STOCHASTIC CONTROL DESIGN FOR STOCHASTIC UNCERTAIN TIME-DELAY SYSTEMS

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Abstract**:** In this paper, we investigate the stochastic output feedback control problem for a class of stochastic non-linear time-delay systems. The aim of this paper is to design a linear, delayless, and independent - uncertainty control for all admissible uncertainties. The designed control ensures stochastically exponential stability in the mean square, independent of the time-delay. The sufficient conditions for the existence of such a control are proposed in terms of certain quadratic matrix inequalities. The simulations of applying the proposed method to a stochastic time-delay system subject to non-linear uncertainties are shown in a simple example.

Keywords: Exponential stability; quadratic matrix inequality; stochastic control; time-delay systems; stochastic systems.

1. INTRODUCTION

Most of the systems, which are encountered in control engineering, contain various non-linearities and are affected by random disturbance signals. Non-linear systems with time-delay constitute basic mathematical models of real phenomena, for instance in biology, mechanics and economics, see (Niculescu, et al., 1997; Hale, 1997). Control of time-delay systems has been a subject of great practical importance, which has attracted a great deal of interest for several decades. On the other hand, it turns out that the delayed state is very often the cause for instability and poor performance of systems. Moreover, considerable attention has been given to both the problems of robust stabilization and robust control for linear systems with unavoidable time-varying parameter uncertainties in modelling dynamical systems and certain types of time-delays by (Malek-Zavarei and Jamshidi, 1987).

Recently, several criteria of input-to-bounded state (IBS) stabilization and bounded-input-bounded-output (BIBO) stabilization in mean square for non-linear and quasi-linear stochastic control systems with time-varying uncertainties has been investigated by (Fu and Liao, 2003), also, another stability concepts in the mean-square sense such as mean-square stability (MSS) and the internal mean-square stability (IMSS) have been studied by (Lu and Skelton, 2002).

The stabilization of stochastic systems with multiplicative noise has been studied since the late sixties, particularly in the context of linear quadratic optimal control, see (Mclane, 1971; Willems, 1983). More recently a number of papers have been published which deal with robust stability and robust stabilization problems in the spirit of H_∞ control or the stability radius approach. Stochastic stability analysis and H_∞ disturbance attenuation for a class of discrete-time non-linear stochastic systems has been investigated by (Cao and Hu, 2001). Wang, et al., (2001) proposed the robust reliable control problem for a class of non-linear time-delay stochastic systems and their attention were focused on the design of *linear state feedback* memoryless controllers.

In the sequel of the work by (Wang, et al., 2001), in this paper, we consider the stochastic control problem for a class of stochastic non-linear time-delay systems. A state-space model with real time-varying norm-bounded parameter uncertainties and non-linear disturbance meeting the boundedness condition describes the class of the stochastic time-delay systems. Here, attention is focused on the design of *stochastic output feedback controller* which for all admissible uncertainties as well as non-linear disturbances ensures stochastically exponentially stable in the mean square, independent of the time- delay. The sufficient conditions for the existence of such a control is proposed in terms of certain quadratic matrix inequality.

2. PRELIMINARIES AND PROBLEM FORMULATION

Consider a class of non-linear uncertain continuous-time state delayed stochastic system described by (Wang, et al., 2001; Wang and Burnham, 2001),

$$\begin{aligned} dx(t) &= [(A+\Delta A(t))x(t)+(A_d+\Delta A_d(t))x(t-h) \\ &\quad + Bu(t)+f(x(t))]dt+E_1\,dw(t) \end{aligned} \tag{1}$$

$$x(t)=\varphi(t), \qquad t\in[-h,0] \tag{2}$$

together with the measurement output

$$dy(t)=[C+\Delta C(t)]x(t)\,dt+E_2\,dw(t) \tag{3}$$

where $x(t)\in\Re^n$ is the state, $u(t)\in\Re^m$ is the control input, $y(t)\in\Re^p$ is the measurement output, $f(.):\Re^n\to\Re^{n_f}$ is the unknown non-linear disturbance input, h is the unknown state delay, $\varphi(t)$ is the continuous vector valued initial function and $w(t)$ is a scalar Brownian motion defined on the probability space $(\Omega, F, \{F_t\}_{t\ge 0}, P)$. A, A_d, B, E_1 are known constant matrices with appropriate dimensions. $\Delta A(t)$, $\Delta A_d(t)$, $\Delta C(t)$ are real-valued time-varying matrix functions representing norm-bounded parameter uncertainties and satisfy

