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SAFETY DRIVING OF THE DUBINS' CAR

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Abstract: In this paper we consider the problem of safe driving of the Dubins' car on a straight road where the steering control is in the form $(1-h)s+hv$ where v represents a free control and s is an automatic control preventing the system from undesired behaviors that may result from the action of v. We design the safety control s steering the car to the center of the road in minimum time which fact implies to consider discontinuous feedbacks, hence to face the problem of de£ning a solution to a system with a discontinuous right hand side, and to deal with the de£nition of regular synthesis used in optimal control. We thus consider generalized solutions in Krasowskii and Filippov sense and guarantee that these solutions have a *good behavior*, i.e., when only s is acting ($h = 0$) the generalized solutions coincide with the optimal synthesis trajectories. Finally, we illustrate our strategy by simulations.

Keywords: Cooperative Controls, Differential Inclusions, Optimal Feedbacks.

1. INTRODUCTION

Consider the system:

$$\dot{x} = F(x) + (1-h)G_1(x)s + hG_2(x)v \quad (1)$$

where $x \in \mathbb{R}^n$, $F : \mathbb{R}^n \to \mathbb{R}^n$, $G_i(x) \in \mathbb{R}^{n\times m_i}$, $i = 1, 2$, $h \in [0, 1]$, $s \in \mathbb{R}^{m_1}$ and $v \in \mathbb{R}^{m_2}$. Here v represents a free control chosen by some controller and s is a safety control that is an automatic control preventing the system from undesired behaviors that may result from the action of v. For example, equation (1) may model a car with a non expert driver choosing the control v, a teacher or an automatic control choosing s, a safety control, and h to be equal to 1 when the driver is driving properly and equal to 0 in case of danger. We choose the model of Dubins' car with controlled steering (see section 2.2) and tackle the problem of safe driving of this car on a straight road. Moreover we want to design the safety control s to steer the car to the center of the road in minimum time. Thus we have to consider discontinuous feedbacks, hence to face the problem of de£ning a solution to (1) with a discontinuous right hand side, and to deal with the de£nition of regular synthesis used in optimal control. The dif£culty of the problem is not bounded to treat the discontinuities of s. Indeed, even if solutions to the equation $\dot{x} = s(x)$ are de£ned (in Caratheodory sense), the presence of the free control v may provoke non existence, see Example 1. We then consider generalized solutions in Krasowskii and Filippov sense. As a drawback, we should guarantee that these solutions have a *good behavior*. More precisely, when only s is acting ($h = 0$) the generalized solutions should coincide with the optimal synthesis trajectories. This is not always the case if the feedback s corresponds to a synthesis in Boltyanskii-Brunovskii sense (see (Boltyanskii 1966, Brunovsky 1980)).

We propose a general strategy to deal with the above problems. This consists of three steps. First we give conditions to ensure that the set of generalized solutions for s have a *good behavior*. These conditions are based on a de£nition of regular strati£ed feedback, modeled on those given by Boltyanskii and Brunovsky and reported in the Appendix.

The second step consists in studying the set of generalized solutions to the system obtained putting the

