is as follows .

Lemma 1. A polynomial $a(z)$ will be Schur stable if and only if its reflection coefficients $k_i, i = 1, ..., n$ lie within the interval $-1 < k_i < 1$.

A polynomial $a(z)$ lies on the stability boundary if some $k_i = \pm 1, i = 1, ..., n$. For monic Schur polynomials, $a_n = 1$, there is an one-to-one correspondence between the vectors $a = (a_0, ..., a_{n-1})$ and $k = (k_1, ..., k_n)$.

The transformation from reflection coefficients k_i to polynomial coefficients $a_{i-1}, i = 1, ..., n$ is multilinear. For monic polynomials we obtain from (1)-(3)

$$\begin{aligned} a_i &= a_{n-i}^{(n)}, \\ a_i^{(i)} &= -k_i, \qquad\qquad i = 1, ..., n; \\ a_j^{(i)} &= a_j^{(i-1)} - k_i a_{i-j}^{(i-1)}, j = 1, ..., i-1. \end{aligned} \tag{4}$$

Lemma 2.(Nurges, 1999) Through an arbitrary stable point $a = [a_0, a_1, ..., a_{n-1}]$ with reflection coefficients $k_i^a \in (-1, 1)$, $i = 1, ..., n$ you can draw n stable line segments

$$A^i(\pm 1) = conv\{a | k_i^a = \pm 1\}$$

where $conv\{a | k_i^a = \pm 1\}$ denotes the convex hull obtained by varying the reflection coefficient k_i^a between -1 and 1.

Now let us introduce the reflection vectors of a monic polynomial $a(z)$. They will be useful for convex stable subsets building in polynomial coefficient space.

Definition. Let us call the vectors

$$a^i(1) = (a | k_i = 1), i = 1, ..., n$$

positive reflection vectors and

$$a^i(-1) = (a | k_i = -1), i = 1, ..., n$$

negative reflection vectors of a monic polynomial $a(z)$.

It means, reflection vectors are the extreme points of the Schur stable line segment $A^i(\pm 1)$ through the point a defined by Lemma 2. Due to the definition and the Lemmas 1 and 2 the following assertions hold:

1) every Schur polynomial has $2n$ reflection vectors $a^i(1)$ and $a^i(-1)$, $i = 1, ..., n$;
2) all the reflection vectors lie on the stability boundary ($k_i = \pm 1$);
3) all the innerpoints of the line segments between reflection vectors $a^i(1)$ and $a^i(-1)$ are Schur stable.

Obviously, at least one of the roots of a reflection vector (polynomial) $a^i(\pm 1)$ must lay on the unit circle.

The following theorem states that the number of unit circle roots is determined by the number i of the reflection vector $a^i(\pm 1)$ and the character of the roots (real or complex) is determined by the sign of the boundary reflection coefficient

$k_i = \pm 1$).

Theorem 1. Reflection vectors $a^i(\pm 1)$, $i = 1, ..., n$ of a monic Schur polynomial $a(z)$ have the following i roots r_j , $j = 1, .., i$ on the stability boundary:

1) the positive reflection vector $a^i(1)$ has
   - for i even $r_1 = 1$, $r_2 = -1$ and $(i-2)/2$ pairs of complex roots on the unit circle,
   - for i odd $r_1 = 1$, and $(i-1)/2$ pairs of complex roots on the unit circle,
2) the negative reflection vector $a^i(-1)$ has
   - for i even $i/2$ pairs of complex roots on the unit circle,
   - for i odd $r_1 = -1$, and $(i-1)/2$ pairs of complex roots on the unit circle.

The proof is given in (Nurges and Rüstern, 2002).

3. STABLE POLYTOPE BUILDING BY REFLECTION VECTORS OF POLYNOMIALS

Two different approaches can be used for stable simplex (or polytope) building via reflection vectors:

1) choose such a stable point that the linear cover of its reflection vectors is stable;
2) choose an arbitrary stable point and build the stable simplex by n edges in directions of reflection vectors of the starting point.

