ROBUST STABILIZATION OF AN INVERTED PENDULUM USING A SLOW-FAST DECOMPOSITION APPROACH

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Abstract: In this paper, robust stabilization of an experimental system is considered. This system consists of a pendulum free to rotate 360 degrees that is attached to a cart. The cart can move in one dimension. The linearized model of the system is used and transformed to a linear diagonal form. The system is separated into slow and fast subsystems. The fast dynamics are treated as a disturbance and this is used to design a H_∞ controller for a system with lower order than the original system.

Keywords: H-infinity control, Decomposition methods, Robust performance.

1. INTRODUCTION

Van der Schaft (1992) indicated that in control of nonlinear systems, if the H_∞ control problem for the linearized system is solvable, then one obtains a local solution to the nonlinear H_∞ control problem. One problem with H_∞ designs is that the order of the controller is at least the order of the plant, and larger if, as is common, weights are included in the design.

An approach to reduced order controller design based on the idea that one can consider the fast dynamics of a system as disturbances is first introduced by Khalil (1996) and then is discussed by Yazdanpanah et al. (1997) and Yazdanpanah and Karimi (2002). In this paper the fast, stable part of the system is considered as uncertainty and then the controller is designed for the remaining part of the system. The remaining slow subsystem has order less than the original one. The only information must be known, is the H_∞ norm of the fast subsystem. The part of the system regarded as uncertainty is not entirely arbitrary since the small gain theorem must hold.
Most systems have a lower gain in high frequencies than in the low frequencies and so this approach has wide applicability. No other dynamical information is

required. This is advantageous since in general the high frequency aspect of a model is not well determined. With this idea, one can use the H_∞ method to design a robust controller using the slow subsystem as the nominal plant. The proposed method is applied to a flexible joint robot manipulator by Amjadifard et al. (2003), and the simulation results showed the desired behavior of system.

In this work the approach to an unstable system is extended. The stabilization of an inverted pendulum-cart is considered. First, the nonlinear part of the system is eliminated since it is stable and small. Then, the linearized model is transformed to Jordan canonical form and the slow and fast modes are separated. The stability of the controlled system was verified on an experimental apparatus. The performance is shown to be superior to a linear quadratic regulator previously implemented by Landry et al. (2003).

2. SYSTEM DEFINITION

A pendulum is attached to the side of a cart by means of a pivot that allows the pendulum to swing in the xy-plane over 360 degrees. (See Fig. 1.) A force $F(t)$ can be applied to the cart in the x direction. In Table 1 there is a complete list of notation.

The equations of motion for the system are (which is mentioned, *e.g.* by Landry et al., 2003)

$$\begin{aligned}(M+m)\ddot{x}+\varepsilon\dot{x}+ml\ddot{\theta}\cos\theta-ml\dot{\theta}^2\sin\theta=F(t),\\ ml\ddot{x}\cos\theta+\tfrac{4}{3}ml^2\ddot{\theta}-mgl\sin\theta=0.\end{aligned} \quad (1)$$

Parameter values for the apparatus that is made by Quanser Consulting Inc. (1996) are given in Table 2.

Based on previous experiments, a value $\varepsilon = 8$ for the friction parameter was used.

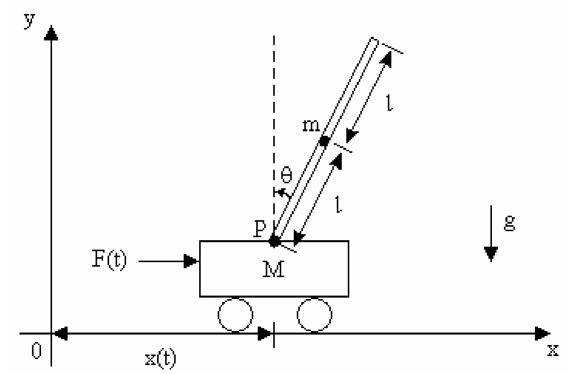

Fig. 1. Inverted pendulum system

Table 1. Notation

$x(t)$	Displacement of the centre of mass of the cart from point O
$\theta(t)$	Angle the pendulum makes with the top vertical
M	Mass of the cart
m	Mass of the pendulum
L	Length of the pendulum
l	Distance from the pivot to the centre of mass of the pendulum
P	Pivot point of the pendulum
$F(t)$	Force applied to the cart

