

LIMITATION OF GENERALIZED DELAYED FEEDBACK CONTROL FOR DISCRETE-TIME SYSTEMS¹

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Abstract: In this paper, the stabilizability problem for chaotic discrete-time systems under the generalized delayed feedback control (GDFC) is addressed. It is proved that $0 < \det(I - A) < 2^{n+m}$ is a necessary and sufficient condition of stabilizability via m -step GDFC for an n -order system with Jacobi A . The condition reveals the limitation of GDFC more exactly than the odd number limitation. An analytical procedure of designing GDFC is proposed and illustrated by an example. *Copyright*©2005 IFAC

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1. INTRODUCTION

The delayed feedback control (DFC) proposed by Pyragas (1992) is an important method in chaos control and has been successfully applied to various systems (Just et al., 1997). The advantage of the DFC is requiring no information of UPOs except their periods. For chaotic discrete-time systems, Ushio used the DFC to stabilize unstable fixed points and found a limitation, called odd number limitation, which can be stated as follows: if the system Jacobi about the target fixed point has an odd number of real eigenvalues greater than unity, then the system is not stabilizable by DFC (Ushio, 1996). A similar limitation exists in control of chaotic continuous-time systems (Nakajima, 1997). Nakajima and Ueda further proved that the odd number limitation holds for a generalized delayed feedback control including

extended time-delayed auto-synchronization (Socolar, et al., 1994), exponential DFC among others (Nakajima and Ueda, 1998). In Tian and Chen (2001), it was showed that the limitation is inherited in an observer-based dynamical DFC for continuous-time systems.

The odd number limitation actually describes a necessary condition for stabilizability via DFC. The problem of finding necessary and sufficient conditions has attracted much attention and remained open for a long time. For the first-order and second-order discrete-time systems, Ushio obtained necessary and sufficient conditions using Jury's stability test (Ushio, 1996). In Ushio and Yamamoto (1998), Ushio and Yamamoto extended DFC to the nonlinear estimation case, and reduced the stabilization problem to solving linear matrix inequalities (LMIs). But the solvability of the LMIs was not addressed.

In this paper, the stabilizability problem for discrete-time systems under the generalized delayed feedback control (GDFC) is addressed. It is proved that $0 < \det(I - A) < 2^{n+m}$ is a necessary

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and sufficient condition of stabilizability via m -step GDFC for an n -order system with Jacobi A . This result shows that the upper bound in the above condition can be enlarged by increasing the number of delays in the feedback. In other words, a system which can not be stabilized by the conventional DFC may still be stabilized by GDFC, while early results show that GDFC has no advantage over the conventional DFC in overcoming the odd number limitation (Nakajima and Ueda, 1998). An analytical procedure of designing GDFC is also proposed and illustrated by an example.

2. NECESSARY AND SUFFICIENT CONDITION FOR STABILIZABILITY VIA GDFC

Let us consider an n -order nonlinear discrete-time system described by

$$\mathbf{x}(k+1) = f(\mathbf{x}(k), u(k)), \quad (1)$$

where $u(k) \in R$ is the control input, $\mathbf{x}(k) \in R^n$ is the state, $f: R^n \times R \rightarrow R^n$ is a smooth mapping. In this paper we consider the generalized delayed feedback control (GDFC)

$$u(k) = \sum_{i=1}^n p_i [\mathbf{x}_i(k) - \sum_{j=1}^m \lambda_{ij} \mathbf{x}_i(k-m-1+j)], \quad (2)$$

where $\lambda_{ij} \in R$, and satisfy

$$\sum_{j=1}^m \lambda_{ij} = 1, \quad i = 1, \dots, n, \quad (3)$$

and $p_i, i = 1, \dots, n$, are control gains to be designed. An equivalent form of (2) can be written as

$$u(k) = \sum_{i=1}^n \sum_{j=1}^m p_{ij} [\mathbf{x}_i(k) - \mathbf{x}_i(k-m-1+j)], \quad (4)$$

which is a linear discrete-time version of the GDFC discussed by Nakajima and Ueda for continuous-time systems (Nakajima and Ueda, 1998). Indeed, if we let $p_i = \sum_{j=1}^m p_{ij}$ and $\lambda_{ij} = p_{ij}/p_i$, then (4) is exactly (2).