regular strati£ed feedback s in equation (1). Finally, the third step amounts to ensuring that the generalized solutions to this complete system do not enter the set of bad con£gurations. Moreover, checking the strategy by simulations is possible if approximate solutions converge to generalized ones, which is the case because we consider Krasowskii and Filippov solutions. Based on this approach we then propose, in section 4, two regular strati£ed feedbacks that provide safety driving for the Dubin's car. The analysis of the two dimensional case, developed in (Marigo and Piccoli 2002), permits to check that generalized solutions remain in the safe zone. The results are tested also by simulations described in the last section 5.
Our problem is similar to that of air traf£c management. considered in (Tomlin *et al.* 1998, Bicchi *et al.* 1998). Another approach would be to consider v and s in competition, but we want rather to design s and h as feedbacks that should reject only some dangerous con£guration towards which v could lead the system. Indeed, the idea is that in some "safe" zone v should act freely either meanwhile learning or because some risky choice can be convenient, e.g. in problems from economics. Even more, in case of danger we would like s to steer the system towards the safe zone minimizing the time or some more complicate cost functional.
The paper is organized as follows. In section 2.1 we state the problem of safety control and highlight the drawbacks in the concept of solution that one has to face when treating this kind of control problem. A description of the safety control problem for the Dubins' car is given in section 2.2. In section 2.3 we give the de£nitions of Krasowskii and Filippov admissible solutions for our safety control problem. In section 3.1 we give condition on a feedback, associated to a synthesis, such that the set of Krasowskii (Filippov) solutions coincide with the trajectory of the synthesis. Next in section 3.2 we show that the safety control system admits solutions in the sense given previously and give the de£nition of safety and stabilizing feedback. Finally in section 3.3 we give further conditions on h ((H1), (H2)) under which a safety (stabilizing) feedback actually is safe (stabilizing) for the system. The problem of convergence for simulations and implementations is also treated. The solution and simulation results for the safety control problem of Dubins' car are given in section 4 and 5 respectively. The Appendix contains the de£nition of regular strati£ed feedback.

2. BASIC DEFINITIONS

2.1 *Basic Model for a Safety Control System*

In the system (1) we assume that: F, G_1, G_2 smooth, $\exists \bar{C} > 0$ s.t. $\|F\|, \|G_1\|, \|G_2\| \le \bar{C}(1 + \|x\|)$ and s, v take values in compact sets.
We want s to bring back the system from a bad region $\mathcal{B}$ to a safe region $\mathcal{S} \subset \mathbb{R}^n$ in the most ef£cient way that is minimizing the time or a more complicate cost. We assume that $\mathcal{S}$ and $\mathcal{B}$ are closed, $\mathcal{S} \cap \mathcal{B} = \emptyset$ and $\mathcal{L} = \mathbb{R}^n \setminus (\mathcal{S} \cup \mathcal{B}) \neq \emptyset$. In order to obtain this behavior automatically we shall design h to be a smooth feedback which is equal to 0 on the bad region $\mathcal{B}$, 1 on the safe region $\mathcal{S}$ and taking values in $(0, 1)$ in the zone $\mathcal{L}$, which we call "learning" zone because in this region the not expert controller v is helped by the "teacher" s. Moreover s is an optimal feedback that in general may be not continuous. We thus obtain a system:

$$\dot{x} = F + (1 - h)G_1 s(x) + h\, G_2 v(t) \qquad (2)$$

which is comprised of an ODE with an autonomous discontinuous term plus a time dependent smooth (in x) one. Hence when h is not constantly equal to 0 or 1 we immediately face the problem of de£ning a solution as illustrated by the following example.

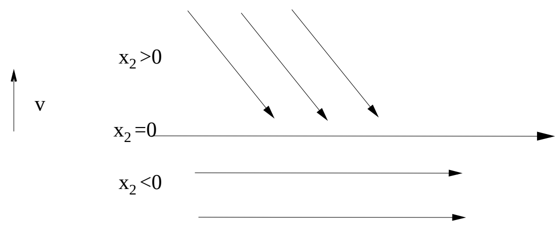

Fig. 1. No Caratheodory solution for the system (3).

Example 1. Let $\dot{x} = (1 - h)s + hv$ where $x = (x_1, x_2) \in \mathbb{R}^2$, $h(x) = 1/2$,

$$s(x_1, x_2) = \begin{cases} (1, 0) & \text{if } x_2 \le 0 \\ (1, -1) & \text{if } x_2 > 0 \end{cases} \qquad (3)$$

and $v(t) = (0, 1/2)$. It is easy to check that there is no Caratheodory solution if we take the initial data on the line $x_2 = 0$ (see £gure 1). ◁

