ADAPTIVE LEARNING CONTROL OF LINEAR SYSTEMS BY OUTPUT ERROR FEEDBACK

Stefano Liuzzo * **Riccardo Marino** * **Patrizio Tomei** *

* *Dept. of Electronic Engineering, University of Rome Tor Vergata, Via del Politecnico 1, 00133 Roma, Italy. e-mail: liuzzo,marino,tomei@ing.uniroma2.it*

Abstract: This paper addresses the problem of designing an output error feedback tracking control for single-input, single-output, minimum phase, observable linear systems. The reference output signal is assumed to be smooth and periodic with known period. By developing in Fourier series expansion a suitable periodic input reference signal, an output error feedback adaptive learning control is designed which 'learns' the input reference signal by identifying its Fourier coefficients: exponential tracking of both the input and the output reference signals is achieved if the Fourier series expansion is finite, while arbitrary small tracking errors are guaranteed otherwise.

Keywords: Learning control, adaptive control, linear systems, output feedback

1. INTRODUCTION

Learning control was conceived in robotics (see (Arimoto *et al.*, 1984)) to design a control law such that the output of an uncertain system tracks a given periodic output reference characterizing repetitive tasks. The key idea is to use the information obtained in the preceding trial to improve the performance in the current one. Several contributions for state feedback learning control have been presented in (Jang *et al.*, 1995; Kim and Ha, 2000; Del Vecchio *et al.*, 2003). Output feedback controllers for linear and nonlinear systems have been proposed: (Owens and Munde, 2000; French *et al.*, 1999; Chien and Yao, 2004*a*; Chien and Yao, 2004*b*). In (Owens and Munde, 2000) an adaptive iterative learning control is proposed for minimum phase linear systems of relative degree one and the convergence to zero of the tracking error in $L_2(0,T)$ is proved, from any initial condition and for any reference in $L_2(0,T)$. This result was extended in (French *et al.*, 1999) to linear systems of any relative degree by resorting to a resetting procedure. Linear systems of known order n and known relative degree ρ are also considered in (Chien and Yao, 2004*b*), where the output tracking error can be made arbitrarily small in $L_2(0,T)$ from any bounded initial resetting error. Iterative learning schemes do not guarantee so far asymptotic output tracking from any initial condition for linear systems with relative degree greater than one. In this paper it is considered (as in (Chien and Yao, 2004*b*)) the class of minimum phase linear systems with known relative degree and high frequency gain sign, for which the problem of tracking a smooth periodic output reference, with known period, by adaptive output error feedback learning control is solved. Exponential tracking of both the input and the output reference or arbitrary small input and output tracking error is achieved under suitable assumptions: this result is obtained from any initial condition. The proposed adaptive learning control is not model based and has a fixed structure which includes a filter of order $\rho - 1$ and a dynamic estimator of order p to estimate p Fourier coefficients: only constant bounds on the system coefficients are required. The adaptive learning control proposed in this paper may be compared with adaptive controls (see (Marino and Tomei, 1995; Kristic *et al.*, 1995)) and the robust regulator (Serrani and Isidori, 2000). The robust regulator in (Serrani and Isidori, 2000) requires the output reference signal to be generated by a known linear exosystem and the corresponding input reference signal to be exactly generated by a known linear finite dimensional internal model. Adaptive controls can track arbitrary smooth bounded reference signals but do not guarantee in general exponential tracking.

2. BASIC ASSUMPTIONS AND PRELIMINARY RESULTS

Consider the following class of linear systems:

$$\begin{aligned} \dot{x} &= A_c x + bu + hy, \qquad x \in \Re^n \\ y &= C_c x, \qquad\qquad\quad\;\; y \in \Re \end{aligned} \tag{1}$$

where $b = [0, \cdots, 0, b_{n-\rho}, \cdots, b_0]^T \in \Re^n$ and $h = [h_1, \cdots, h_n]^T \in \Re^n$ are unknown constant vectors,

$$A_c = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \qquad C_c = [1, 0, \cdots, 0] \ .$$

The following hypotheses are made:

(A) The vector h belongs to the known compact set $S_h = \{h \in \Re^n : ||h|| \leq h_M\}$.
(B) The system is of known relative degree ρ, $1 \leq \rho \leq n$, all the zeros of the polynomial $p(s) = b_{n-\rho}s^{n-\rho} + \cdots + b_1 s + b_0$ have negative real part and the vector b belongs to the known compact set $S_b = \{[0, \cdots, 0, b_{n-\rho}, \cdots, b_0]^T \in \Re^n : 0 < b_{i,m} \leq b_i \leq b_{i,M},\ 0 \leq i \leq n-\rho\}$.
(C) The reference output signal $y_r(t) \in C^N$ (with $N > 3\rho + 1/2$) is periodic with known period T and is such that $|y_r(t)| \leq R_1$, $\forall t \in [0, T]$. Moreover $|y_r^{(1)}(t)| \leq R_{1d}$ with R_{1d} a known constant.