$$\begin{bmatrix}\Delta A(t)\\ \Delta C(t)\end{bmatrix}=\begin{bmatrix}M_1\\ M_2\end{bmatrix}\Xi(t)N_1\,,\ \Delta A_d(t)=M_1\,\Xi(t)\,N_2 \tag{4}$$

where $\Xi(t)\in\Re^{i\times j}$ is an unknown time-varying matrix function satisfying

$$\Xi^T(t)\,\Xi(t)\le I \qquad \forall\, t. \tag{5}$$

It is assumed that all the elements of $\Xi(t)$ are Lebesgue measurable and M_1, M_2, N_1, N_2 are known real constant matrices of appropriate dimensions that specify how the uncertain parameters in $\Xi(t)$ enter the nominal matrices A, A_d, C. The parameter uncertainties $\Delta A(t)$, $\Delta A_d(t)$, $\Delta C(t)$ are said to be admissible if both (4) and (5) hold.

Assumption 1 (Wang and Burnham, 2001): There exists a known real constant matrix $H\in\Re^{n\times n}$ such that $|f(x(t))|\le|H\,x(t)|$ for any $x(t)\in\Re^n$.

The stochastic output feedback control problem that we address in this paper is of the form

$$\begin{cases} d\hat{x}(t)=A_K\,\hat{x}(t)\,dt+B_K\,dy(t)\\ \quad\ u(t)=C_K\,\hat{x}(t)\end{cases} \tag{6}$$

where $\hat{x}(t)\in\Re^{\hat{n}}$ and the constant matrices of $A_K\in\Re^{\hat{n}\times\hat{n}}$, $B_K\in\Re^{\hat{n}\times p}$ and $C_K\in\Re^{m\times\hat{n}}$ are controller parameters to be designed. By using the control (6) and (1-3), we obtain the following augmented closed-loop system:

$$\begin{aligned} d\bar{x} &= [(\bar{A}+\Delta\bar{A}(t))\bar{x}(t)+(\bar{A}_d+\Delta\bar{A}_d(t))x(t-h) \\ &\quad +\bar{f}(S\,\bar{x}(t))]dt+\bar{E}\,dw(t) \end{aligned} \tag{7}$$

where

$$\bar{x}(t):=\begin{bmatrix}x(t)\\ \hat{x}(t)\end{bmatrix},\ \bar{A}:=\begin{bmatrix}A & BC_K\\ B_KC & A_K\end{bmatrix},\ \bar{A}_d:=\begin{bmatrix}A_d & 0\\ 0 & 0\end{bmatrix},$$

$$\Delta\bar{A}(t):=\begin{bmatrix}\Delta A(t) & 0\\ B_K\,\Delta C(t) & 0\end{bmatrix}=\bar{M}_1\,\Xi(t)\,\bar{N}_1\ ,\ x(t)=S\,\bar{x}(t)\,,$$

$$\Delta\bar{A}_d(t):=\begin{bmatrix}\Delta A_d & 0\\ 0 & 0\end{bmatrix}=\bar{M}_2\,\Xi(t)\bar{N}_2\ ,\ S:=\begin{bmatrix}I & 0\end{bmatrix},$$

$$\bar{f}(.):=\begin{bmatrix}f(.)\\ 0\end{bmatrix},\ \bar{E}:=\begin{bmatrix}E_1\\ B_K\,E_2\end{bmatrix},\ \bar{M}_1:=\begin{bmatrix}M_1\\ B_KM_2\end{bmatrix},$$

$$\bar{N}_1:=\begin{bmatrix}N_1 & 0\end{bmatrix},\ \bar{M}_2:=\begin{bmatrix}M_1\\ 0\end{bmatrix}\ \textit{and}\ \bar{N}_2:=\begin{bmatrix}N_2 & 0\end{bmatrix}.$$

Next, observe the augmented system (7) and let $\bar{x}(t;\zeta)$ denote the state trajectory from the initial data $\bar{x}(\theta)=\zeta(\theta)$ on $-h\le\theta\le 0$ in $L^2_{F_0}([-h,0];\,\Re^{2n})$. Clearly, the system (7) admits a trivial solution $\bar{x}(t;0)\equiv 0$ corresponding to the initial data $\zeta=0$. We introduce the following stability and stabilizability concepts.