In the following the first approach will be used.The next lemma follows immediately from the Cohn stability criterion

$$\sum_{i=0}^{n-1} |a_i| < 1.$$

Lemma 3. The innerpoints of the polytope R^0 generated by reflection vectors of the origin $a = 0$

$$R^0 = conv\{0^i(\pm 1), \quad i = 1, ..., n\} \tag{5}$$

are Schur stable.

Lemma 3 (or Cohn stability condition) is quite conservative. The question is: is it possible to relax the initial condition of Lemma 3 in some neighborhood of the origin? The answer is given by the following proposition.

Theorem 2. Let $k_1^a \in (-1, 1)$, $k_n^a \in (-1, 1)$ and $k_2^a = ... = k_{n-1}^a = 0$. Then the innerpoints of the polytope R^a generated by the reflection vectors of the point a

$$R^a = conv\{a^i(\pm 1), \quad i = 1, ..., n\} \tag{6}$$

are Schur stable.

The proof is given in (Nurges, 2001).

Example 1. Let $a(z) = z^3 - 0.75z^2$. The reflection coefficients and reflection vectors of the polynomial $a(z)$ are following:

$$\begin{array}{ll} k_1^a = 0.75, & a^1(1) = [1 \ -1 \ 0 \ 0 \ ]^T, \\ k_2^a = 0, & a^2(1) = [1 \ 0 \ -1 \ 0 \ ]^T, \\ k_3^a = 0, & a^3(1) = [1 \ -0.75 \ 0.75 \ -1 \ ]^T, \end{array}$$

$$\begin{array}{l} a^1(-1) = [1 \ 1 \ 0 \ 0 \ ]^T, \\ a^2(-1) = [1 \ -1.5 \ 1 \ 0 \ ]^T, \\ a^3(-1) = [1 \ -0.75 \ -0.75 \ 1 \ ]^T. \end{array}$$

By Theorem 2 the polytope

$$R^a = conv \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & -0.75 & 0.75 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1.5 & 1 & 0 \\ 1 & -0.75 & -0.75 & 1 \end{pmatrix}$$

is stable.

For generating a "nicely stable" simplex we have to choose $n+1$ most suitable vertices of the polytope R^a. It is well known that the poles with positive real part are preferred to those with negative real part (Ackermann, 1993). Thus, according to Theorem 1, the positive reflection vectors $a^{i(1)}$ with i odd and negative reflection vectors $a^i(-1)$ with i even will be chosen. It gives us n vertices. The $(n+1)$-th vertex of the simplex S will be chosen as the mean of the remaining reflection vectors.

4. ROBUST OUTPUT CONTROLLER DESIGN

Let us return now to the problem of robust modal control . We are looking for a robust output controller such that the closed-loop characteristic polynomial will be placed in the stable polytope (linear cover) of reflection vectors. Three different problems will be studied:

1) the nicely stable simplex of reflection vectors is preselected according to Section 3,
2) the initial point for generating a nicely stable simplex will be chosen in the procedure of robust controller design,
3) a stable polytope (with $2n$ vertices) of reflection vectors will be chosen in the procedure of robust controller design.

Consider a discrete-time linear SISO system. Let the plant transfer function $G(z)$ of dynamic order m and the controller transfer function $C(z)$ of dynamic order r be given respectively by

$$G(z) = \frac{b(z)}{a(z)} = \frac{b_{m-1}z^{m-1} + \cdots + b_1 z + b_0}{a_m z^m + \cdots + a_1 z + a_0}$$

and

$$C(z) = \frac{q(z)}{r(z)} = \frac{q_r z^r + \cdots + q_1 z + q_0}{r_r z^r + \cdots + r_1 z + r_0}.$$

It means that the closed loop characteristic polynomial

$$f(z) = a(z)r(z) + b(z)q(z)$$

is of degree $m + r$.

Let us require that the polynomial $f(z)$ will be placed in a simplex S of coefficient space . Without any restrictions we can assume that $a_m = r_r = 1$ and deal with monic polynomials.