Using the state variables

$$X=(x_1,x_2,x_3,x_4)=(x,\dot{x},\theta,\dot{\theta})^T \quad (2)$$

equations (1) can be written in first-order form as

$$\dot{X}=f(X)=\begin{bmatrix} x_2 \\ \dfrac{4F(t)-4\varepsilon x_2+4mlx_4^2\sin x_3-3mg\sin x_3\cos x_3}{4(M+m)-3m\cos^2 x_3} \\ x_4 \\ \dfrac{(M+m)g\sin x_3-(F(t)-\varepsilon x_2)\cos x_3-mlx_4^2\sin x_3\cos x_3}{l(\frac{4}{3}(M+m)-m\cos^2 x_3)}\end{bmatrix} \quad (3)$$

The force $F(t)$ on the cart is due to a voltage *V(t)* applied to a motor:

$$F(t)=\alpha V(t)-\beta\dot{x}(t). \quad (4)$$

The second term is due to electrical resistance in the motor. The physical constants are

$$\alpha=\frac{K_m K_g}{Rd},\quad \beta=\alpha^2 R.$$

The voltage $V(t)$ can be varied and is used to control the system. The model for the controlled system linearized about the upright position is

$$\dot{X}=AX+bV(t) \quad (5)$$

Where

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & \dfrac{-4(\varepsilon+\beta)}{m+4M} & \dfrac{-3mg}{m+4M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \dfrac{3(\varepsilon+\beta)}{l(m+4M)} & \dfrac{3(m+M)g}{l(m+4M)} & 0 \end{bmatrix},$$

$$b=\begin{bmatrix} 0 \\ \dfrac{4\alpha}{m+4M} \\ 0 \\ \dfrac{-3\alpha}{l(m+4M)} \end{bmatrix} \quad (6)$$

Table 2. Values of parameters

Parameter	Value	Description
w_M	0.360 Kg	Weight mass
M	0.455 Kg+ w_M	Mass of the cart
m	0.210 Kg	Mass of the pendulum
L	0.61 m	Length of the pendulum
g	9.8 m/s	Acceleration due to gravity
ε	Unknown	Viscous friction
K_m	0.00767 V/(rad/sec)	Motor torque and back emf constant
K_g	3.7	Gearbox ratio
R	2.6 Ω	Motor armature resistance
d	0.00635 m	Motor pinion diameter

It is well-known (e.g. as indicated by Landry et al., 2003) that for the uncontrolled system (*V(t)=0*), the cart-pendulum at rest in any upright position *(x,0,n*π*,* 0) is at an unstable equilibrium point.

3. H_∞ CONTROLLER DESIGN

A similarity transformation $X = Ty$ is used, where T contains the system eigenvectors, to transform A into Jordan canonical form. Equation (5) becomes

$$\dot{y} = Jw + BV(t) \tag{7}$$

where $J = T^{-1}AT$ is a diagonal matrix of system eigenvalues and $B = T^{-1}b$.

The system of equations (7) can be decomposed into fast and slow subsystems

$$\begin{aligned} \dot{X}_1 &= \Lambda_{11} X_1 + B_1 u \\ \dot{X}_2 &= \Lambda_{22} X_2 + B_2 u \end{aligned} \tag{8}$$

where $\Lambda_{11} = diag\{0,5,-4.74\}$, and $\Lambda_{22} = -18.33$. The vector B is a permutation of elements of B_1 and B_2. Here X_1 indicates the slow dynamics of system and X_2 the fast dynamics of system. The nominal system with no disturbance can be written

$$P \sim \begin{bmatrix} \Lambda_{11} & 0 & B_1 \\ C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{bmatrix}$$

where it is assumed full information, so that the controlled output, Z, is

$$Z = C_1 X_1 + D_{12} u := \begin{bmatrix} I \\ 0 \end{bmatrix} X_1 + \begin{bmatrix} 0 \\ I \end{bmatrix} u.$$

and the measured output is

$$Y = C_2 X_1 + D_{21} X_2$$

where C_2 is chosen to be *I*.

As mentioned earlier the stable subsystem with fast dynamics will be considered as uncertainty Δ. This means rewriting the system (8) is needed so that the fast dynamics appear as disturbance to the nominal system.