It follows from assumptions (B) and (C) that there exist a suitable initial condition x_{r0} and a bounded periodic reference input $u_r(t)$ of period T such that

$$\begin{aligned} \dot{x}_r &= A_c x_r + bu_r + hy_r, \quad x_r(0) = x_{r0}, \quad x_r \in \Re^n \\ y_r &= C_c x_r, \qquad\qquad\qquad\qquad\qquad\quad\; y_r \in \Re \ . \end{aligned}$$

Since the vectors h and b are unknown the reference input is unknown as well. If $\rho > 1$ define the periodic reference signal $\xi_{r,1}(t)$ generated by the following stable filter of order $\rho - 1$ with suitable initial conditions $\xi_r(0) = [\xi_{r,1}(0), \cdots, \xi_{r,\rho-1}(0)]^T$

$$\begin{aligned} \dot{\xi}_{r,1} &= -\lambda_1 \xi_{r,1} + \xi_{r,2} \\ &\vdots \\ \dot{\xi}_{r,\rho-1} &= -\lambda_{\rho-1}\xi_{r,\rho-1} + u_r \end{aligned}$$

with $\lambda_i > 0$, $1 \leq i \leq \rho - 1$. Let the generalized reference input be defined as $\mu_r(t) = u_r(t)$ if $\rho = 1$ and $\mu_r(t) = \xi_{r,1}(t)$ if $\rho > 1$.

In this section the bounds for the reference signal $\mu_r(t)$ will be computed. Let us transform system (1) into a relative-degree-one system with respect to a new input μ. Consider the filter $\dot{\xi} = \Lambda\xi + b_c u$

$$\begin{aligned} \dot{\xi}_1 &= -\lambda_1 \xi_1 + \xi_2 \\ &\vdots \\ \dot{\xi}_{\rho-1} &= -\lambda_{\rho-1}\xi_{\rho-1} + u \end{aligned} \tag{2}$$

with arbitrary initial conditions $\xi(0) = [\xi_1(0), \cdots, \xi_{\rho-1}(0)]^T$ and define $d[\rho] = b$, $d[i] = [d_1[i], \cdots, d_n[i]]^T = [A_c + \lambda_i I]d[i+1]$ $(i = \rho - 1, \cdots, 1)$, $\gamma_i = d_i[1]/d_1[1]$ $(2 \leq i \leq n)$. The filtered transformation

$$\begin{bmatrix} y \\ \eta \end{bmatrix} = \begin{cases} \Omega x & \rho = 1 \\ \Omega\left(x - \sum_{i=1}^{\rho-1} d[i+1]\xi_i(t)\right) & \rho > 1 \end{cases} \tag{3}$$

is introduced where

$$\Omega = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -\gamma_2 & 1 & 0 & \cdots & 0 \\ -\gamma_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\gamma_n & 0 & 0 & \cdots & 1 \end{bmatrix}$$

with $\sup_{b \in S_b} ||\Omega|| \triangleq \Omega_M$. In the new coordinates, system (1) becomes

$$\begin{aligned} \dot{y} &= \eta_1 + \gamma_2 y + d_1[1]\mu + h_1 y \\ \dot{\eta} &= \Gamma\eta + \beta y + \nu y \end{aligned} \tag{4}$$

where $\mu(t) = u(t)$ if $\rho = 1$, $\mu(t) = \xi_1(t)$ if $\rho > 1$, $\beta = [\gamma_3 - \gamma_2^2, \gamma_4 - \gamma_3\gamma_2, \cdots, -\gamma_n\gamma_2]^T$, $\nu = [h_2 - \gamma_2 h_1, h_3 - \gamma_3 h_1, \cdots, h_n - \gamma_n h_1]^T$ and

$$\Gamma = \begin{bmatrix} -\gamma_2 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -\gamma_{n-1} & 0 & \cdots & 1 \\ -\gamma_n & 0 & \cdots & 0 \end{bmatrix} .$$

By virtue of assumption (B), since $\lambda_i > 0$, the zeros of the polynomial $d_1[1](s^{n-1} + \gamma_2 s^{n-2} + \cdots + \gamma_n) = (b_{n-\rho}s^{n-\rho} + \cdots + b_1 s + b_0)\prod_{i=1}^{\rho-1}(s +$