Let us now introduce a stability measure p in accordance with the simplex S

$$p = c^T c$$

where

$$c = S^{-1} f$$

and S is the $(m + r + 1)$x$(m + r + 1)$ matrix of vertices of the target simplex. Obviously, for monic polynomials

$$\sum_{i=1}^{n+1} c_i = 1$$

where $n = m + r$. If all coefficients $c_i > 0$, $i = 1, ..., n + 1$ then the point f is placed inside the simplex S.

It is easy to see that the minimum of p is obtained by

$$c_1 = c_2 = ... = c_{n+1} = \frac{1}{n+1}.$$

Then the point f is placed in the center of the simplex S.

Now we can formulate the following problem of controller design : find a controller $C(z)$ such that the stability measure p is minimal. In other words, we are looking for a controller which places the closed loop characteristic polynomial $f(z)$ as close as possible to the center of the target simplex S.

In matrix form we have

$$f = Gx \qquad (7)$$

where G is the plant Sylvester matrix

$$G = \begin{bmatrix} a_0 & 0 & ... & 0 & b_0 & 0 & ... & 0 \\ a_1 & a_0 & ... & 0 & b_1 & b_0 & ... & 0 \\ . & . & . & . & . & . & . & . \\ a_{n-1} & a_{n-2} & ... & a_0 & b_{n-1} & b_{n-2} & ... & b_0 \\ 0 & a_{n-1} & ... & a_1 & 0 & b_{n-1} & ... & b_1 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & ... & a_{n-1} & 0 & 0 & ... & b_{n-1} \end{bmatrix}$$

of dimensions $(m + r + 1)$x$(2r + 1)$ and x is the $(2r + 1)$-vector of controller parameters $x = [q_0, ..., q_{r-1}, r_0, ..., r_r]^T$.

The above controller design problem is equivalent to the quadratic programming problem: find x such that the minimum

$$J = \min_x x^T G^T (SS^T)^{-1} Gx$$

is obtained by the linear restrictions

$$S^{-1}Gx > 0$$

$$1^T S^{-1} Gx = 1$$

where $1^T = [1...1]$ is an n vector.

Let us now consider the case where the plant is subject to parameter uncertainty. We represent this by supposing that the given plant transfer function coefficients $a_0, ..., a_{m-1}$ and $b_0, ..., b_{m-1}$ are placed in a polytope P with vertices $p^1, ..., p^M$

$$P = conv\{p^j, j = 1, ..., M\}.$$

Because the relations (7) are linear in plant parameters we can claim that for an arbitrary fixed controller x the vector f of closed loop characteristic polynomial coefficients is placed in a polytope F with vertices $f^1, ..., f^M$

$$F = conv\{f^j, j = 1, ..., M\}$$

where

$$f^j = P^j x$$

and P^j is a $(m + r + 1)$x$(2r + 1)$ Sylvester matrix composed by the vertex plant $p^j = [a_0^j, ..., a_{m-1}^j, b_0 j, ..., b_{m-1}^j]$.

The problem of robust controller design can be formulated as follows : find a controller x such that all vertices f^j , $j = 1, ..., M$ are placed inside the simplex S.

This problem can be solved by quadratic programming task : find x which minimizes

$$J = \min_x x^T \tilde{P}^T (I \otimes ((S^T)^{-1})(I \otimes S^{-1})\tilde{P}x$$

by linear restrictions

$$S^{-1} P^j x > 0,$$

$$1^T S^{-1} P^j x = 1, \qquad j = 1, ..., M.$$

Here I is the unit matrix, $\otimes$ denotes the Kronecker product and $\tilde{P}^T = [P_1^T, ..., P_M^T]$.

Up to now the "nicely stable" simplex S has been fixed (preselected). According to Theorem 2 the reflection coefficients k_1^a and k_n^a of the generating point a can be freely chosen from the interval $-1, 1$.