A state transformation $[X_1, X_2]^T = M[\overline{X}_1, \overline{X}_2]^T$ is applied to the system, where M has the structure

$$M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}.$$

The equations of system (8) after transformation are (see Fig. 2)

$$\dot{\overline{X}} = \overline{\Lambda}\,\overline{X} + \overline{B} u \tag{9}$$

where

$$\overline{\Lambda} = M^{-1} \begin{bmatrix} \Lambda_{11} & 0 \\ 0 & \Lambda_{22} \end{bmatrix} M = \begin{bmatrix} \overline{\Lambda}_{11} & \overline{\Lambda}_{12} \\ 0 & \overline{\Lambda}_{22} \end{bmatrix},$$
$$\overline{B} = M^{-1} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix}.$$

The new equation for the fast sub-system, or uncertainty block, is

$$\dot{\overline{X}}_2 = \overline{\Lambda}_{22} \overline{X}_2 + \overline{B}_2 u .$$

Note that the coefficient of $\overline{X}_1$ in the fast sub-system is zero. Also, the equations for the slow sub-system, or nominal block, become

$$\overline{P} \sim \begin{bmatrix} \overline{\Lambda}_{11} & \overline{\Lambda}_{12} & \overline{B}_1 \\ \overline{C}_1 & \overline{D}_{11} & D_{12} \\ \overline{C}_2 & \overline{D}_{21} & 0 \end{bmatrix},$$
$$Z = \overline{C}_1 \overline{X}_1 + D_{11} \overline{X}_2 + D_{12} u,$$
$$Y = \overline{C}_2 \overline{X}_1 + \overline{D}_{21} \overline{X}_2$$

where

$$\overline{C}_1 = C_1 M_{11}^{-1},\ \overline{C}_2 = C_2 M_{22}^{-1},$$
$$D_{11} = -C_1 M_{11}^{-1} M_{12} M_{22}^{-1},$$
$$\overline{D}_{21} = D_{21} - C_2 M_{11}^{-1} M_{12} M_{22}^{-1}.$$

In Fig. 2, Z is the input to the uncertainty block.

The fast dynamics are exponentially stable. Indicating the transfer function by

$$\Delta(s) = (sI - \Lambda_{22})^{-1} B_2, \quad \|\Delta\|_\infty \le \gamma_1,$$

where $\gamma_1 = 0.301$.

Since $\overline{\Delta}(s) = \overline{C_2}(sI - \overline{\Lambda}_{22})^{-1}\overline{B}_2 = C_2(sI - \Lambda_{22})^{-1}B_2$, the H_∞-norm of the uncertainty block is $\overline{\gamma}_1 = 0.301$.

The H_∞ controller design problem for the system shown in Fig. 2, will lead to a H_∞ controller for the slow sub-system. The H_∞ controller will be designed here via state feedback (or full-information). The next step is to determine $\overline{\gamma}_2 = \min \gamma$, where $\overline{\gamma}_2$ indicates the H_∞-norm of the controlled slow sub-system, and a corresponding controller that achieves this. The transformation M must be chosen so $\overline{\gamma}_1 . \overline{\gamma}_2 < 1$. By trial and error, a suitable transformation was found:

$$M = \begin{bmatrix} I_3 & M_{12} \\ 0_{3\times 1} & 1 \end{bmatrix}, \quad M_{12} = 0.0095[1\ 1\ 1]^{\mathrm{T}}.$$

It is straightforward to verify that the slow sub-system is stablizable and detectable. As indicated by Doyle et al. (1989), it then follows that $\overline{\gamma}_2$ is the smallest value of γ such that that the eigenvalues of the Hamiltonian matrix

$$H = \begin{bmatrix} \overline{\Lambda}_{11} & \gamma^{-2}\overline{\Lambda}_{12}\overline{\Lambda}_{12}^T - \overline{B}_1\overline{B}_1^T \\ -\overline{C}_1^T\overline{C}_1 & -\overline{\Lambda}_{11}^T \end{bmatrix}$$

are not on the imaginary axis. Doyle et al. (1989) also have shown that for any $\gamma > \overline{\gamma}_2$, there is an