λ_i) have negative real part. The reference system associated to system (4) can be written as

$$\begin{aligned}\dot{y}_r &= \eta_{r,1} + \gamma_2 y_r + d_1[1]\mu_r + h_1 y_r \\ \dot{\eta}_r &= \Gamma\eta_r + \beta y_r + \nu y_r \end{aligned} \quad (5)$$

from which it follows that $|\mu_r| \leq (|\dot{y}_r| + |\eta_{r,1}| + |\gamma_2||y_r| + |h_1||y_r|)/|d_1[1]|$. Then consider the function $V_{\eta_r} = \eta_r^T P \eta_r$, with P the symmetric positive definite solution of $P\Gamma + \Gamma^T P = -(\rho+3)I$, whose time derivative satisfies the inequality

$$\begin{aligned}\dot{V}_{\eta_r} \leq & -(\rho+3)\left\|\eta_r(t)\right\|^2 + 2\left\|\eta_r\right\| P_M \beta_M \left|y_r(t)\right| \\ & +2\left\|\eta_r\right\| P_M \Omega_M h_M |y_r(t)| \end{aligned} \quad (6)$$

where $P_M = \sup_{b\in S_b} ||P||$ and $\beta_M = \sup_{b\in S_b} ||\beta||$. From (6), since the reference signal $\eta_r(t)$ is periodic, it follows

$$||\eta_r(t)|| \leq \zeta |y_r(t)| \quad (7)$$

with $\zeta = (2P_M\beta_M + 2P_M\Omega_M h_M)/(\rho+3)$. By assumptions (B) and (C) we have that $d_1[1] \in [b_{n-\rho,m}, b_{n-\rho,M}]$, $\gamma_2 \in [\gamma_{2,m}, \gamma_{2,M}]$ and

$$|\mu_r(t)| \leq \frac{(\zeta + \gamma_{2,M} + h_M)R_1 + R_{1d}}{b_{n-\rho,m}}. \quad (8)$$

Let $\theta = [\theta_1, \theta_2, \cdots, \theta_p]^T$ be the vector of the first p Fourier coefficients of the Fourier series expansion of the periodic function $\mu_r(t)$ (p is an odd number). A positive real ϵ_p exists such that (see (Körner, 1988)) $\mu_r(t) = \sum_{k=1}^{p} \theta_k \phi_k(t) + \epsilon(t) = \phi^T(t)\theta + \epsilon(t)$ with $|\epsilon(t)| \leq \epsilon_p$, $\phi(t) = [\phi_1(t), \cdots, \phi_p(t)]^T$ and $\phi_1(t) = 1$, $\phi_{2i}(t) = \sqrt{2}\sin(2\pi i t/T)$, $\phi_{2i+1}(t) = \sqrt{2}\cos(2\pi i t/T)$ ($i = 1, \ldots, (p-1)/2$). Since by assumption (C) $y_r(t) \in C^N$ then $\mu_r(t) \in C^{N-1}$ and ϵ_p is given by (Kŏrner, 1988)

$$\epsilon_p = \begin{cases} 4\left(\dfrac{T}{2\pi}\right)^{N-1} \dfrac{(N-1)B_{N-1}}{N-2}, & p = 1 \\ 4\left(\dfrac{T}{2\pi}\right)^{N-1} \dfrac{2^{N-2}B_{N-1}}{(N-2)(p-1)^{N-2}}, & p > 1 \end{cases} \quad (9)$$

where $N > 2$ and $B_{N-1} = \sup_{0\leq t\leq T}(|\mu_r^{(N-1)}(t)|)$. The reference signal $\mu_r(t)$ has a known upper bound B, defined in (8), so that by virtue of the Bessel inequality we have $\sum_{i=1}^{p} \theta_i^2 \leq (1/T)\int_{-T/2}^{T/2} \mu_r^2(\tau)d\tau$ and, consequently,

$$||\theta|| \leq B \ . \quad (10)$$

3. CONTROLLER DESIGN

Since the reference signal $\mu_r(t)$ is unknown, the estimate $\hat{\mu}_r(t) = \sum_{k=1}^{p} \hat{\theta}_k(t)\phi_k(t) = \phi^T(t)\hat{\theta}(t)$ is introduced with $\hat{\theta}(t) = [\hat{\theta}_1(t), \cdots, \hat{\theta}_p(t)]^T$. Since, by (10), θ is bounded by a known bound, the projection algorithm $\text{proj}(\chi, \hat{\theta})$ considered in (Pomet and Praly, 1992) is used so that the estimate $\hat{\theta}(t)$ is constrained to belong to a suitable region. Define $\dot{\hat{\theta}} = c_0 \ \text{proj}(\chi, \hat{\theta})$, in which c_0 is a positive adaptation gain, χ is a suitable function and $\text{proj}(\chi, \hat{\theta})$ is given by

$$\text{proj}\left(\chi, \hat{\theta}\right) = \begin{cases} \chi, & \text{if} \quad p(\hat{\theta}) \leq 0 \\ \chi, & \text{if} \quad p(\hat{\theta}) > 0 \quad \text{and} \\ & \quad \chi^T \text{grad}(p(\hat{\theta})) \leq 0 \\ \chi_p, & \text{if} \quad p(\hat{\theta}) > 0 \quad \text{and} \\ & \quad \chi^T \text{grad}(p(\hat{\theta})) > 0 \end{cases}$$