2.2 *The Dubins' Car in the Safety Control Model*

We introduce in this section the Dubins' model of a car moving on the plane and the safe driving problem. We consider a car moving only forward on a plane at constant velocity, that we may assume is equal to 1, with controlled steering. The position of the car is given by $Z = (x, y, \theta) \in \mathbb{R}^2 \times S^1$, where $(x, y) \in \mathbb{R}^2$ indicates the position of the baricenter of the car and $\theta \in S^1$ is the angle formed by the car axis with the positive x axis. The equation of motions are: $\dot{x} = \cos(\theta),\ \dot{y} = \sin(\theta),\ \dot{\theta} = u,\ |u| \le 1$, where u represents the control. We assume to have a street described, up to change of coordinates, by the strip $\mathcal{X} = \{(x, y) :\ |y| \le L\} \subset \mathbb{R}^2$, where $L > 1$. The aim is to stay inside the street without hitting the boundaries. The not expert driver will be modeled by a random control v and the safety control is a feedback again indicated by s, so the complete system becomes:

$$\dot{Z} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ (1-h(Z))s(Z) + h(Z)v(t) \end{pmatrix},$$

where $h, s, v \in [0, 1]$. We restrict to the case where $\theta \in [-\pi/2, \pi/2]$ which means that we are not allowed driving in the negative direction for x. We de£ne the bad zones as:
$\mathcal{B}_1 = \{(x, y, \theta) : -1 + \cos(\theta) \leq y - L \leq 0, \ \theta \in [0, \pi/2]\}$
$\mathcal{B}_2 = \{(x, y, \theta) : 0 \leq y + L \leq 1 - \cos(\theta), \ \theta \in [-\pi/2, 0]\}$.
Indeed being in a point of $\mathcal{B}_1 \cup \mathcal{B}_2$ implies that the trajectories corresponding to extremal velocities (hence all trajectories) steer the car to hit the boundary of $\mathcal{X}$ in time less than or equal to time $t = |\theta|$. Observe that the symmetric sets
$\mathcal{B}_1' = \{(x, y, \theta) : -1 + \cos(\theta) \leq y - L \leq 0, \theta \in [-\pi/2, 0]\}$
$\mathcal{B}_2' = \{(x, y, \theta) : 0 \leq y + L \leq 1 - \cos(\theta), \theta \in [0, \pi/2]\}$,
are not accessible indeed the system could be in one of these regions only if, some time before, its con£guration was outside $\mathcal{X}$. Hence we can de£ne the set $\mathcal{B}$ to be the union of both the bad zones and the not accessible zones: $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_1' \cup \mathcal{B}_2 \cup \mathcal{B}_2'$. Finally we de£ne the safe zone as
$\mathcal{S} = \{(x, y, \theta) : -L' + 1 - \cos(\theta) \leq y \leq L' - 1 + \cos(\theta)\}$
where $L' < L$. The learning zone $\mathcal{L}$ is given by $\mathcal{L} = \mathcal{X} \times [-\pi/2, \pi/2] \setminus (\mathcal{B} \cup \mathcal{S})$.

2.3 *Good De£nition of Solution for a Safety Control System*

To solve the problem of de£ning a solution we will use the concept of Krasowskii and Filippov solutions, see (Filippov 1988).

De£nition 1. Given a function $s : \mathbb{R}^n \to \mathbb{R}^n$ we associate two multifunctions S_K and S_F in the following way:

$$S_K(x) = \bigcap_{\delta > 0} \overline{\text{co}}\, s(x + \delta B_n)$$

$$S_F(x) = \bigcap_{\delta > 0} \bigcap_{\text{meas}(N)=0} \overline{\text{co}}\, s((x + \delta B_n) \setminus N)$$

where $\overline{\text{co}}$ indicates the closed convex hull, B_n is the unit ball of $\mathbb{R}^n$, $N \subset \mathbb{R}^n$ and meas is the Lebesgue measure on $\mathbb{R}^n$.

De£nition 2. A Krasowskii (resp. Filippov) solution to $\dot{x} = s(x)$ is a solution to the differential inclusion $\dot{x} \in S_K(x)$ (resp. $\dot{x} \in S_F(x)$).