Assume \mathbf{x}^* is the unstable fixed point of the open-loop system, i.e., $\mathbf{x}^* = f(\mathbf{x}^*, 0)$. Write $A = D_{\mathbf{x}}f(\mathbf{x}^*, 0)$ and $b = D_u f(\mathbf{x}^*, 0)$. Then the linearized system of (1) around \mathbf{x}^* is

$$x(k+1) = Ax(k) + bu(k), \quad (5)$$

where $x(k) = \mathbf{x}(k) - \mathbf{x}^* \in R^n$ is the state variation about the fixed point. The GDFC defined by (2) can also be represented by state variation as

$$u(k) = \sum_{i=1}^n p_i [x_i(k) - \sum_{j=1}^m \lambda_{ij} x_i(k-m-1+j)]. \quad (6)$$

The aim of this paper is to find a necessary and sufficient condition under which there exist $p = [p_1, \dots, p_m]$ and λ_{ij} satisfying (3) such that the closed-loop system composed by (5) and (6) is asymptotically stable.

In this paper, we assume that system (5) is controllable. In this case, by linear system theory (Chen, 1984), system (5) can always be transformed into the following canonical form:

$$\begin{aligned} x_1(k+1) &= x_2(k) \\ x_2(k+1) &= x_3(k) \\ &\vdots \\ x_{n-1}(k+1) &= x_n(k) \\ x_n(k+1) &= \sum_{i=1}^n a_i x_i(k) + u(k) \end{aligned} \quad (7)$$

Let $y_1(k) = x_1(k-m), y_2(k) = x_1(k-m+1), \dots, y_{m+n}(k) = x_1(n+k-1) = x_n(k)$. Then the closed-loop system (7) with (6) can be written as

$$\begin{aligned} y_1(k+1) &= y_2(k), \\ &\vdots \\ y_{n+m-1}(k+1) &= y_{n+m}(k), \\ y_{n+m}(k+1) &= \sum_{i=1}^n (a_i + p_i) y_{m+i}(k) \\ &\quad - p_1 \lambda_{11} y_1 \\ &\quad - (p_2 \lambda_{21} + p_1 \lambda_{12}) y_2 \\ &\vdots \\ &\quad - (p_n \lambda_{n1} + p_{n-1} \lambda_{(n-1)2} + \dots + p_1 \lambda_{1n}) y_n \\ &\vdots \\ &\quad - (p_n \lambda_{n(m-n+1)} + \dots + p_1 \lambda_{1m}) y_m \\ &\quad - (p_n \lambda_{n(m-n+2)} + \dots + p_2 \lambda_{2m}) y_{m+1} \\ &\vdots \\ &\quad - p_n \lambda_{nm} y_{m+m-1}. \end{aligned} \quad (8)$$

For the simplicity of statement, here and below, we give the matrix formulae only for the case when $m \geq n$. But all the discussions and results hold for the case when $m < n$. The difference between two cases is in notations of the subscripts of matrix elements. The characteristic polynomial of system (8) is

$$\begin{aligned} g(s) &= s^{n+m} + g_{n+m-1} s^{n+m-1} \\ &\quad + \dots + g_1 s + g_0, \end{aligned} \quad (9)$$

where

$$\begin{aligned}
g_0 &= p_1 \lambda_{11}, \\
g_1 &= p_2 \lambda_{21} + p_1 \lambda_{12}, \\
&\vdots \\
g_{n-1} &= p_n \lambda_{n1} + p_{n-1} \lambda_{(n-1)2} + \cdots + p_1 \lambda_{1n}, \\
&\vdots \\
g_m &= p_n \lambda_{n(m-n+1)} + \cdots + p_1 \lambda_{1m}, \\
g_{m+1} &= p_n \lambda_{n(m-n+2)} + \cdots + p_2 \lambda_{2m} - a_1, \\
&\vdots \\
g_{n+m-2} &= \lambda_{nm} p_n - p_{n-1} - a_{n-1}, \\
g_{n+m-1} &= -p_n - a_n.
\end{aligned} \tag{10}$$