where $p(\hat{\theta}) = (||\hat{\theta}||^2 - r_\theta^2)/(\alpha^2 + 2\alpha r_\theta)$, $\chi_p = [I - (p(\hat{\theta})\text{grad}(p(\hat{\theta}))\text{grad}^T(p(\hat{\theta})))/||\text{grad}(p(\hat{\theta}))||^2]\chi$, α is an arbitrary positive constant and r_θ is the radius of the region $S_\theta \subset \Re^p$, centred at the origin, in which θ is assumed to be. According to (8) and (10), $r_\theta = B$ in our case. By definition, $\text{proj}(\chi, \hat{\theta})$ is Lipschitz continuous and if $\hat{\theta}(0) \in S_\theta$ then the following properties hold (Pomet and Praly, 1992) $\forall t \geq 0$:

$$\left\|\hat{\theta}(t)\right\| \leq \alpha + r_\theta, \quad \forall t \geq 0 \quad (11)$$

$$\left\|\text{proj}(\chi, \hat{\theta})\right\| \leq \|\chi\| \quad (12)$$

$$\tilde{\theta}^T(t)\text{proj}(\chi, \hat{\theta}(t)) \geq \tilde{\theta}^T(t)\chi \quad (13)$$

with $\tilde{\theta} = \theta - \hat{\theta}$.

Subtracting (5) from (4) we obtain the error system

$$\begin{aligned}\dot{e} &= \tilde{\eta}_1 + \gamma_2 e + d_1[1]\mu + d_1[1]\left(-\mu_r\right) + h_1 e \\ \dot{\tilde{\eta}} &= \Gamma\tilde{\eta} + \beta e + \nu e \end{aligned} \quad (14)$$

where $e = y - y_r$, $\tilde{\eta} = \eta - \eta_r$ and, by assumption (A), $|h_1| \leq h_M$ and $||\nu|| \leq \Omega_M h_M$.

Before enunciating the main theorem the following periodic signals are introduced:

$$\begin{aligned}\varphi_1(t) &= \phi(t) \\ \varphi_j(t) &= \lambda_{j-1}\varphi_{j-1}(t) + \dot{\varphi}_{j-1}(t) \end{aligned} \quad (15)$$

$2 \leq j \leq \rho$, with $\phi(t)$ given in Section 2.

Theorem 1. Consider system (1) satisfying assumptions (A), (B) and a reference output signal $y_r(t)$ satisfying assumption (C). Consider the dynamic control algorithm

$$\begin{aligned}\dot{\xi}(t) &= \Lambda\xi(t) + b_c\xi_\rho^*(t) \ , \\ \xi_j^*(t) &= -\varphi_j^T(t)\hat{\theta}(t) \\ & \quad -\sigma_j g_{j,1}(c,k)\left(y(t) - y_r(t)\right) \end{aligned}$$

$$
\begin{aligned}
& \quad - \sum_{i=1}^{j-1} g_{j,i+1}(c,k)\,(\xi_i(t) - \xi_i^*(t)) \ , \\
& \dot{\hat{\theta}}(t) = c_0 \text{ proj}\,[\varphi_1(t)\,(y(t) - y_r(t)) \\
& \quad +2\varphi_1(t)\frac{b_{n-\rho,m}}{\rho} \cdot \\
& \quad \cdot \sum_{i=1}^{\rho-1} \frac{G_i(c,k)}{E_i^2(c,k)}\,(\xi_i(t) - \xi_i^*(t)) \Bigg] \ , \\
& u(t) = \xi_\rho^*(t) \qquad (16)
\end{aligned}
$$

in which $\xi(0) = \xi_0$, $\hat{\theta}(0) = \hat{\theta}_0$, $1 \le j \le \rho$, $c_0 > 0$ is the adaptation gain, $g_{j,1}(c,k)$, $g_{j,i+1}(c,k)$, σ_j, $G_i(c,k)$ and $E_i(c,k)$ are suitable control gains, $||\hat{\theta}_0|| \le B$ and $\xi_0 \in \Re^{\rho-1}$. If $c \ge c^*$ and $k > 0$ with $c^* = h_M + \gamma_{2,M} + 3/2 + P_M^2 \beta_M^2 + P_M^2 \Omega_M^2 h_M^2 + (\rho - 1)(1 + \gamma_{2,M}^2 + h_M^2)$ then:

- (i) All closed loop signals are bounded and there exist two class K functions $g_1(x)$, $g_2(x)$ such that $\forall\ t \ \ge\ t_0\ \int_{t_0}^t e^2(\tau) d\tau\ \le\ g_1(x(t_0)) + b_{n-\rho,M}^2\ (\tau_p/k) \int_{t_0}^t (\epsilon^2\ (\tau) + ||\theta - \hat{\theta}(\tau)||^2) d\tau$ and $|e(t)| \le (g_2(x(t_0))\cdot \cdot e^{-(t-t_0)/(2\tau_p)} + b_{n-\rho,M}(\tau_p/k)^{1/2}(\epsilon_p + \sup_{t_0 \le \tau \le t} ||\theta - \hat{\theta}(\tau)||)$ with $\tau_p = \max\{1/2, P_M\}$ and ϵ, ϵ_p given in (9).
- (ii) $\limsup_{t\to\infty} ||\theta - \hat{\theta}(\tau)|| \le r_{\tilde{\theta}}$ with $r_{\tilde{\theta}} = O\left(1/p^{N-3\rho}\right)$ as $p \to \infty$,
  $\limsup_{t\to\infty} |\mu(t) - \mu_r(t)| \le r_{\tilde{\mu}}$ with $r_{\tilde{\mu}} = O\left(1/p^{N-3\rho-1/2}\right)$ as $p \to \infty$,
  $\limsup_{t\to\infty} |y(t) - y_r(t)| \le r_e$ with $r_e = O\left(1/p^{N-3\rho}\right)$ as $p \to \infty$.
- (iii) If $\epsilon(t) = 0$, $\forall t \ge 0$, the equilibrium point $(e, \tilde{\eta}, \tilde{\xi}_1, \cdots, \tilde{\xi}_{\rho-1}, \tilde{\theta}) = 0$, with $\tilde{\xi}_i = \xi_i - \xi_i^*$ $(1 \le i \le \rho-1)$, of the closed loop system (14), (16) is globally exponentially stable and $x - x_r$, $\xi - \xi^*$, $\mu - \mu_r$, $\theta - \hat{\theta}$ converge exponentially to zero. □

The expressions of the gains $g_{j,1}(c,k)$, $g_{j,i+1}(c,k)$, σ_j, $G_i(c,k)$ and $E_i(c,k)$ which appear in (16) are omitted for short. A block diagram of the proposed controller, for a linear system with $\rho = 3$, is shown in Figure 1: the controller is an output error feedback controller and contains an estimator which learns a suitable input reference signal.

Proof. *Property (i).* Consider the following function

$$V(e, \tilde{\eta}) = \tilde{\eta}^T P \tilde{\eta} + \frac{1}{2} e^2 \qquad (17)$$

and the input control signal $u(t) = \xi_1^*(t) = -\varphi_1^T(t)\hat{\theta}(t) - g_{1,1} e(t)/b_{n-\rho,m}$ where $g_{1,1}(c,k) = c + \rho\, k(1+p)$. Considering that $d_1[1](-u_r - \phi^T\hat{\theta})e - k(1 + ||\phi||^2)e^2 \le b_{n-\rho,M}^2(||\tilde{\theta}||^2 + \epsilon_p^2)/(2k)$ and completing the squares, $\dot{V}$ satisfies the inequality

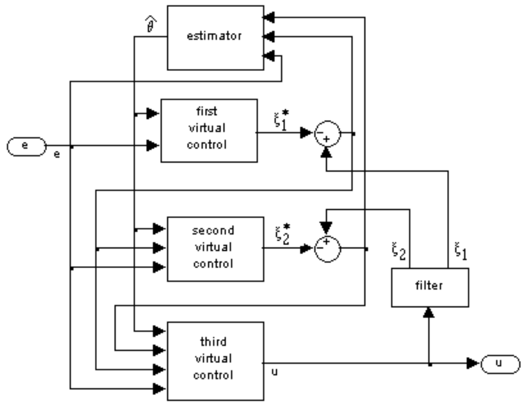

Fig. 1. Block diagram for the controller when $\rho = 3$.

$\dot{V} \le -V/\tau_p + b_{n-\rho,M}^2(||\tilde{\theta}||^2 + \epsilon_p^2)/(2k)$ which implies property (i) with $g_1(x(t_0)) = 2\tau_p V(t_0)$ and $g_2(x(t_0)) = (2V(t_0))^{1/2}$.

Property (ii). Consider the function

$$W(e, \tilde{\eta}, \tilde{\theta}) = V(e, \tilde{\eta}) + \frac{d_1[1]}{2c_0}\tilde{\theta}^T\tilde{\theta} \ : \qquad (18)$$

completing the squares, recalling (13) and considering that $d_1[1]\epsilon e - k(1+||\phi||^2)e^2 \le b_{n-\rho,M}^2 \epsilon^2/(4kp)$ and $c \ge c^*$, its time derivative becomes

$$\dot{W} = -\|\tilde{\eta}\|^2 - e^2 + \frac{\epsilon^2}{4kp}\, b_{n-\rho,M}^2 \ . \qquad (19)$$