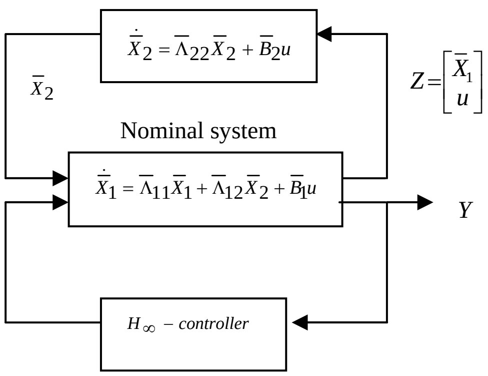

Fig. 2. Block diagram of system with fast dynamics as uncertainty

internal stabilizing controller such that $\|T_{z\overline{X}_2}\|_\infty \le \gamma$ if and only if there is a positive semi-definite solution X_∞ of the algebraic Riccati equation:

$$\overline{\Lambda}_{11}^T X_\infty + X_\infty \overline{\Lambda}_{11} + X_\infty(\gamma^{-2}\overline{\Lambda}_{12}\overline{\Lambda}_{12}^T - \overline{B}_1\overline{B}_1^T)X_\infty + \overline{C}_1^T\overline{C}_1 = 0$$

In this case, a suitable feedback is

$$u(t) = K\overline{X}_1(t), \quad K = -\overline{B}_1^T X_\infty \overline{X}_1(t). \tag{10}$$

4. EXPERIMENTAL RESULTS

The H_∞ controller of equation (10) is applied to the pendulum system. Only the position of the cart and the pendulum angle can be measured. An observer is required to obtain $\dot{x}$ and $\dot{\theta}$. In order to compare to the results shown by Landry et al. (2003), the same Luenberger observer was used. The same linear-quadratic regulator (LQR) used by Landry et al. (2003) was used as a comparison for the H_∞ state-feedback controller designed using the slow-fast approach in this paper (or for simplicity, the 'slow-fast controller').

The controlled pendulum angle and the cart position are shown in Fig. 3. The equilibrium x_1 (cart position) is arbitrary, as can be seen from equation (3).

In Fig. 4, the input controller signals, produced by the slow-fast and the LQR controllers, are shown. Although the performance of the two controlled systems is similar, the slow-fast controller achieves this performance with a smaller controller signal and also with a system of lower degree.
The response of the controlled pendulum system to a disturbing “tap” on the pendulum controller was investigated for each controller. Fig. 5 shows the angular position of pendulum and the cart position under this disturbance.

In Fig. 6, the behavior of the two controlled systems with a time delay of 0.035 seconds in the controller output is shown.

The performance of the slow-fast controller is superior to that of the LQR controller for both the disturbed and delayed systems.

5. CONCLUSIONS

In this paper the robust stabilization of an experimental pendulum systems using slow-fast decomposition approach is considered. First using the linearized model, it was transformed to a diagonal form and the slow and fast modes of system were separated. Then considering the fast dynamics as norm-bounded uncertainty, a H_∞ controller for the reduced order system (slow subsystem) was designed. The resulting controller was implemented.

Experimental results indicate that the performance is superior to the full-order LQR controller previously used.

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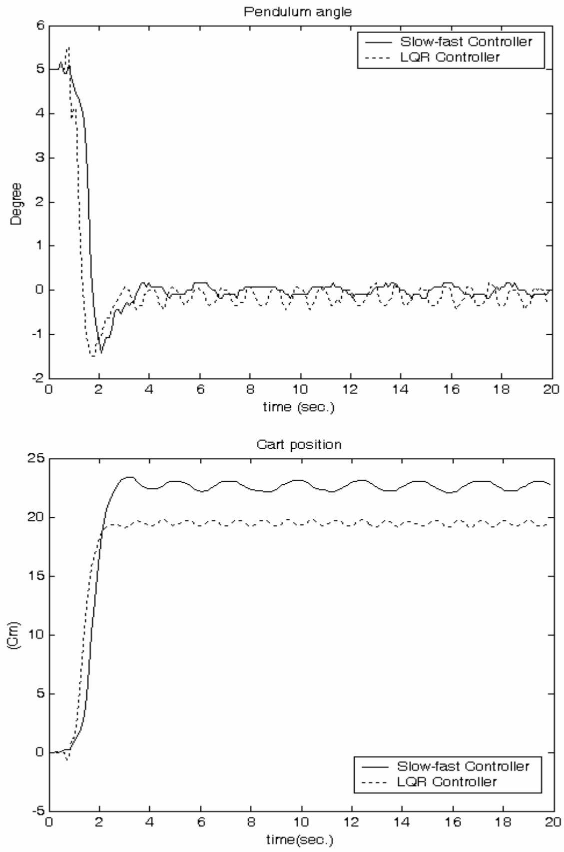

Fig. 3. System behavior via the two controllers without any additional disturbance or noise.