Definition 1 (Wang, et al., 2001): For the system (7) and every $\zeta\in L^2_{F_0}([-h,0];\,\Re^{2n})$, the trivial solution is asymptotically stable in the mean square if

$$\lim_{t\to\infty} E|\bar{x}(t;\zeta)|^2=0$$

and is exponentially stable in the mean square if there exist constants $\alpha>0$ and $\beta>0$ such that

$$E|\bar{x}(t;\zeta)|^2\le\alpha\,e^{-\beta t}\sup_{-h\le\theta\le 0}E|\zeta(\theta)|^2.$$

Definition 2 (Wang, et al., 2001): we say that the system (1-3) is exponentially stabilizable in mean square if, for every $\zeta \in L^2_{F_0}([-h,0];\Re^{2n})$, there exists a linear stochastic control law (6) such that the resulting closed-loop system is exponentially stable in mean square.

The objective of this paper is to design an exponential control for the stochastic uncertain non-linear time-delay system (1-3). More specifically, we are interested in seeking the control parameters A_K, B_K and $\boldsymbol{C_K}$ such that for all admissible parameter uncertainties $\Delta A(t)$, $\Delta A_d(t)$, $\Delta C(t)$ and the non-linear disturbance input $f(x(t))$, the augmented system (7) is exponentially stable in mean square, independent of the unknown time-delay h.

3. MAIN RESULTS

We first give the following lemma, which will be used in the proof of our main results.

Lemma 1 (Zhou and Khargonekar, 1988): For any matrices X and Y with appropriate dimensions and for any constant $\eta > 0$, we have:

$$X^T Y + Y^T X \le \eta X^T X + \frac{1}{\eta} Y^T Y . \tag{8}$$

3.1 Stochastic Stability Analysis

In this section, assuming that the stochastic control structure is known and we will study the conditions under which the closed-loop system is stochastically exponentially stable in the mean square. The following theorem will play a key role in the stability analysis of closed-loop system and design of the expected stochastic control.

Theorem 1: Let the stochastic control parameters A_K, B_K and C_K be given. If there exists positive scalars $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ and a positive definite matrix P such that the following matrix inequality

$$\begin{aligned}&\overline{A}^T P + P\overline{A} + P[(\varepsilon_1+\varepsilon_2+\varepsilon_4)I + \varepsilon_3 \overline{M}_1 \overline{M}_1^T]P + \varepsilon_1^{-1}\overline{A}_d^T \overline{A}_d \\ &+ \varepsilon_3^{-1}\overline{N}_1^T\overline{N}_1 + \varepsilon_4^{-1}(HS)^T(HS) + \lambda_{\max}(\overline{M}_2^T\overline{M}_2)\,\varepsilon_2^{-1}\overline{N}_2^T\overline{N}_2 < 0.\end{aligned} \tag{9}$$

Holds, then the augmented closed-loop (7) is exponentially stable in the mean square for all admissible parameter uncertainties and independent of the unknown time-delay h.

Proof : For simplicity, we make the definitions

$$\overline{A}_1(t) := \overline{A} + \Delta\overline{A}(t) = \overline{A} + \overline{M}_1\,\Xi(t)\,\overline{N}_1 \tag{10}$$

$$\overline{A}_{1d}(t) := \overline{A}_d + \Delta\overline{A}_d(t) = \overline{A}_d + \overline{M}_2\,\Xi(t)\,\overline{N}_2 \tag{11}$$

and then the augmented closed-loop system (7) can be rewritten as

$$d\overline{x} = [\overline{A}_1(t)\overline{x}(t) + \overline{A}_{1d}(t)\overline{x}(t-h) + \bar{f}(S\,\overline{x}(t))]dt + \overline{E}\,dw(t) \tag{12}$$

Fix $\zeta \in L^2_{F_0}([-h,0];\Re^{2n})$ arbitrarily, and write $\overline{x}(t;\zeta) = \overline{x}(t)$. We define the Lyapunov function candidate

$$\Upsilon(\overline{x}(t),t) = \overline{x}^T(t)P\overline{x}(t) + \int_{t-h}^{t}\overline{x}^T(s)Q\overline{x}(s)\,ds \tag{13}$$

where P is the positive definite solution to the matrix inequality (9) and $Q > 0$ is defined by

$$Q := \varepsilon_1^{-1}\overline{A}_d^T\,\overline{A}_d + \lambda_{\max}(\overline{M}_2^T\,\overline{M}_2)\varepsilon_2^{-1}\,\overline{N}_2^T\,\overline{N}_2 \tag{14}$$

