STATISTICAL ANALYSIS OF STABLE FDLCP SYSTEMS DESCRIBED BY HIGHER ORDER DIFFERENTIAL EQUATIONS

B.P. Lampe [*], **E.N. Rosenwasser** [**]

[*] *University of Rostock, D-18051 Rostock, Germany*
bernhard.lampe@etechnik.uni-rostock.de
[**] *St. Petersburg State University of Ocean Technology*
k10@smtu.ru

Abstract: The paper investigates the response of stable finite dimensional linear continuous-time periodic (FDLCP) systems to white noise input. The FDLCP system is described by differential equations of higher order. Closed formulae for calculating the matrix variance of the output, as well as the mean variance and the $\mathcal{H}_2$-norm of the system are derived on basis of the parametric transfer matrix. These formulae only employ matrices of finite dimensions. An example was computed with MATLAB.

Keywords: Linear systems, Time-varying systems, Periodic motion, $\mathcal{H}_2$-norm

1. INTRODUCTION

One of the most fundamental problems in control theory consists in calculating the response of various types of systems on stationary stochastic inputs. In particular, the solution of this problem proves to be actual for FDLCP systems, which arise in connection with numerous theoretical questions and practical applications (Bittanti and Colaneri 2001). In analogy to the theory of continuous-time LTI systems (Kwakernaak and Sivan 1972, Green and Limebeer 1995), presently two basic approaches exist for the solution of the statistical analysis problem for FDLCP systems:

(1) Time-domain methods, based on the solution of the periodic Lyapunov differential equation (Bittanti 1986, Bolzern and Colaneri 1988, Colaneri 2000).
(2) Frequency-domain methods, using the Laplace transformation and the transfer function concept.

The practical realization of the methods described in (Araki *et al.* 1996, Wereley and Hall 1990, Zhang and Zhang 1997, Zhou *et al.* 2001, Zhou *et al.* 2003, Cantoni and Glover 2000, Möllerstedt and Bernhardsson 2000) leads to operations with infinite-dimensional matrices and determinants, what results in substantial difficulties. The approach presented in (Rosenwasser 1977, Lampe and Rosenwasser 2001, Lampe and Rosenwasser 2003) describes FDLCP systems in state space, and it allows to overcome this difficulty, only operations with finite-dimensional matrices are used there.

In (Lampe and Rosenwasser 2004) the authors generalized this approach to FDLCP systems, which are described by differential equations of higher order, among them are systems with differentiated input. Hereby, the general formulae in (Lampe and Rosenwasser 2004) are applicable to the wide class of e-dichotomic systems, which do not possess multipliers on the periphery of the unit circle. Thus unstable systems are included.

In practical applications the FDLCP system often has to be stable. In this important special case the general relations of the paper (Lampe and Rosenwasser 2004) might be substantially simplified, what leads to essentially simpler computing procedures and new propositions. The usage of the method is illustrated by an example.

2. QUASI-STATIONARY MOTION

1. The paper considers FDLCP systems described by a matrix differential equation of the form

$$\frac{\mathrm{d}^\ell y}{\mathrm{d}t^\ell} - a_1(t)\frac{\mathrm{d}^{\ell-1} y}{\mathrm{d}t^{\ell-1}} - \ldots - a_\ell(t)y = b_1(t)\frac{\mathrm{d}^{\ell-1} u}{\mathrm{d}t^{\ell-1}} + \ldots + b_\ell(t)u \qquad (1)$$

where $\ell > 1$, $y(t)$, $u(t)$ are matrices of dimensions $p \times n$ and $m \times n$, respectively. The coefficients $a_i(t) = a_i(t+T)$, $b_i(t) = b_i(t+T)$ are periodic matrices of appropriate dimensions. Suppose all $a_i(t)$ to be continuous, and every matrix $b_i(t)$ should have bounded derivatives up to the order $\ell - i + 1$. For $n = 1$ equation (1) is called vectorial.