If s is bounded, the multifunctions S_K and S_F are upper semicontinuous, with compact convex values. It follows (see (Aubin and Cellina 1984)) the following:

Proposition 1. If s is bounded, then for every $x \in \mathbb{R}^n$ and every $T > 0$, the set of Krasowskii (Filippov) solutions de£ned on $[0, T]$ starting at x is a nonempty, connected, compact subset of $\mathcal{C}([0, T], \mathbb{R}^n)$.

We want to use the concept of Krasowskii (Filippov) solutions to de£ne a solution to our problem (2) with a discontinuous bounded s and measurable bounded v.

De£nition 3. Given a bounded measurable control v, a Krasowskii admissible solution to equation (2) is a solution to the differential inclusion

$$\dot{x} \in F + (1-h)G_1 S_K(x) + h\, G_2 v(t) \qquad (4)$$

Given $T, C > 0$, the set of Krasowskii admissible solutions for controls in $L^1([0, T], CB_n)$ is the set of solutions de£ned in $[0, T]$ for the differential inclusion

$$\dot{x} \in F + (1-h)G_1 S_K(x) + Ch\, G_2 B_n. \qquad (5)$$

We de£ne Filippov admissible solutions in the same way by replacing S_K with S_F in equations (4) and (5).

3. GENERAL STRATEGY

To construct a safety feedback we use the de£nition of admissible Krasowskii (Filippov) solutions. Our strategy is explained by the following scheme.

Step 1. Introduce a de£nition of feedback such that the corresponding Krasowskii (Filippov) solutions steer the system to a prescribed set possibly with minimal cost.
Step 2. Construct a safety control s satisfying the assumptions of **Step 1** and steering the system to the safe zone $\mathcal{S}$ possibly with minimal cost. Prove that for every bounded control v there is a solution to (4).
Step 3. Choose h and use **Step 2.** to prove that solutions do not enter the dangerous zone and tend to $\mathcal{S}$. Moreover prove that approximate solutions converge to generalized solutions.

3.1 *Step 1: Regular Strati£cations*

Now we give suf£cient conditions for *good behavior* of solutions, based on concepts of regular strati£ed feedback and strati£ed solutions which are given in the Appendix and are modeled on the de£nitions given by Boltyanskii and Brunovsky, see (Boltyanskii 1966, Brunovsky 1980).

De£nition 4. The regular strati£ed feedback $\Xi = (\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2, \Pi, \Sigma, s)$ is Krasowskii admissible if the following holds:

1. If P is a cell of type I then

Iα) If M is a cell such that $\partial M \supset P$ then M is a cell of type I.

Iβ) For every $x \in P$, $S_K(x) \cap T_x P = s(x)$.

Iγ) For each cell M such that $\partial M \supset P$ (hence M is of type I) the vector £eld s on M can be prolonged continuously to P and we call X_M the obtained vector £eld on P. For each $x \in P$ let $N(x)$ be the space normal to T_xP, then there exists $v \in N(x)$ such that for every sequence $\{y_n\}$ in M with $y_n \to x$, if $\omega = \lim_n \frac{x-y_n}{\|x-y_n\|}$ then $X_M(x) \cdot v > 0$ and $\omega \cdot v \geq 0$.

2. If P is of type II then

IIα) For each $x \in P$ let $N(x)$ be the space normal to T_xP, then there exist $v_x \in N(x)$ and $\epsilon_x > 0$ such that for every $u \in S_K(x)$, $u \cdot v_x \geq \epsilon_x$. Moreover v_x and ϵ_x are continuous with respect to x, $\inf\{\epsilon_x : x \in P\} > 0$ and v_x can be continuously prolonged to ∂P.

IIβ) For every $x \in P$, $S_K(x) \cap T_xP = \emptyset$.

IIγ) For each cell M of type I such that $\partial M \supset P$ and $M \notin \Sigma(P)$ the assumption Iγ) holds.

The same de£nition applies to Filippov admissible regular strati£ed feedback.

Theorem 1. (Marigo and Piccoli 2002) Let $\Xi = (\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2, \Pi, \Sigma, s)$ be a Krasowskii admissible regular strati£ed feedback then Krasowskii solutions to s (not passing through the origin) coincide with strati£ed solutions. The same conclusion holds for Filippov solutions.