The stochastic differential of Υ along a given trajectory is obtained as

$$\begin{aligned}d\Upsilon(\overline{x}(t),t) &= \{\overline{x}^T(t)(\overline{A}_1^T(t)P + P\overline{A}_1(t) + Q)\overline{x}(t) \\ &+ \overline{x}^T(t-h)\overline{A}_{1d}{}^T(t)P\overline{x}(t) + \overline{x}^T(t)P\overline{A}_{1d}(t)\overline{x}(t-h) \\ &- \overline{x}^T(t-h)Q\overline{x}(t-h) + \bar{f}^T(S\overline{x}(t))P\overline{x}(t) \\ &+ \overline{x}^T(t)P\bar{f}(S\overline{x}(t))\}dt + 2\overline{x}^T(t)P\overline{E}dw(t) \\ &= \{\overline{x}^T(t)(\overline{A}^TP + P\overline{A} + Q)\overline{x}(t) + \overline{x}^T(t)[(\Delta\overline{A}(t))^TP \\ &+ P(\Delta\overline{A}(t))]\overline{x}(t) + \overline{x}^T(t-h)\overline{A}_d{}^T(t)P\overline{x}(t) \\ &+ \overline{x}^T(t)P\overline{A}_d(t)\overline{x}(t-h) + \overline{x}^T(t-h)(\Delta\overline{A}_d(t))^TP\overline{x}(t) \\ &+ \overline{x}^T(t)P(\Delta\overline{A}_d(t))\overline{x}(t-h) - \overline{x}^T(t-h)Q\overline{x}(t-h) \\ &+ \bar{f}^T(S\overline{x}(t))P\overline{x}(t) + \overline{x}^T(t)P\bar{f}(S\overline{x}(t))\}dt + 2\overline{x}^T(t)P\overline{E}dw(t)\end{aligned} \tag{15}$$

Now, by Lemma 1 and assumption 1, it is trivial to show that for any positive scalars of $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ the following matrix inequalities hold:

$$\begin{aligned}&\overline{x}^T(t-h)\overline{A}_d{}^T(t)P\overline{x}(t) + \overline{x}^T(t)P\overline{A}_d(t)\overline{x}(t-h) \\ &\le \varepsilon_1\overline{x}^T(t)P^2\overline{x}(t) + \varepsilon_1^{-1}\overline{x}^T(t-h)\overline{A}_d{}^T(t)\overline{A}_d(t)\overline{x}(t-h)\end{aligned} \tag{16}$$

and

$$\begin{aligned}&\overline{x}^T(t-h)(\Delta\overline{A}_d(t))^TP\overline{x}(t) + \overline{x}^T(t)P(\Delta\overline{A}_d(t))\overline{x}(t-h) \\ &\le \varepsilon_2\overline{x}^T(t)P^2\overline{x}(t) + \varepsilon_2^{-1}\lambda_{\max}(\overline{M}_2^T\overline{M}_2)\overline{x}^T(t-h)\overline{N}_2^T\overline{N}_2\overline{x}(t-h)\end{aligned} \tag{17}$$

and

$$\begin{aligned}&\overline{x}^T(t)[(\Delta\overline{A}(t))^TP + P(\Delta\overline{A}(t))]\overline{x}(t) \\ &\le \varepsilon_3\overline{x}^T(t)P\overline{M}_1\overline{M}_1^TP\overline{x}(t) + \varepsilon_3^{-1}\overline{x}^T(t)\overline{N}_1{}^T\overline{N}_1\overline{x}(t)\end{aligned} \tag{18}$$

and

$$\begin{aligned}&\bar{f}^T(S\overline{x}(t))P\overline{x}(t) + \overline{x}^T(t)P\bar{f}(S\overline{x}(t)) \\ &\le \varepsilon_4^{-1}\overline{x}^T(t)(HS)^T(HS)\overline{x}(t) + \varepsilon_4\overline{x}^T(t)P^2\overline{x}(t).\end{aligned} \tag{19}$$