2. Consider the homogeneous matrix equation

$$\frac{\mathrm{d}^\ell y}{\mathrm{d}t^\ell} - a_1(t)\frac{\mathrm{d}^{\ell-1} y}{\mathrm{d}t^{\ell-1}} - \ldots - a_\ell(t) y = O_{pp}\,. \quad (2)$$

In (2) and furthermore, O_{ik} means the $i \times k$ zero matrix. In what follows, we propose that equation (2) is asymptotically stable, that means all its solutions tend to zero for $t \to \infty$. Let the $p \times p$ matrix $h(t,\tau)$ be the solution of (2) satisfying the initial conditions

$$\begin{aligned} h(t,\tau)|_{t=\tau} &= \frac{\partial h(t,\tau)}{\partial t}\Big|_{t=\tau} = \ldots \\ &= \frac{\partial^{\ell-2} h(t,\tau)}{\partial t^{\ell-2}}\Big|_{t=\tau} = O_{pp} \\ \frac{\partial^{\ell-1} h(t,\tau)}{\partial t^{\ell-1}}\Big|_{t=\tau} &= I_p \end{aligned} \quad (3)$$

where I_p stands for the $p \times p$ unit matrix. Then, the $p \times m$ matrix

$$r(t,\tau) = \sum_{\eta=1}^{\ell} (-1)^{\ell-\eta} \frac{\partial^{\ell-\eta}}{\partial \tau^{\ell-\eta}} \left[h(t,\tau) b_\eta(\tau)\right] \quad (4)$$

is called the *weighting matrix* of equation (1).

3. Consider the integral expression

$$y_\infty(t) = \int_{-\infty}^{t} r(t,\tau) u(\tau)\,\mathrm{d}\tau \quad (5)$$

where $u(t)$ is a $m \times 1$ vector. It is directly verified that in the case where the integral in (5) converges and allows a satisfying number of differentiations with respect to t, then it defines a solution of the vector differential equation (1), which is called *quasi-stationary*. On the set of inputs $u(t)$ for which quasi-stationary solutions exist, the expression (5) defines a linear periodic integral operator

$$y_\infty(t) = L[u(t)] \quad (6)$$

which is regular in the sense of (Rosenwasser and Lampe 2000). Thus, for $Re\lambda > -\gamma$, where γ is a certain positive number, there exists the parametric transfer matrix (PTM) of the operator (5) $W(\lambda, t)$, which is given by the formulae

$$\begin{aligned} W(s,t) &= L[\mathrm{e}^{st}]\,\mathrm{e}^{-st} = \int_{-\infty}^{t} r(t,\tau)\,\mathrm{e}^{-s(t-\tau)}\,\mathrm{d}\tau \\ W(s,t+T) &= W(s,t)\,. \end{aligned} \quad (7)$$

4. Assume $u(t)$ in (5) to be a centralized white noise with the spectral density $\Phi(s) = I_m$. Then, as follows from (Lampe and Rosenwasser 2001), (Lampe and Rosenwasser 2003), the covariance matrix of the output of the regular operator (5) is determined by the formula

$$\begin{aligned} &K_y(t_1,t_2) = \\ &\frac{1}{2\pi \mathrm{j}} \int_{-\mathrm{j}\infty}^{\mathrm{j}\infty} W(-s,t_1) W'(s,t_2)\,\mathrm{e}^{s(t_2-t_1)}\,\mathrm{d}s \end{aligned} \quad (8)$$

where the prime is written for transposition, and $\mathrm{j} = \sqrt{-1}$. Under the taken assumptions both sides of relation (8) are continuous in both variables t_1, t_2. Therefore, in (8) we can assume $t_1 = t_2 = t$. Thus with the notation

$$K(t) = K_y(t_1,t_2)|_{t_1=t_2=t} \quad (9)$$

we obtain

$$\begin{aligned} K(t) &= \frac{1}{2\pi \mathrm{j}} \int_{-\mathrm{j}\infty}^{\mathrm{j}\infty} W(-s,t) W'(s,t)\,\mathrm{d}s \\ &= K(t+T)\,. \end{aligned} \quad (10)$$

In what follows, the matrix $K(t)$ is called variance matrix of the quasi-stationary output, or shortly, *variance matrix.* The scalar function

$$d_y(s) = \operatorname{trace} K(t) = d_y(t+T) \quad (11)$$

is named the *variance* of the quasi-stationary output, and the positive number

$$\bar{d}_y = \frac{1}{T}\int_0^T d_y(t)\,\mathrm{d}t \quad (12)$$

its *mean variance.* The quantity

$$||L||_2 = +\sqrt{\bar{d}_y} \quad (13)$$

is called $\mathcal{H}_2$*-norm* of the operator (6) (of the equation (1)). The goal of the present paper consists in constructing closed formulae for determining the matrix variance $K(t)$ and the statistical characteristics (11)-(13).