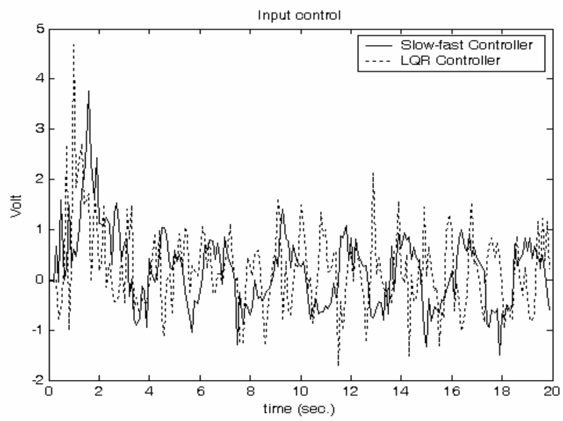

Fig. 4. The input controller signal, produced by the two controllers in a condition without any additional disturbance or noise.

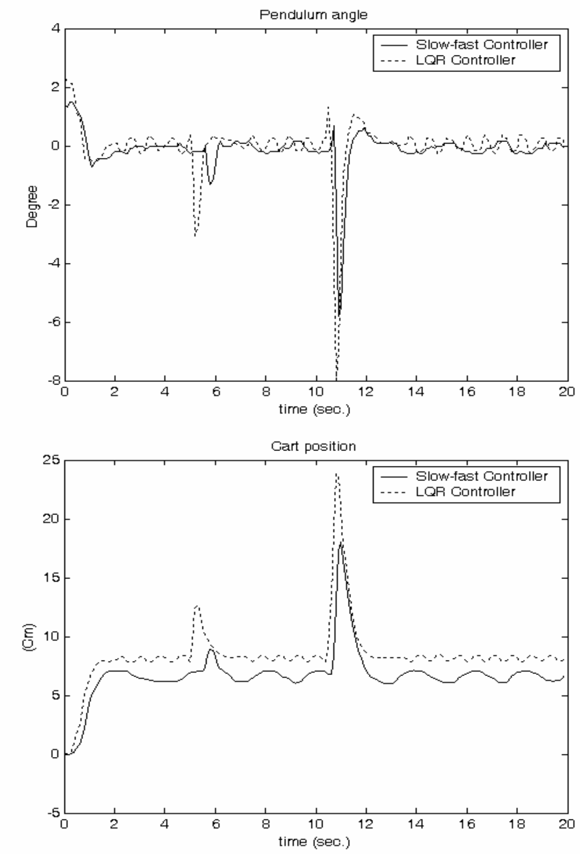

Fig. 5. The behavior of pendulum system with an additional disturbance on pendulum via the two controllers.

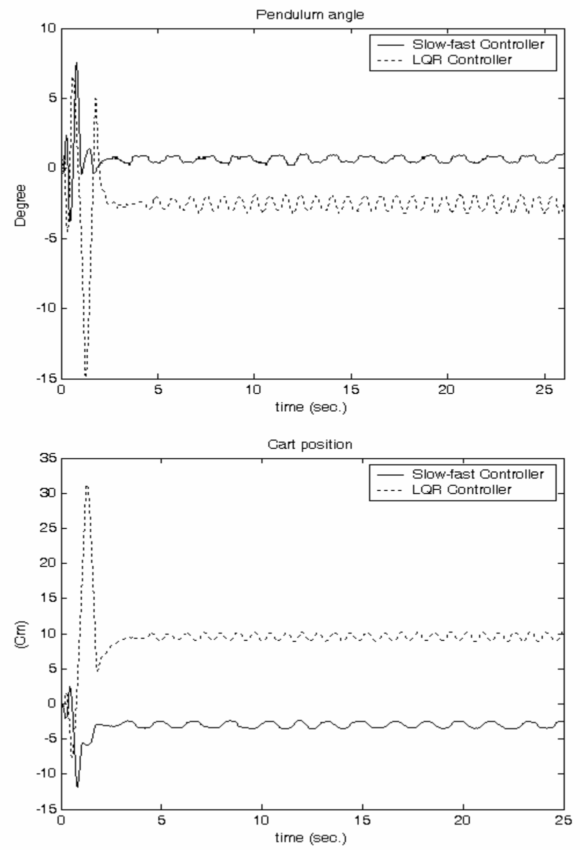

Fig. 6. Pendulum system behavior with a transport delay of 0.035 sec. in the controller output.