Clearly, the system (8) is asymptotically stable if and only if all the zeros of (9) can be rendered in the unit circle of the complex plane. To this end, we suppose the desired stable characteristic polynomial of $(n+m)$ -order is

$$d(s) = s^{n+m} + d_{n+m-1}s^{n+m-1} + \cdots + d_1s + d_0. \tag{11}$$

Comparing coefficients of (9) with those of (11) we can get

$$\begin{aligned}
&\begin{bmatrix} \lambda_{11} & & & & 0 \\ \lambda_{12} & \lambda_{21} & & & \\ \vdots & & \ddots & & \\ \lambda_{1n} & \cdots & \cdots & \cdots & \lambda_{n1} \\ \vdots & & & & \vdots \\ \lambda_{1m} & \cdots & \cdots & \cdots & \lambda_{n(m-n+1)} \\ -1 & \lambda_{2m} & \cdots & \cdots & \lambda_{n(m-n+2)} \\ & \ddots & \ddots & & \vdots \\ & & & -1 & \lambda_{nm} \\ 0 & & & & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix} \\
&= \begin{bmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \\ \vdots \\ d_{m-1} \\ d_m + a_1 \\ \vdots \\ d_{n+m-2} + a_{n-1} \\ d_{n+m-1} + a_n \end{bmatrix}. \tag{12}
\end{aligned}$$

In both sides of the above equation, we carry out such row transformations: add row 1 to row 2, then add the new row 2 to row 3, \cdots . Finally, taking Eq.(3) into account we get

$$\begin{aligned}
&\begin{bmatrix} \lambda_{11} & & & & 0 \\ \lambda_{11} + \lambda_{12} & \lambda_{21} & & & \\ \vdots & & \ddots & & \\ \sum_{i=1}^n \lambda_{1i} & \cdots & \cdots & \cdots & \lambda_{n1} \\ \vdots & & & & \vdots \\ 1 & \cdots & \cdots & \cdots & \sum_{i=1}^{m-n+1} \lambda_{ni} \\ 0 & 1 & \cdots & \cdots & \sum_{i=1}^{m-n+2} \lambda_{ni} \\ & \ddots & \ddots & & \vdots \\ & & & 0 & 1 \\ 0 & & & & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix} \\
&= \begin{bmatrix} d_0 \\ d_0 + d_1 \\ \vdots \\ \sum_{j=0}^{n-1} d_j \\ \vdots \\ \sum_{j=0}^{m-1} d_j \\ \sum_{j=0}^m d_j + a_1 \\ \vdots \\ \sum_{j=0}^{n+m-2} d_j + \sum_{i=1}^{n-1} a_i \\ \sum_{j=0}^{n+m-1} d_j + \sum_{i=1}^n a_i \end{bmatrix}. \tag{13}
\end{aligned}$$

The linear equations (13) has solutions $p_i (i = 1, 2, \cdots, n)$ only if

$$\sum_{j=0}^{n+m-1} d_j + \sum_{i=1}^n a_i = 0. \tag{14}$$

Since $\det(I - A) = 1 - \sum_{i=1}^n a_i$, and $d(1) = 1 + \sum_{j=0}^{n+m-1} d_j$. Thus, Eq.(14) can be equivalently rewritten as

$$\det(I - A) = d(1). \tag{15}$$

So we have actually proved the necessity part of the following theorem.