3. PRELIMINARIES

1. The $p\ell \times p\ell$ matrix

$$A_c(t) = \begin{bmatrix} O_{pp} & I_p & O_{pp} & \ldots & O_{pp} & O_{pp} \\ O_{pp} & O_{pp} & I_p & \ldots & O_{pp} & O_{pp} \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ O_{pp} & O_{pp} & O_{pp} & \ldots & O_{pp} & I_p \\ a_\ell(t) & a_{\ell-1}(t) & a_{\ell-2}(t) & \ldots & a_2(t) & a_1(t) \end{bmatrix} \quad (14)$$

is called the *accompanying matrix* for equation (1), and the $p\ell \times p\ell$ matrix $H(t)$ defined by

$$\frac{\mathrm{d}H(t)}{\mathrm{d}t} = A_c(t) H(t), \qquad H(0) = I_{p\ell} \quad (15)$$

is named *transition matrix.* We assume that $H(t)$ is known on the interval $0 \le t \le T$, what might be achieved by numerical integration.

2. The matrix

$$M = H(T) \quad (16)$$

is called the *monodromy matrix.* The eigenvalues of the monodromy matrix, that are the roots of the equation

$$\det(sI_{p\ell} - M) = 0 \tag{17}$$

are referred to as the *multipliers* of equation (1). Let $\mu_1, \dots, \mu_\lambda$, $(\lambda \le p\ell)$ be the different multipliers of equation (1), and $\nu_1, \dots, \nu_\lambda$ their corresponding multiplicities such that $\nu_1 + \dots + \nu_\lambda = p\ell$. Then, equation (2) turns out to be stable if and only if

$$|\mu_i| < 1, \quad (i = 1, \dots, \lambda) \tag{18}$$

is true.

3. Consider the matrix

$$G_T(s,t,\tau) = \begin{cases} H(t)\left(I_{p\ell} - \mathrm{e}^{-sT} M\right)^{-1} H^{-1}(\tau) \\ \qquad 0 \le \tau < t \le T \\ H(t)\left(I_{p\ell} - \mathrm{e}^{-sT} M\right)^{-1} \mathrm{e}^{-sT} MH^{-1}(\tau) \\ \qquad 0 \le t < \tau \le T. \end{cases} \tag{19}$$

By construction for $t \ne \tau$ the relations

$$\frac{\partial G_T(s,t,\tau)}{\partial t} = A_c(t) G_T(s,t,\tau) \tag{20}$$

$$\frac{\partial G_T(s,t,\tau)}{\partial \tau} = -G_T(s,t,\tau) A_c(\tau) \tag{21}$$

hold. Moreover for $t = \tau$ we obtain

$$G_T(s,t+0,t) - G_T(s,t-0,t) = I_{p\ell}\,. \tag{22}$$

4. Using the identity

$$\left(I_{p\ell} - \mathrm{e}^{-sT} M\right)^{-1} = I_{p\ell} + \left(I_{p\ell} - \mathrm{e}^{-sT} M\right)^{-1} \mathrm{e}^{-sT} M \tag{23}$$

the matrix (19) might be represented in the form

$$G_T(s,t,\tau) = \tilde{G}_T(s,t,\tau) + G(t,\tau) \tag{24}$$

where for $0 \le t, \tau \le T$

$$\tilde{G}(s,t,\tau) = H(t)\left(I_{p\ell} - \mathrm{e}^{-sT} M\right)^{-1} \mathrm{e}^{-sT} MH^{-1}(\tau) \tag{25}$$

with

$$G(t,\tau) = \begin{cases} H(t)H^{-1}(\tau), & t > \tau \\ O_{p\ell,p\ell}\,, & t < \tau\,. \end{cases} \tag{26}$$

5. Furthermore, we use the block representation

$$\begin{aligned} G_T(s,t,\tau) &= \left[\, g_{T,ik}(s,t,\tau) \,\right] \\ \tilde{G}(s,t,\tau) &= \left[\, \tilde{g}_{T,ik(s,t,\tau)} \,\right] \\ G(t,\tau) &= \left[\, g_{ik}(t,\tau) \,\right] \end{aligned} \tag{27}$$

where $i,k = 1, \dots, \ell$, and $g_{T,ik}(s,t,\tau)$, $\tilde{g}_{T,ik}(s,t,\tau)$, $g_{ik}(t,\tau)$ are $p \times p$ matrices. Moreover, introduce the matrix

$$B(s) = \left(I_{p\ell} - \mathrm{e}^{-sT} M\right)^{-1} \mathrm{e}^{-sT} M \tag{28}$$

and its block form

$$B(s) = \left[\, b_{ik}(s) \,\right], \qquad (i,k = 1, \dots, \ell) \tag{29}$$

where the $b_{ik}(s)$ are $p \times p$ matrices. Thus we find

$$\tilde{G}_T(s,t,\tau) = H(t)B(s)H^{-1}(\tau), \quad 0 \le t, \tau \le T\,. \tag{30}$$