From (19) and from Lemma 2 (in Appendix), if $p > 1$, the inequality

$$
\begin{aligned}
& \limsup_{t\to\infty} ||[e, \tilde{\eta}^T, \tilde{\theta}]^T|| \le r(p) \\
& \triangleq \frac{c_1}{(p-1)^{N-2}} \sqrt{\frac{a_6 M_1}{a_1 m_1}} \qquad (20)
\end{aligned}
$$

is obtained where $c_1 = 4(T/2\pi)^{N-1}(2^{N-2}/(N-2)) \sup_{0 \le t \le T} (|\mu_r^{(N-1)}(t)|)$ and a_6, M_1 and m_1 are given by Lemma 2. As p tends to ∞, $a_6 = O(1/p)$, $a_1 = O(1)$, $M_1 = O(1)$ and $m_1 = O(1/p^3)$ so that from (20) we obtain that $r(p) = O(p^{-(N-3)})$ which implies $r_{\tilde{\theta}} = O(p^{-(N-3)})$ and $r_e = O(p^{-(N-3)})$. Since $|u_r(t) - \hat{u}_r(t)| = |\phi^T\tilde{\theta} + \epsilon| \le \epsilon_p + \sqrt{p}\ ||\tilde{\theta}|| \triangleq r_{\tilde{\mu}}$ it follows that $r_{\tilde{\mu}} = O(p^{-(N-7/2)})$.

Property (iii). The persistency of excitation Lemma 2 implies that if $\epsilon(t) = 0$, the equilibrium point $(e, \tilde{\eta}, \tilde{\theta}) = 0$ of the closed loop system (14), (16) is globally exponentially stable and $x - x_r$, $u(t) - u_r(t)$, $\theta - \hat{\theta}$ converge exponentially to zero.

If $\rho > 1$, properties (i), (ii) and (iii) can be proved employing the functions: $V = \tilde{\eta} P \tilde{\eta} + e^2/2 + \sum_{i=1}^{\rho-1} \delta_i \tilde{\xi}_i^2/2$, $W = V + d_1[1]\tilde{\theta}^T\tilde{\theta}/(2c_0)$ where $\delta_i = 4b_{n-\rho,m}^2/(4\ E_i^2)$ and P is such that $P\Gamma + \Gamma^T P = -(\rho+3)I$. □

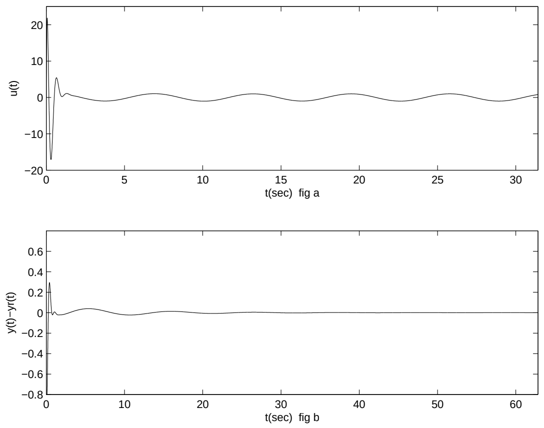

Fig. 2. proposed control: (a):$u(t)$, (b): $e(t)$.

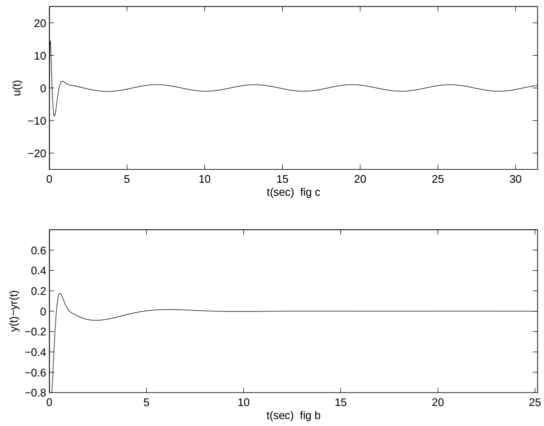

Fig. 3. robust regulator: (a):$u(t)$, (b): $e(t)$.

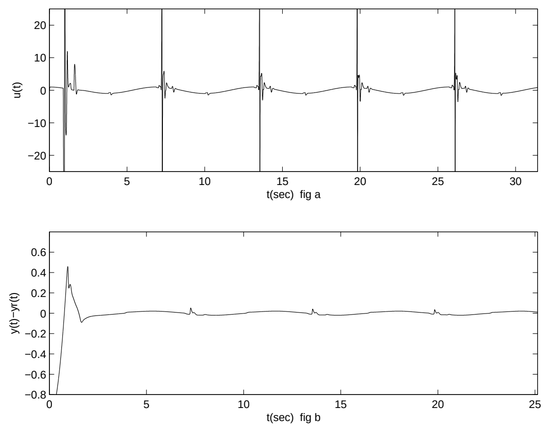

Fig. 4. ILC control: (a):$u(t)$, (b): $e(t)$.

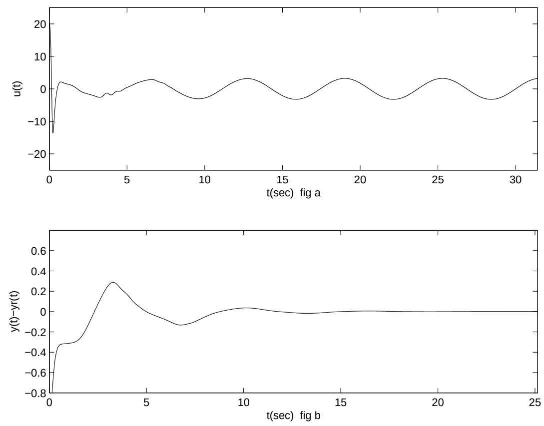

Fig. 5. adaptive control: (a):$u(t)$, (b): $e(t)$.