Then, noticing the definition (14), substituting (16-19) into (15) results in

$$\begin{aligned}&d\Upsilon(\overline{x}(t),t) \le \overline{x}^T(t)\Pi\overline{x}(t)dt + 2\overline{x}^T(t)P\overline{E}\,dw(t) \\ &\le -\lambda_{\min}(-\Pi)\overline{x}^T(t)\overline{x}(t)dt + 2\overline{x}^T(t)P\overline{E}\,dw(t)\end{aligned} \tag{20}$$

where

$$\begin{aligned}\Pi :=& \overline{A}^TP + P\overline{A} + P[(\varepsilon_1+\varepsilon_2+\varepsilon_4)I + \varepsilon_3\overline{M}_1\overline{M}_1^T]P \\ &+ \varepsilon_1^{-1}\overline{A}_d^T\overline{A}_d + \varepsilon_3^{-1}\overline{N}_1^T\overline{N}_1 + \varepsilon_4^{-1}(HS)^T(HS) \\ &+ \lambda_{\max}(\overline{M}_d^T\overline{M}_d)\,\varepsilon_2^{-1}\overline{N}_2^T\overline{N}_2 .\end{aligned} \tag{21}$$

Then, according to the inequality (9), we find

$$\Pi < 0 . \quad (22)$$

Consequently, the inequalities (20) and (22) mean that the non-linear uncertain stochastic time-delay augmented closed-loop system (7) is asymptotically stable (in the mean square) by the stochastic control law (6).

The expected exponential stability (in the mean square) of the augmented closed-loop system (7) can be proved by making some standard manipulation on (20), see (Mao, 1996). Let β be the unique root of the equation

$$\lambda_{\min}(-\Pi) - \beta\,\lambda_{\max}(P) - \beta\, h\,\lambda_{\max}(Q) e^{\beta h} = 0 \quad (23)$$

where Π and Q are defined, respectively, in (21) and (14) and P is the positive definite solution to (9) and h is the unknown time-delay. Then, by (Wang and Burnham, 2001), we have:

$$E\left|\bar{x}(t)\right|^2 \le \lambda_{\min}^{-1}(P)\,([\lambda_{\max}(P) + h\,\lambda_{\max}(Q)] + \beta\,\lambda_{\max}(q) h^2 e^{\beta h}) \sup_{-h\le\theta\le 0} E\left|\zeta(\theta)\right|^2 e^{-\beta t} . \quad (24)$$

Notice that, according to (24), the definition of exponential stable in *Definition 1* is satisfied and this complete the proof of Theorem 1.

The result of Theorem 1 may be conservative due to using of inequalities (16-19). However, such conservativeness can be significantly reduced by appropriate choices of the parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ in a matrix norm sense.

3.2 Stochastic Control Design

This subsection is devoted to the design of stochastic control parameters A_K, B_K and C_K by using the result in Theorem 1. We will show that the design of stochastic control parameters problem can be solved via the resolution of matrix inequalities. Our approach follows the one developed by (Gahinet and Apkarian, 1994) for the deterministic case. The key tool, which makes this possible, is the stochastic version of the Bounded Real Lemma. From deterministic H_∞ control theory we will need the following, so called, Projection Lemma.

Lemma 2 (Xu and Chen, 2002): Given a symmetric matrix $\mathrm{H} \in \Re^{m\times m}$ and two matrices $\mathrm{N} \in \Re^{l\times m}$ and $\mathrm{M} \in \Re^{n\times m}$, consider the problem of finding some matrix X such that

$$\mathrm{H} + \mathrm{N}^T \mathrm{X}^T \mathrm{M} + \mathrm{M}^T \mathrm{X} \mathrm{N} < 0 \quad (25)$$

Then, (25) is solvable for X if and only if

$$\mathrm{N}^{T\perp} H\, \mathrm{N}^{T\perp T} < 0, \quad \mathrm{M}^{T\perp} H\, \mathrm{M}^{T\perp T} < 0$$

here, if $\Sigma \in \Re^{n\times m}$ and rank $\Sigma = r$, the orthogonal complement $\Sigma^{\perp}$ is defined as a possibly nonunique $(n-r)\times n$ matrix with rank $n-r$, such that $\Sigma^{\perp}\Sigma = 0$.