From (23)-(30) the existence of a representation in the shape

$$g_{T,ik}(s,t,\tau) = \tilde{g}_{T,ik}(s,t,\tau) + g_{ik}(t,\tau) \tag{31}$$

emerge, where

$$\tilde{g}_{T,ik}(s,t,\tau) = \sum_{\mu=1}^{\ell} \sum_{\nu=1}^{\ell} \alpha_{\mu\nu}^{ik}(t) b_{\mu\nu}(s) \beta_{\mu\nu}^{ik}(\tau) \tag{32}$$

is valid with known matrices $\alpha_{\mu\nu}^{ik}(t)$, $\beta_{\mu\nu}^{ik}(\tau)$. Hereby, the matrices $\alpha_{\mu\nu}^{ik}(t)$ are uniquely determined by the matrix $H(t)$, and the matrices $\beta_{\mu\nu}^{ik}(\tau)$ are uniquely determined by the matrix $H^{-1}(\tau)$. Moreover, if the block representation

$$H(t) = \left[\, h_{ik}(t) \,\right], \quad H^{-1}(\tau) = \left[\, d_{ik}(\tau) \,\right] \quad i,k = 1, \dots, \ell \tag{33}$$

with $p \times p$ matrices $h_{ik}(t)$, $d_{ik}(\tau)$ is used, then we obtain

$$g_{ik}(t,\tau) = \begin{cases} \displaystyle\sum_{j=1}^{\ell} h_{ij}(t) d_{jk}(\tau), & t > \tau \\ O_{pp}\,, & t < \tau\,. \end{cases} \tag{34}$$

4. CALCULATION OF VARIANCE MATRIX

1. Introduce the notation

$$g_\eta(s,t,\tau) = g_{T,1\ell-\eta+1}(s,t,\tau) \tag{35}$$

and consider the matrix

$$r_T(s,t,\tau) = \sum_{\eta=1}^{\ell} (-1)^{\ell-\eta} \frac{\partial^{\ell-\eta}}{\partial \tau^{\ell-\eta}} \left[g_1(s,t,\tau) b_\eta(\tau) \right]. \tag{36}$$

As was shown in (Lampe and Rosenwasser 2004), after transforming (10) the variance matrix could be expressed by

$$K(t) = \frac{T}{2\pi \mathrm{j}} \int_{-\mathrm{j}\omega/2}^{\mathrm{j}\omega/2} \int_0^T r_T(-s,t,\tau) r'(s,t,\tau)\, \mathrm{d}\tau\, \mathrm{d}s \tag{37}$$

where $\omega = 2\pi/T$.

2. Below we formulate a number of statements which allow to find the matrix $K(t)$ with the help of the transition matrix $H(t)$, which is precalculated on the interval $0 \le t \le T$.

Lemma 1. The presentation

$$\frac{\partial^\rho g_1(t,s,\tau)}{\partial \tau^\rho} = \sum_{k=1}^{\rho+1} g_k(s,t,\tau) d_k(\tau), \quad \rho = 1, \dots, \ell-1 \tag{38}$$

takes place, where $d_k(\tau)$ are known $p \times p$ matrices.

Proof. Relation (38) could be held by equalization of the elements of the first rows of the left and right side of matrix equation (21), and substitution of the undesired variables.

Lemma 2. Matrix (36) allows a representation of the form

$$r_T(s,t,\tau) = \tilde{r}_T(s,t,\tau) + r_0(t,\tau) \tag{39}$$

with

$$\tilde{r}_T(s,t,\tau) = \sum_{i=1}^{\ell}\sum_{k=1}^{\ell} \phi_{ik}(t) b_{ik}(s) \psi_{ik}(\tau) \tag{40}$$

where $\phi_{ik}(t)$, $\psi_{ik}(\tau)$ are known matrices. Moreover, $r_0(t,\tau)$ is a known matrix, such that

$$r_0(t,\tau) = O_{pm}\,, \qquad t < \tau \tag{41}$$

is true.

Proof. Relations (39)-(41) could be derived by inserting (38) into (36), and regarding formulae (31)-(34).