4. SIMULATIONS

In this section the proposed controller is compared to the controllers proposed in (Serrani and Isidori, 2000; Marino and Tomei, 1995; Chien and Yao, 2004a): consider the linear system

$$F(s) = \frac{2s+1}{s^3 - s^2 + 2s + 1}$$

and the periodic reference signal generated by the exosystem $\dot{w}_1 = w_2$, $\dot{w}_2 = -w_1$, $y_r = w_2$, $w_1(0) = 0$, $w_2(0) = 1$: the period of the reference signal is assumed to be known though in some practical applications this could be a strong assumption. The proposed controller is given by: $\xi_1^*(t) = -\phi^T(t)\hat{\theta}(t) - 5.2e(t)$, $u(t) = -(3\phi^T(t) + \dot{\phi}^T(t))\hat{\theta}(t) - 15.6e(t) - 26.12(\xi_1(t) - \xi_1^*(t))$, $\dot{\xi}_1(t) = -3\xi_1(t) - (3\phi^T(t) + \dot{\phi}^T(t))\hat{\theta}(t) - 15.6e(t) - 26.12(\xi_1(t) - \xi_1^*(t))$, $\xi_1(0) = -5.7972e(0)$, $\dot{\hat{\theta}}(t) = 5\text{proj}(\chi, \hat{\theta})$, $\chi = \phi(t)e(t) + 0.1923\phi(t)$ $(\xi_1(t) - \xi_1^*(t))$, $\hat{\theta}(0) = 0$ where $\phi(t)$ is given in Section 2 and $e(t) = y(t) - y_r(t)$. The robust regulator is given by $\alpha_1(e) = -5.2e$, $u(t) = 29.12\eta_1(t) + \eta_2(t) - 15.6e(t) - 26.12(\xi_1(t) - \alpha_1(e))$, $\dot{\xi}_1(t) = -3\xi_1(t) + 29.12\eta_1(t) + \eta_2(t) - 15.6e(t) - 26.12(\xi_1(t) - \alpha_1(e))$, $\xi_1(0) = -5.7972e(0)$, $\dot{\eta}_1(t) = -\eta_1(t) + \eta_2(t) + \xi_1(t)$, $\eta_1(0) = 0$, $\dot{\eta}_2(t) = \eta_1(t)$, $\eta_2(0) = 0$. Both the robust regulator and the proposed controller use the same values of the feedback gains and, from Figures 2 and 3, it can be seen that they guarantee similiar transient performances. Both the controllers guarantee that the output tracking error converges to zero. Also the controller proposed in (Chien and Yao, 2004a) can be used to control the proposed linear system; the control parameters are $\tau = 0.01$, $\bar{\zeta} = 100$, $\epsilon_1 = 5$, $\delta = 0.01$, $\alpha = 3$, $\gamma = 10$, $\lambda(s) = s^3 + 18s^2 + 107s + 210$: figure 4 shows that the input signal is not regular and the tracking error doesn't converge to zero. Finally, Figure 5 shows the results obtained by the adaptive controller proposed in (Marino and Tomei, 1995) with $\lambda_c = 2$, $\lambda_o = 5$, $k_1 = 5$, $k_{\alpha_1} = 5$, $\eta = 5$, $\dot{\phi}_1(t) = -\phi_1(t) + u$, $\dot{\mu}[1](t) = -\mu[1](t) + \phi_1(t)$, $\dot{\mu}_1[2](t) = -\mu_1[2](t) + \mu_2[2]$, $\dot{\mu}_1[2](t) = -2\mu_2[2] + \phi_1(t)$ and with adaptation gains equal to 5. Even though the reference output is not periodic, adaptive controls guarantees that the output tracking error converges asymptotically to zero, but the controller depends on the order of the system to be controlled and does not guarantee exponential tracking. On the other hand the proposed control and the robust regulator depend only on the relative degree and on the number of frequencies of the input reference. While the robust regulator can only track reference signals generated by a known linear exosystem, the proposed controller doesn't have such a limitation. Also the ILC approach doesn't have such a limitation but it requires the knowledge of the order of the system to be controlled.

5. CONCLUSIONS

For linear systems (1) the problem of tracking a smooth periodic output reference with known period by feeding back the output tracking error has been solved. The designed dynamic controller (16) has order $p+\rho-1$ which depends on the relative degree ρ and on p estimated Fourier coefficients: it has a fixed structure which is independent on the system order n. When the reference input has a finite Fourier series expansion, exponential tracking of both the input and the output reference is achieved, so that the required reference input is learned. If the reference input Fourier series expansion is not finite, the tracking errors can be arbitrarily reduced by increasing the number p of the estimated Fourier coefficients in the control. Some simulations have been carried out showing the performance of the proposed controller and those of the adaptive control, the robust regulator and the ILC control.