So we have obtained the following , more general, optimization task for solving the robust output controller design problem:
find x and $k_1^a \in (-1,1)$, $k_n^a \in (-1,1)$ such that the criterion

$$J = \min_{x,k} x^T \tilde{P}^T (I \otimes ((S^T(k_1,k_n))^{-1})(I \otimes S(k_1,k_n)^{-1})\tilde{P}x$$

will be minimized by linear restrictions

$$S(k_1,k_n)^{-1}P^j x > 0,$$
$$1^T S(k_1,k_n)^{-1}P^j x = 1, \qquad j = 1,...,M.$$

Of course, we can choose instead of a "nicely stable" simplex S the polytope R of reflection vectors as a convex approximation of the stability region. Then the square $(n+1)$x$(n+1)$ matrix $S(k_1,k_n)$ will be replaced by a $(n+1)$x$2n$ matrix $R(k_1,k_n)$. It means , we have $n-1$ degrees of freedom in the previous optimization task and so we can choose $n-1$ additional restrictions to obtain a single solution. Additional restrictions can be given via reflection coefficient placement of the nominal closed loop characteristic polynomial $f(z)$. A reasonably damped system has reflection coefficients with following properties:

a) k_1^f is nonnegative, $k_1 \in (0,1)$;
b) the sign of successive reflection coefficients is alternating $sign(k_i^f) = sign(-1)^{i+1}$, $i = 1,...,n$;
c) the absolute values of successive reflection coefficients are decreasing $|k_i^f| \geq |k_{i+1}^f|$, $i = 1,...,n$.

Example 2. Let us consider an uncertain second order interval plant

$$G(z) = \frac{b_0}{z^2 + a_1 z + a_0}$$

with parameters in the intervals $1.78 \leq b_0 \leq 2.02$, $-1.62 \leq a_1 \leq -1.38$, $0.43 \leq a_0 \leq 0.67$ and we are looking for a first order robust controller

$$C_0(z) = \frac{x_4 z + x_3}{x_2 z + x_1}.$$

For preselected $k_1^f = 0.5$, $k_2^f = k_3^f = 0$ we obtain the "nicely stable" simplex according to Theorem 2 as follows

$$S = \begin{bmatrix} 0 & 0 & -1 & 0.3333 \\ 0 & 1 & 0.5 & -0.5 \\ -1 & -1 & -0.5 & 0.1667 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

By the use of MATLAB Optimization Toolbox and above quadratic programming formulation we have find the robust controller

$$C(z) = \frac{0.4538z - 0.2656}{z + 0.7281}$$

with $J_{min}(k_1 = 0.5) = 6.5987$.

It can be easily checked that the closed loop polytope

$$P = \begin{bmatrix} -0.056 & -0.2162 & -0.056 & -0.2162 \\ 0.519 & 0.3529 & 0.3972 & 0.1772 \\ -0.6519 & -0.6519 & -0.8919 & -0.8919 \\ 1 & 1 & 1 & 1 \\ 0.0078 & -0.1524 & 0.0078 & -0.1524 \\ 0.463 & 0.243 & 0.2882 & 0.0682 \\ -0.6519 & -0.6519 & -0.8919 & -0.8919 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

is placed inside the simplex S.

A simple search for optimal k_1 gives

$$J_{min}(k_1^* = 0.1) = 3.3509.$$

5. CONCLUSIONS

A constructive procedure for generating stable polytopes in polynomial coefficients space is given. This procedure of stable polytope (or simplex) building is quite straightforward because you need to choose only one stable point with some restrictions for reflection coefficients of it. Then all the vertices of the polytope (or simplex) will be generated by reflection vectors of this point.

It is shown, first, that reflection vectors are placed on the stability boundary with specific roots depending on the reflection vector number and the argument sign and, second, that the line segments between an arbitrary Schur polynomial and its reflection vectors are Schur stable.

The procedure of robust output controller design by quadratic programming is based on a stability measure p which indicates the placement of vertices of the polytope of closed loop system against the stable simplex of reflection vectors of a (optimally) selected stable point .

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