3. It follows the main result of the paper.

Theorem 1. Construct the representation (39)-(40). Denote

$$\gamma_{ik\eta\rho} = \int_0^T \psi_{ik}(\tau)\psi'_{\eta\rho}(\tau)\,\mathrm{d}\tau \tag{42}$$

$$\delta_{ik\eta\rho} = \frac{T}{2\pi \mathrm{j}} \int_{-\mathrm{j}\omega/2}^{\mathrm{j}\omega/2} b_{ik}(-s)\gamma_{ik\eta\rho} b'_{\eta\rho}(s)\,\mathrm{d}s\,. \tag{43}$$

Then, the variance matrix $K(t)$ for $0 \le t \le T$ might be written in the form

$$K(t) = \sum_{i=1}^{\ell}\sum_{k=1}^{\ell}\sum_{\eta=1}^{\ell}\sum_{\rho=1}^{\ell} \phi_{ik}(t)\delta_{ik\eta\rho}\phi'_{\eta\rho}(t) + \int_0^t r_0(t,\tau) r'_0(t,\tau)\,\mathrm{d}\tau\,. \tag{44}$$

Proof. From (40) follows

$$r_T(-s,t,\tau) = \sum_{\eta=1}^{\ell}\sum_{\rho=1}^{\ell} \phi_{\eta\rho}(t) b_{\eta\rho}(-s)\psi_{\eta\rho}(\tau) + r_0(t,\tau)\,. \tag{45}$$

Inserting (40) and (45) into (37) yields

$$K(t) = K_1(t) + K_2(t) + K_3(t) + K_4(t) \tag{46}$$

with

$$K_1(t) = \frac{T}{2\pi \mathrm{j}} \int_{-\mathrm{j}\omega/2}^{\mathrm{j}\omega/2} \int_0^T \tilde{r}_T(-s,t,\tau)\tilde{r}'_T(s,t,\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$

$$K_2(t) = \frac{T}{2\pi \mathrm{j}} \int_{-\mathrm{j}\omega/2}^{\mathrm{j}\omega/2} \int_0^T \tilde{r}_T(-s,t,\tau) r'_0(t,\tau)\,\mathrm{d}\tau\,\mathrm{d}s \tag{47}$$

$$K_3(t) = \frac{T}{2\pi \mathrm{j}} \int_{-\mathrm{j}\omega/2}^{\mathrm{j}\omega/2} \int_0^T r_0(t,\tau)\tilde{r}'_T(s,t,\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$

$$K_4(t) = \int_0^t r_0(t,\tau) r'_0(t,\tau)\,\mathrm{d}\tau\,.$$

We will show that $K_2(t) = O_{pp}$. For that reason we present the second relation in (47) as

$$K_2(t) = \sum_{i=1}^{\ell}\sum_{k=1}^{\ell} \int_0^T r_0(t,\tau)\phi_{ik}(t) \cdot \left[\frac{T}{2\pi \mathrm{j}} \int_{-\mathrm{j}\omega/2}^{\mathrm{j}\omega/2} b_{ik}(-s)\,\mathrm{d}s\right]\psi_{ik}(\tau)\,\mathrm{d}\tau\,. \tag{48}$$

Owing to the residue theorem according to (18) we obtain

$$\frac{T}{2\pi j} \int_{-\mathrm{j}\omega/2}^{\mathrm{j}\omega/2} \left(\mathrm{e}^{sT} I_{p\ell} - M\right)^{-1}\,\mathrm{d}s = O_{p\ell,p\ell}\,. \tag{49}$$

Hence, for all $i,k = 1,\ldots,\ell$

$$\frac{T}{2\pi \mathrm{j}} \int_{-\mathrm{j}\omega/2}^{\mathrm{j}\omega/2} b_{ik}(-s)\,\mathrm{d}s = O_{pp} \tag{50}$$

is valid, and from (48) follows $K_2(t) = O_{pp}$. Analogously, $K_3 = O_{pp}$ could be demonstrated. As a result from (46) with the help of (47) and (40) we arrive at formula (44).