REFERENCES

Arimoto, S, S Kawamura and F Miyazaki (1984). Bettering operation of robots by learning. *J. Robotic Systems* **1**, 123–140.

Chien, C-J and C-Y Yao (2004*a*). Iterative learning of model reference adaptive controller for uncertain nonlinear systems with only output measurement. *Automatica* **40**, 855–864.

Chien, C-J and C-Y Yao (2004*b*). An output-based adaptive iterative learning controller for high relative degree uncertain linear systems. *Automatica* **40**, 145–153.

Del Vecchio, D, R Marino and P Tomei (2003). Adaptive learning control for feedback linearizable systems. *European journal of control* **9**, 479–492.

French, M, G Munde, E Rogers and D H Owens (1999). Recent developments in adaptive iterative learning control. In: *Proceedings of the 38th Conference on Decision and Control*. Vol. 1. Phoenix,Arizona,USA. pp. 264–269.

Jang, T J, C H Choi and H S Ahn (1995). Iterative learning control in feedback systems. *Automatica* **31**, 243–248.

Körner, T W (1988). *Fourier Analysis*. Cambridge university press.

Kim, Y H and I J Ha (2000). Asymptotic state tracking in a class of nonlinear systems via learning-based inversion. *IEEE Trans. Automatic Control* **45**, 2011–2027.

Kristic, M, I Kanellakopoulos and P V Kokotovic (1995). *Nonlinear and adaptive control design*. Wiley, New York.

Marino, R and P Tomei (1995). *Non linear control design-Geometric, Adaptive and Robust*. Prentice Hall, London.

Owens, D H and G Munde (2000). Error convergence in an adaptive iterative learning controller. *Int. J. of Control* **73**, 851–857.

Pomet, J and L Praly (1992). Adaptive nonlinear regulation: estimation from lyapunov equation. *IEEE Trans. Automatic Control* **37**, 729–740.

Serrani, A and A Isidori (2000). Global robust output regulation for a class of nonlinear systems. *Systems and Control Letters* **39**, 133–139.

APPENDIX

In the following a persistency of excitation result is recalled.

Lemma 2. Given the nonlinear time varying system

$$\begin{aligned} \dot{x} &= f(x,t) + \Omega^T(t)z + C(t)\omega(t), & x \in \Re^n \\ \dot{z} &= g(x,t), & z \in \Re^p \end{aligned} \quad (21)$$

with bounded input $\omega \in \Re^m$, assume that all the solutions $(x(t), z(t))$ belong to a region $S \subseteq \Re^{n+p}$ where the following properties hold, $\forall t \geq t_0$:

- (i) $f(x,t)$ and $g(x,t)$ are continuous and uniformly bounded in t with $||f(x,t)|| \leq k_f||x||$, $||g(x,t)|| \leq k_g||x||$;
- (ii) The matrices $\Omega(t)$ and $C(t)$ are continuous and uniformly bounded with $||\Omega(t)|| \leq \Omega_M$, $||C(t)|| \leq C_M$, $\dot{\Omega}(t)$ is uniformly bounded with $||\dot{\Omega}(t)|| \leq \dot{\Omega}_M$;
- (iii) There exists a smooth proper function $V(x,z,t)$ such that $a_1(||x||^2 + ||z||^2) \leq V(x,z,t) \leq a_2(||x||^2 + ||z||^2)$ and $\dot{V}(x,z,t) \leq -a_3||x||^2 + a_4||\omega||^2$ for suitable reals $a_i > 0$, $1 \leq i \leq 4$;
- (iv) There exist two positive reals T_p and k_p such that $\int_t^{t+T_p} \Omega(\tau)\Omega^T(\tau)d\tau \geq k_p I > 0$.

Then, $\limsup_{t\to\infty} ||[x(t)^T, z(t)^T]^T|| \leq ((a_6 M_1)/(a_1 m_1))^{1/2} \sup_{\tau\in[t_0,\infty)} ||\omega(\tau)||$ with $a_6 = a\Omega_M^2 C_M^2 + a_4$, $M_1 = a_2 + a\max\{\Omega_M^2, \left(k_p + \Omega_M^2\right)^2\}$, $m_1 = \frac{1}{2}\min\{a_3, \frac{1}{2}ak_p e^{-2T_p}\}$, $a = a_3/(2(\Omega_M + \dot{\Omega}_M + \Omega_M k_f + (k_p + \Omega_M^2)k_g)^2 + \Omega_M^2)$. If $\omega(t) = 0$, $\forall t \geq t_0$, all the solutions $(x(t), z(t))$ converge exponentially to the origin.

The proof follows from a similar result given in (Del Vecchio *et al.*, 2003) with minor modifications.