By using the Schur - complement formula, inequality (9) is equivalent to

$$\begin{bmatrix} \overline{A}^T P + P\overline{A} + \Psi^T \Psi & P\Phi \\ \Phi^T P & -I \end{bmatrix} < 0 \quad (26)$$

where

$$\Psi := \begin{bmatrix} \varepsilon_1^{-\frac{1}{2}} \overline{A}_d \\ \varepsilon_3^{-\frac{1}{2}} \overline{N}_1 \\ \varepsilon_4^{-\frac{1}{2}} HS \\ \lambda_{\max}^{\frac{1}{2}} (\overline{M}_2^T \overline{M}_2) \varepsilon_2^{-\frac{1}{2}} \overline{N}_2 \end{bmatrix} \equiv \begin{bmatrix} \varepsilon_1^{-\frac{1}{2}} A_d & 0 \\ 0 & 0 \\ \varepsilon_3^{-\frac{1}{2}} N_1 & 0 \\ \varepsilon_4^{-\frac{1}{2}} H & 0 \\ \lambda_{\max}^{\frac{1}{2}} (\overline{M}_2^T \overline{M}_2) \varepsilon_2^{-\frac{1}{2}} N_2 & 0 \end{bmatrix}$$

and

$$\Phi := \begin{bmatrix} \varepsilon^{\frac{1}{2}} I & \varepsilon_3^{\frac{1}{2}} \overline{M}_1 \end{bmatrix}, \varepsilon := \varepsilon_1 + \varepsilon_2 + \varepsilon_4 .$$

Consider the following partition of P and P^{-1}:

$$P := \begin{bmatrix} R & M \\ M^T & \widetilde{R} \end{bmatrix}, P^{-1} := \begin{bmatrix} R_0 & M_0 \\ M_0^T & \widetilde{R}_0 \end{bmatrix} \quad (27)$$

such that

$R R_0 + M M_0^T = I$, $M^T R_0 + \widetilde{R} M_0^T = 0$, $R M_0 + M \widetilde{R}_0 = 0$ and $M^T M_0 + \widetilde{R}\widetilde{R}_0 = I$, where $\{R, R_0\} \in \Re^{n\times n}$, $\{M, M_0\} \in \Re^{n\times \hat{n}}$ and $\{\widetilde{R}, \widetilde{R}_0\} \in \Re^{\hat{n}\times \hat{n}}$. Then using expressions (7), the inequality (26) becomes:

$$\Gamma := \left\lfloor \Gamma_{ij} \right\rfloor_{i,\, j=1,\ldots,5} < 0 \quad (28)$$

where the elements of the symmetric matrix Γ are

$$\Gamma_{11} = A^T R + RA + C^T B_K^T M^T + M B_K C + \varepsilon_1^{-1} A_d^T A_d + \varepsilon_3^{-1} N_1^T N_1 + \varepsilon_4^{-1} H^T H + \lambda_{\max}(\overline{M}_2^T \overline{M}_2)\, \varepsilon_2^{-1} N_2^T N_2$$

$$\Gamma_{12} = A^T M + M A_K + C^T B_K^T \widetilde{R} + R B C_K$$

$$\Gamma_{13} = \varepsilon^{\frac{1}{2}} R$$

$$\Gamma_{14} = \varepsilon^{\frac{1}{2}} M$$

$$\Gamma_{15} = \varepsilon_3^{\frac{1}{2}} R M_1 + \varepsilon_3^{\frac{1}{2}} M B_K M_2$$

$$\Gamma_{22} = A_K^T \widetilde{R} + \widetilde{R} A_K + M^T B C_K + C_K^T B^T M$$

$$\Gamma_{23} = \varepsilon^{\frac{1}{2}} M^T$$

$$\Gamma_{24} = \varepsilon^{\frac{1}{2}} \widetilde{R}$$

$$\Gamma_{25} = \varepsilon_3^{\frac{1}{2}} M^T M_1 + \varepsilon_3^{\frac{1}{2}} \widetilde{R} B_K M_2$$

$$\Gamma_{33} = \Gamma_{44} = \Gamma_{55} = -I$$

$$\Gamma_{34} = \Gamma_{35} = \Gamma_{45} = 0 .$$

The condition (28) can be expressed as:

$$\hat{\Gamma} + \hat{\mathrm{N}}^T \Omega \hat{\mathrm{M}} + \hat{\mathrm{M}}^T \Omega^T \hat{\mathrm{N}} < 0 \quad (29)$$