4. The next statement yields a constructive procedure for computing the constant matrix (43).

Theorem 2. Denote

$$\begin{aligned} C_{ik}(z) &= b_{ik}(s)|_{\mathrm{e}^{sT}=z} \\ \tilde{C}_{ik}(z) &= z^{-1} b_{ik}(-s)|_{\mathrm{e}^{sT}=z}\,. \end{aligned} \tag{51}$$

The matrices (51) are rational. Besides, the poles of the matrices $C_{ik}(z)$ are contained in the set of multipliers $\mu_1,\ldots,\mu_\lambda$ of equation (1), and hence are located inside the unit circle. The poles of the matrix $\tilde{C}_{ik}(z)$ are contained in the set of numbers $\mu_1^{-1},\ldots,\mu_\lambda^{-1}$, and thus are located all outside the unit circle. Therefore, matrix (43) can be computed by the formula

$$\delta_{ik\eta\rho} = \sum_{j=1}^{\lambda} \operatorname*{Res}_{z=\mu_j} \left[\tilde{C}_{ik}(z)\gamma_{ik\eta\rho} C'_{\eta\rho}(z)\right]. \tag{52}$$

In particular, when all multipliers $\mu_1,\ldots,\mu_{p\ell}$ are simple, so formula (52) takes the form

$$\delta_{ik\eta\rho} = \sum_{j=1}^{p\ell} \tilde{C}_{ik}(\mu_j)\gamma_{ik\eta\rho} \operatorname*{Res}_{z=\mu_j} \left[C'_{\eta\rho}(z)\right]\,. \tag{53}$$

Proof. From (28) we obtain for $\mathrm{e}^{sT} = z$

$$C(z) = B(s)|_{\mathrm{e}^{sT}=z} = (zI_{p\ell} - M)^{-1} M \tag{54}$$

the poles of which are contained in the set of multipliers of equation (1). The same is also true for the elements $C_{ik}(z)$ of the block matrix (54). Analogously, the properties of the matrices $\tilde{C}_{ik}(z)$

emerge. Furthermore, applying in (43) $e^{sT} = z$, so we obtain owing to (51)

$$\delta_{ik\eta\rho} = \frac{1}{2\pi j} \oint \tilde{C}_{ik}(z)\gamma_{ik\eta\rho}C'_{\eta\rho}(z)\,dz \quad (55)$$

where the contour integral has to be taken over the unit circle in positive direction (anti-clockwise). Calculating the integral (55) by applying the residue theorem yields (52). If in addition all multipliers are simple, we obtain

$$\operatorname*{Res}_{z=\mu_j}\left[\tilde{C}_{ik}(z)\gamma_{ik\eta\rho}C'_{\eta\rho}(z)\right] = \tilde{C}_{ik}(\mu_j)\gamma_{ik\eta\rho}\operatorname*{Res}_{z=\mu_\rho}C'_{\eta\rho}(z) \quad (56)$$

what together with (52) leads to formula (53).

5. NUMERICAL EXAMPLE

1. Consider the equation [1]

$$\frac{d^2y}{dt^2} + (3-\cos t)\frac{dy}{dt} + (2-\cos t)y = \frac{du}{dt} + u \quad (57)$$

such that $\ell = 2$, $T = 2\pi$, $\omega = 1$. The accompanying matrix (14) takes the form

$$A_c(t) = \begin{bmatrix} 0 & 1 \\ \cos t - 2 & \cos t - 3 \end{bmatrix}. \quad (58)$$

2. The transition matrix $H(t)$ might be written in the form

$$H(t) = \begin{bmatrix} h_1(t) & h_2(t) \\ h_3(t) & h_4(t) \end{bmatrix} = \quad (59)$$
$$\begin{bmatrix} e^{-t} + f(t) & f(t) \\ -e^{-t} - f(t) + e^{-2t}e^{\sin t} & -f(t) + e^{-2t}e^{\sin t} \end{bmatrix}$$

with

$$f(t) = e^{-t}\int_0^t e^{-\nu}e^{\sin\nu}\,d\nu\,. \quad (60)$$

3. We write the matrix $H^{-1}(\tau)$ in the form

$$H^{-1}(\tau) = \begin{bmatrix} d_1(\tau) & d_2(\tau) \\ d_3(\tau) & d_4(\tau) \end{bmatrix} \quad (61)$$
$$= \begin{bmatrix} -e^{3\tau}e^{-\sin\tau}f(\tau) + e^{\tau} & -e^{3\tau}e^{-\sin\tau}f(\tau) \\ e^{2\tau}e^{-\sin\tau} + e^{3\tau}e^{-\sin t}f(\tau) - e^{\tau} & e^{2\tau}e^{-\sin\tau} + e^{3\tau}e^{-\sin\tau}f(\tau) \end{bmatrix}.$$

4. The monodromy matrix (16) then emerge as

$$M = \begin{bmatrix} e^{-2\pi} + V & V \\ -e^{-2\pi} + e^{-4\pi} - V & e^{-4\pi} - V \end{bmatrix} \quad (62)$$

with

$$V = e^{-2\pi}\int_0^{2\pi} e^{-\nu}e^{\sin\nu}\,d\nu\,. \quad (63)$$

5. In the given case, using (62) we obtain

$$\det(zI_2 - M) = \left(z - e^{-2\pi}\right)\left(z - e^{-4\pi}\right) \quad (64)$$

such that equation (57) has the two simple multipliers $\mu_1 = e^{-2\pi}$ and $\mu_2 = e^{-4\pi}$, and turns out to be asymptotically stable.