Theorem 1. For any $(n+m)$ -order polynomial given by (11), there exist p and λ_{ij} satisfying (3) such that the characteristic polynomial of the closed-loop system (8) is exactly (11) if and only if

such that the closed-loop system composed by (5) with (6) is asymptotically stable if and only if

$$0 < \det(I_n - A) < 2^{n+m}. \quad (22)$$

Proof. Without loss of generality, let the system be of the form (7). We consider the closed-loop system (8).

(Necessity) Assume there exists p such that (8) is asymptotically stable. Then all the roots of the characteristic polynomial of system (8) lie inside the unit circle. Let the characteristic polynomial of (8) be (11). Then Theorem 1 and Lemma 1 imply (22).

(Sufficiency) From (22), there exists $\eta \in R$ such that

$$0 < \det(I_n - A) < \eta < 2^{n+m}. \quad (23)$$

Set

$$\begin{aligned} \theta &= 1 - \left(\frac{\eta}{2}\right)^{1/(n+m-1)}, \\ \beta &= 1 - \frac{2 \det(I - A)}{\eta}. \end{aligned} \quad (24)$$

By (23), it is easy to check that $-1 < \theta < 1$, $-1 < \beta < 1$. Let

$$d(s) = (s - \theta)^{n+m-1}(s - \beta). \quad (25)$$

Then we have

$$\begin{aligned} d(1) &= (1 - \theta)^{n+m-1}(1 - \beta) \\ &= \frac{\eta}{2} \cdot \frac{2 \det(I - A)}{\eta} \\ &= \det(I - A). \end{aligned} \quad (26)$$

Therefore, by Theorem 1, there exists p such that the characteristic polynomial of (8) is (25), namely, there exists p such that (8) is asymptotically stable. \square

Remark 2. Eq. (23), (24) and (25) actually give a method for determining a stable polynomial $d(s)$ satisfying (14). Combining these equations with Eq. (16) and (18), one can analytically design a GDFC to locally asymptotically stabilize nonlinear system (1).

Remark 3. The assumption that the system is in the controllable canonical form (7) does not impact the generality of the results, because, by linear system theory, any single-input controllable system (A, b) can be converted into $(T^{-1}AT, T^{-1}b)$ so that the equivalently transformed system is in the controllable canonical form. Obviously, we have $\det(I - A) = \det(I - T^{-1}AT)$. So Theorem 2 holds for all controllable systems.

Remark 4. If A has an eigenvalue equal to 1 or has an odd number of real eigenvalues greater than 1, then $\det(I_n - A) \leq 0$. Thus Theorem 2 implies that there does not exist p such that (8) is asymptotically stable, which is just the odd number limitation appeared in Ushio (1996). However, the upper bound 2^{n+m} for $\det(I - A)$ has not been revealed before. The upper bound can be enlarged by increasing the number m of delays in the feedback (2). In other words, a system which can not be stabilized by the conventional DFC may still be stabilized by GDFC, while early results show that GDFC has no advantage over the conventional DFC in overcoming the odd number limitation (Nakajima and Ueda, 1998).

Example 1. As a numerical example, let us consider a simple chaotic system described by

$$\mathbf{x}(k+1) = a\mathbf{x}(k)(1 - \mathbf{x}^2(k)) + u(k), \quad (27)$$

where $a = -3$ (see, e.g., (Sprott, 2003)). The linearized system around the fixed point $x_f = 0$ is

$$x(k+1) = ax(k).$$

So $\det(I - A) = 1 - a = 4$. By Theorem 2, the system is not stabilizable under one-step delayed feedback

$$u(k) = p(\mathbf{x}(k) - \mathbf{x}(k-1)).$$

However, since $\det(I - A) < 2^{1+2}$, it can be stabilized by the following GDFC.

According to Remark 2, the design procedure is sketched as follows.