where

$$\Omega := \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix}, \; \hat{\mathrm{M}} := \begin{bmatrix} C & 0 & 0 & 0 & \varepsilon_3^{\frac{1}{2}} M_2 \\ 0 & I & 0 & 0 & 0 \end{bmatrix},$$

$$\hat{\mathrm{N}}^T := \begin{bmatrix} RB & M \\ M^T B & \widetilde{R} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} R & M & 0 & 0 & 0 \\ M^T & \widetilde{R} & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix} \begin{bmatrix} B & 0 \\ 0 & I \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and the elements of the symmetric matrix $\hat{\Gamma} := \left[\hat{\Gamma}_{ij}\right]_{i,\,j=1,\ldots,5}$ are defined as

$$\hat{\Gamma}_{11} = A^T R + RA + \varepsilon_1^{-1} A_d^T A_d + \varepsilon_3^{-1} N_1^T N_1 + \varepsilon_4^{-1} H^T H + \lambda_{\max}(\overline{M}_2^T \overline{M}_2)\, \varepsilon_2^{-1} N_2^T N_2$$

$$\hat{\Gamma}_{12} = A^T M$$

$$\hat{\Gamma}_{13} = \varepsilon^{\frac{1}{2}} R$$

$$\hat{\Gamma}_{14} = \varepsilon^{\frac{1}{2}} M$$

$$\hat{\Gamma}_{15} = \varepsilon_3^{\frac{1}{2}} R M_1$$

$$\hat{\Gamma}_{23} = \varepsilon^{\frac{1}{2}} M^T$$

$$\hat{\Gamma}_{24} = \varepsilon^{\frac{1}{2}}\, \widetilde{R}$$

$$\hat{\Gamma}_{25} = \varepsilon_3^{\frac{1}{2}} M^T M_1$$

$$\hat{\Gamma}_{33} = \hat{\Gamma}_{44} = \hat{\Gamma}_{55} = -I$$

$$\hat{\Gamma}_{22} = \hat{\Gamma}_{34} = \hat{\Gamma}_{35} = \hat{\Gamma}_{45} = 0\,.$$

Based on the *Projection lemma,* it follows that (29) has a solution if and only if

$$\hat{\mathrm{N}}^{T\perp} \hat{\Gamma}\, \hat{\mathrm{N}}^{T\perp T} < 0 \tag{30}$$

$$\hat{\mathrm{M}}^{T\perp} \hat{\Gamma}\, \hat{\mathrm{M}}^{T\perp T} < 0\,. \tag{31}$$

Then, by some calculation, we have

$$\hat{\mathrm{N}}^{T\perp} = \begin{bmatrix} V_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix} \begin{bmatrix} R_0 & M_0 & 0 & 0 & 0 \\ M_0^T & \widetilde{R}_0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, \tag{32}$$

$$\hat{\mathrm{M}}^{T\perp} = \begin{bmatrix} U_1 & 0 & 0 & 0 & U_2 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \end{bmatrix} \tag{33}$$

where $\begin{bmatrix} U_1 & U_2 \end{bmatrix} = \begin{bmatrix} C & M_2 \end{bmatrix}^{T\perp}$ and $V_1 = B^{\perp}$.

Using the Schur - complement formula, it is easy to see that (30) and (31) are equivalent to

$$R_0 A^T + A R_0 + R_0(\varepsilon_1^{-1} A_d^T A_d + \varepsilon_3^{-1} N_1^T N_1 + \varepsilon_4^{-1} H^T H + \lambda_{\max}(\overline{M}_2^T \overline{M}_2)\, \varepsilon_2^{-1} N_2^T N_2) R_0 + \varepsilon I < 0 \tag{34}$$

and

$$\hat{\Gamma}_{11} + \hat{\Gamma}_{13}\hat{\Gamma}_{13}^T + \hat{\Gamma}_{14}\hat{\Gamma}_{14}^T + \hat{\Gamma}_{15}\hat{\Gamma}_{15}^T < 0 \tag{35}$$

or

$$A^T R + RA + R(\varepsilon I + \varepsilon_3 M_1 M_1^T) R + \varepsilon M M^T + \varepsilon_1^{-1} A_d^T A_d + \varepsilon_3^{-1} N_1^T N_1 + \varepsilon_4^{-1} H^T H + \lambda_{\max}(\overline{M}_2^T \overline{M}_2)\, \varepsilon_2^{-1} N_2^T N_2 < 0. \tag{36}$$