6. Using the above results, the matrix $B(s)$ (28) might be represented in the form

$$B(s) = \begin{bmatrix} b_1(s) & b_2(s) \\ b_3(s) & b_4(s) \end{bmatrix}. \quad (65)$$

In (65) we have

$$b_i(s) = \frac{b_{i1}}{e^{sT} - \mu_1} + \frac{b_{i2}}{e^{sT} - \mu_2} \qquad i = 1,2,3,4 \quad (66)$$

with

$$\begin{aligned} b_{11} &= \mu_1(1+\alpha), & b_{12} &= -\mu_2\alpha \\ b_{21} &= \mu_1\alpha, & b_{22} &= -\mu_2\alpha \\ b_{31} &= -\mu_1(1+\alpha), & b_{32} &= \mu_2(1+\alpha) \\ b_{41} &= -\mu_1\alpha, & b_{42} &= \mu_2(1+\alpha) \end{aligned} \quad (67)$$

where α is the constant

$$\alpha = \frac{V}{\mu_1 - \mu_2}\,. \quad (68)$$

7. Now let us construct for the given example the function $r_T(s,t,\tau)$. Assume

$$G_T(s,T,\tau) = \begin{bmatrix} g_2(s,t,\tau) & g_1(s,t,\tau) \\ \dots & \dots \end{bmatrix}. \quad (69)$$

The dots in (69) are written for functions having no influence on the calculations. From (36) and (57) we obtain

$$r_T(s,t,\tau) = g_1(s,t,\tau) - \frac{\partial g_1(s,t,\tau)}{\partial\tau}. \quad (70)$$

Furthermore, (21) and (58) lead to

$$\frac{\partial g_1(s,t,\tau)}{\partial\tau} = -g_2(s,t,\tau) - g_1(s,t,\tau)(\cos\tau - 3)\,. \quad (71)$$

From (70) and (71) follows

$$r_T(s,t,\tau) = g_2(s,t,\tau) + g_1(s,t,\tau)(\cos\tau - 2)\,. \quad (72)$$

8. Now, let us construct for the function (72) the representation of the form (39)-(40). Using the above calculations we obtain

$$r_T(s,t,\tau) = \sum_{i=1}^{4} r_i(t,\tau)b_i(s) + r_0(t,\tau) \quad (73)$$

where

$$r_1(t,\tau) = h_1(t)f_1(\tau) \quad r_2(t,\tau) = h_1(t)f_2(\tau)$$
$$r_3(t,\tau) = h_2(t)f_1(\tau) \quad r_4(t,\tau) = h_2(t)f_2(\tau)$$

and, moreover

[1] The example was calculated by dr. V. Rybinskii, to whom the authors gratefully acknowledge.

$$r_0(t,\tau) = \begin{cases} h_1(t)f_1(\tau) + h_2(t)f_2(\tau) & t > \tau \\ 0 & t < \tau \end{cases}$$

with

$$f_1(\tau) = d_1(\tau) + d_2(\tau)(\cos\tau - 2)$$
$$f_2(\tau) = d_3(\tau) + d_4(\tau)(\cos\tau - 2)\,.$$

9. With the introduced notation the formula for the output variance (44) takes the form

$$d_y(t) = \sum_{k=1}^{4}\sum_{i=1}^{4} \delta_{ik} \int_0^{2\pi} r_i(t,\tau) r_k(t,\tau)\,\mathrm{d}\tau + \int_0^t r_0^2(t,\tau)\,\mathrm{d}\tau$$

with

$$\delta_{ik} = \frac{T}{2\pi \mathrm{j}} \int_{-\mathrm{j}\omega/2}^{\mathrm{j}\omega/2} b_i(s) b_k(-s)\,\mathrm{d}s\,. \tag{74}$$

This result, after inserting (66) into (74) and evaluating the integral yields

$$\delta_{ik} = \frac{b_{i1}b_{k1}}{1-\mu_1^2} + \frac{b_{i1}b_{k2} + b_{i2}b_{k1}}{1-\mu_1\mu_2} + \frac{b_{i2}b_{k2}}{1-\mu_2^2}$$
$$i,k = 1,2,3,4\,.$$