By Eq. (23) we choose $\eta = 6$ so that

$$4 = \det(I - A) < \eta < 2^3$$

holds. According to (24) we can set

$$\begin{aligned} \theta &= 1 - \left(\frac{\eta}{2}\right)^{1/2} = 1 - \sqrt{3}, \\ \beta &= 1 - \frac{2 \det(I - A)}{\eta} = 1 - 8/6 = -1/3. \end{aligned}$$

Now by Eq.(25) we get a stable polynomial as

$$\begin{aligned} d(s) &= (s - \theta)^2(s - \beta) \\ &= s^3 + (2\sqrt{3} - \frac{5}{3})s^2 + (\frac{10}{3} - \frac{4}{3}\sqrt{3})s \\ &\quad + \frac{4}{3} - \frac{2}{3}\sqrt{3}. \end{aligned}$$

From (18) we have

$$\begin{aligned} p_1 &= d_0 + d_1 = \frac{4}{3} - \frac{2}{3}\sqrt{3} + \frac{10}{3} - \frac{4}{3}\sqrt{3} \\ &= \frac{14}{3} - 2\sqrt{3}, \end{aligned}$$

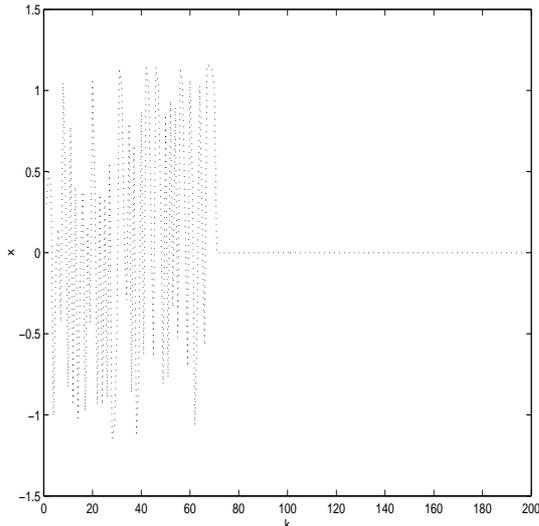


Fig. 1. Controlled behavior of the state in the example.

$$\lambda_{11} = d_0/p_1 = \frac{2 - \sqrt{3}}{7 - 3\sqrt{3}},$$

$$\lambda_{12} = d_1/p_1 = \frac{5 - 2\sqrt{3}}{7 - 3\sqrt{3}}.$$

So the feedback control is designed as

$$u(k) = \left(\frac{14}{3} - 2\sqrt{3} \right) \times \left(\mathbf{x}(k) - \left(\frac{2 - \sqrt{3}}{7 - 3\sqrt{3}} \mathbf{x}(k-2) + \frac{5 - 2\sqrt{3}}{7 - 3\sqrt{3}} \mathbf{x}(k-1) \right) \right). \quad (28)$$

Since stabilization is guaranteed only in a neighborhood of the fixed point, we adopt the following small control law:

$$u_s(k) = \begin{cases} u(k), & \text{if } u(k) < \varepsilon, \\ 0, & \text{otherwise,} \end{cases} \quad (29)$$

where ε is a sufficiently small positive number. Shown in Fig. 1 is the behavior of the controlled system under (29) with $\varepsilon = 0.002$.

$$u(k) = p_1(\mathbf{x}(k) - (\lambda_{11}\mathbf{x}(k-2) + \lambda_{12}\mathbf{x}(k-1))),$$

$$\lambda_{11} + \lambda_{12} = 1.$$

3. CONCLUSION

In the paper we have considered the stabilization of single-input chaotic discrete-time systems by the generalized delayed feedback control. A necessary and sufficient condition for stabilizability via GDFC has been obtained, which includes the

odd number limitation as a special case. It is also shown that by adding the delay steps in the GDFC, the upper bound of the limitation can be arbitrarily enlarged.

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