According to the (1,1) block of P and P^{-1} in (27), we have

$$R - R_0^{-1} = M\, \widetilde{R}^{-1} M^T R_0 \geq 0. \tag{37}$$

This further implies

$$rank\,(R - R_0^{-1}) \leq \hat{n}\,. \tag{38}$$

In addition, when (34), (36) and (38) are satisfied, $\hat{n}$-order controller (6) corresponding to a feasible solution can be obtained by using the result of matrix inequality (29). Then, we obtain the following result:

Theorem 2: If there exist positive scalars $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ such that the *quadratic matrix inequalities* (QMIs) (34), (36), respectively, have positive definite solutions R_0 and R and

1. the matrices R_0, R, M satisfying the following conditions $R R_0 + M M_0^T = I$, $M^T R_0 + \widetilde{R} M_0^T = 0$, $R M_0 + M \widetilde{R}_0 = 0$ and $M^T M_0 + \widetilde{R}\widetilde{R}_0 = I$.
2. the matrices R_0 and R satisfying the non-convex inequality (38),

then, the stochastic control (6) with parameters

$$\Omega := \begin{bmatrix} 0 & C_K \\ B_K & A_K \end{bmatrix}$$

can be easily obtained by solving (29) and will be such that the augmented closed-loop system (7) is exponentially stable in the mean square for all admissible parameter uncertainties, and the non-linear disturbance input $f(.)$ and independent of the unknown time-delay h.

3.3 Example

Consider a second-order non-linear stochastic system with time-delay in the state as

$$\begin{bmatrix} dx_1 \\ dx_2 \end{bmatrix} = [(\begin{bmatrix} -4.3 & 0.8 \\ 0.9 & -1.2 \end{bmatrix} + \Delta A(t)) x(t) + (\begin{bmatrix} 0.5 & 0.2 \\ 0 & 0.7 \end{bmatrix} + \Delta A_d(t)) x(t-h) + \begin{bmatrix} 0.1 & 0 \\ 0 & -1 \end{bmatrix} u(t) + \begin{bmatrix} f_1(x(t)) \\ f_2(x(t)) \end{bmatrix}] dt + \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix} dw$$

where $x(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$ is the state vector and the uncertainty terms $f_i\,(x(t))$; $i = 1, 2$, are assumed to be norm-bounded such that the matrix H has been considered as $diagonal\ \{0.4, 0.3\}$.

Consider $h = 2$ seconds as the time-delay parameter and $\hat{n} = 2$ is the order of stochastic feedback control (6). The required stochastic feedback control (6) is obtained according to Theorem 2. Robust stability of the dynamics in the presence of disturbance has been depicted in Figure 1. Therefore, we conclude that system (27) can be stabilized by the control law (6) which has been depicted in Figure 2.

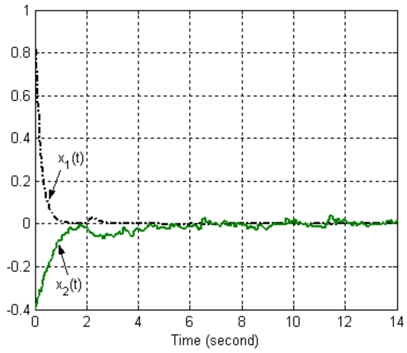

Fig. 1. Regulation of state dynamics under a delay of 2 seconds

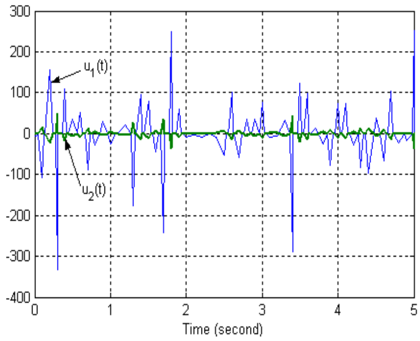

Fig. 2. Control input signals

4. CONCLUSION

In this paper, we have studied the problem of stochastic output feedback control for a class of non-linear time-delay stochastic systems. An LMI approach has been developed to design a linear, delayless, uncertainty- independent control such that for all admissible uncertainties as well as non-linear disturbances ensures stochastically exponentially stability in the mean square, independent of the time-delay.

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