10. The computation results for the variance $d_y(t)$ by the above formula are shown in Figure 1. The numerical value for the mean variance is

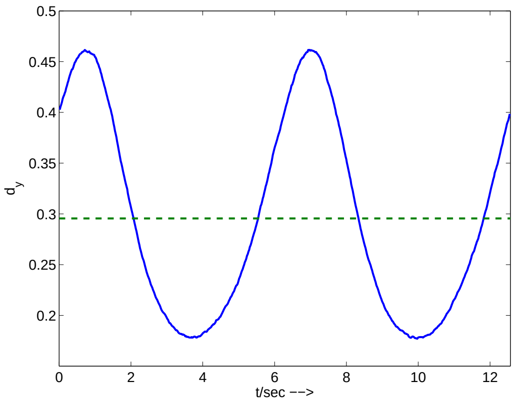

Figure 1. Output variance $d_y(t)$

$\bar{d}_y = 0.2943$, and the $\mathcal{H}_2$-norm (57) takes the value $\|L\|_2 = \sqrt{\bar{d}_y} = 0.5425\,.$

REFERENCES

Araki, M., Y. Ito and T. Hagiwara (1996). Frequency-response of sampled-data systems. *Automatica* **32**, 483–497.

Bittanti, S. (1986). The periodic Riccati equation. In: *Deterministic and stochastic linear-periodic systems* (S. Bittanti, Ed.). pp. 141–182. Time Series and Linear Systems. Springer-Verlag. Berlin, New York.

Bittanti, S. and Colaneri, P., Eds.) (2001). *Preprints: IFAC Workshop on Periodic Control Systems*. Como, Italy.

Bolzern, P. and P. Colaneri (1988). The periodic Lyapunov equation. *SIAM J. Matrix Anal. Appl.* **9**, 499–512.

Cantoni, M. and K. Glover (2000). Existence of right and left representations of the graph for linear periodically time-varying systems. *SIAM J. Contr. Optim.* **38**(3), 786–802.

Colaneri, P. (2000). Continuous-time periodic systems in H_2 and H_∞. Part I: Theoretical aspects; Part II: State feedback control. *Kybernetika* **36**(3), 211–242; 329–350.

Green, M. and D.J.N. Limebeer (1995). *Linear Robust Control*. Prentice-Hall. Englewood Cliffs, NJ.

Kwakernaak, H. and R. Sivan (1972). *Linear Optimal Control Systems*. Wiley-Interscience. New York.

Lampe, B.P. and E.N. Rosenwasser (2001). Statistical analysis and H_2-norm of finite dimensional linear time-periodic systems. In: *Proc. IFAC Workshop on Periodic Control Systems*. Como, Italy. pp. 9–14.

Lampe, B.P. and E.N. Rosenwasser (2003). Operational description and statistical analysis of linear periodic systems on the unbounded interval $-\infty < t < \infty$. *European J. Control* **9**(5), 508–521.

Lampe, B.P. and E.N. Rosenwasser (2004). Transfer functions and analysis of FDLCP systems described by differential equations of higher order. *European J. Control*. (subm. Aug).

Möllerstedt, E. and B. Bernhardsson (2000). Out of control because of harmonics. *IEEE Control Systems Magazine* **20**(4), 70–81.

Rosenwasser, E.N. (1977). *Lyapunov indices in linear control theory*. Nauka. Moscow. (in Russian).

Rosenwasser, E.N. and B.P. Lampe (2000). *Computer Controlled Systems - Analysis and Design with Process-orientated models*. Springer-Verlag. London Berlin Heidelberg.

Wereley, N. and S. Hall (1990). Frequency response of linear time-periodic systems. In: *Proc. 29th IEEE Conf. Decision Contr.* Honolulu, Hawaii. pp. 3650–3655.

Zhang, C. and J. Zhang (1997). H_2-performance of continuous periodically time-varying controllers. *Syst. Contr. Lett* **32**, 209–221.

Zhou, J., T. Hagiwara and M. Araki (2001). Trace formulas for the H_2 norm of linear continuous-time periodic systems. In: *Prepr. IFAC Workshop on Periodic Control Systems*. Como, Italy. pp. 3–8.

Zhou, J., T. Hagiwara and M. Araki (2003). Trace formula of linear continuous-time periodic systems via the harmonic Lyapunov equation. *Int. J. Control* **76**(5), 488–500.