3.2 *Step 2: Krasowskii and Filippov Solutions*

Using standard tools of theory of differential inclusions, it is easy to prove that for every £xed bounded measurable control v, we have a solution to (4). More precisely we have following:

Proposition 2. If s is bounded and v is measurable and bounded then the set of Krasowskii (Filippov) solutions to equation (4) is not empty.

Using **Step 1** we give also the de£nition of safety and stabilizing feedback for equation (2).

De£nition 5. We say that Ξ is a safety feedback, if Ξ is a Krasowskii (Filippov) admissible regular strati£ed feedback, all trajectories of $\dot{x} \in F(x) + G_1(x)S_K(x)$ (for Filippov $\dot{x} \in F(x) + G_1(x)S_F(x)$) starting outside $\mathcal{B}$, do not enter $\mathcal{B}$. If, moreover, the solutions reach $\mathcal{S}$ (possibly with minimal cost) we say that Ξ is a stabilizing feedback to $\mathcal{S}$.

3.3 *Step 3: Safe Evolution*

Given a safety feedback, we can choose h in the following way

(H1) There exists $U(\mathcal{B})$, open neighborhood of $\mathcal{B}$, such that $h(x) = 0$ for every $x \in U(\mathcal{B})$.

We are now ready to prove that the set of Krasowskii (Filippov) admissible solutions do not enter the bad zone $\mathcal{B}$.

Proposition 3. Assume that Ξ is a safety feedback and h is chosen as in (H1). Then every solution to (5) with initial state $x \notin \mathcal{B}$ does not enter the set $\mathcal{B}$.

Proof. Assume by contradiction that $x(\cdot)$ is a solution to (5) with $x(0) \notin \mathcal{B}$ and there exists $t > 0$ such that $x(t) \in \mathcal{B}$. Let $\bar{t}$ be the £rst time such that $x(\bar{t}) \in \mathcal{B}$. From (H1) there exists $\delta > 0$ such that and $h(x(t)) = 0$ for every $t \in [\bar{t} - \delta, \bar{t}]$. But then, on the set $[\bar{t} - \delta, \bar{t}]$, x is a solution to $\dot{x} \in F(x) + G_1(x)S_K(x)$, hence it does not reach $\mathcal{B}$ at time $\bar{t}$. □

Given a stabilizing feedback to $\mathcal{S}$, we can choose h in the following way

(H2) There exists U, open neighborhood of $\mathrm{cl}(\mathbb{R}^n \setminus \mathcal{S})$, such that $h(x) = 0$ for every $x \in U$.

With arguments similar to those used in the proof of the previous proposition we immediately get:

Proposition 4. Assume that Ξ is a stabilizing feedback to $\mathcal{S}$ and h is chosen as in (H2). Then every solution to (5) with initial state $x \notin \mathcal{B}$ does not enter the set $\mathcal{B}$ and reach $\mathcal{S}$ (possibly with minimal cost).

When implementing or simulating the feedback s, we have to consider approximations of s. Indeed in the £rst case we have to use some sampling strategy, while in the second we should consider some numerical methods. In both cases we have convergence to Krasowskii solutions as guaranteed by the following theorem (see Theorem 1 of Chapter 1.4 of (Aubin and Cellina 1984)).

Theorem 2. Let $S : \mathbb{R}^n \to \mathbb{R}^n$ be an upper semicontinuous multifunction with compact convex values. Let $x_k, y_k : [a, b] \to \mathbb{R}^n$ be measurable bounded such that for a.e. $t \in [a, b]$ and every $\delta > 0$, there exists $\bar{k} = \bar{k}(t, \delta)$ such that for every $k \geq \bar{k}$, there exists $\hat{x}, \hat{y}$ satisfying $\hat{y} \in S(\hat{x})$, $|x_k(t) - \hat{x}| < \delta$ and $|y_k(t) - \hat{y}| < \delta$.
If x_k converges a.e. to x and y_k converges weakly in $L^1([a, b], \mathbb{R}^n)$ to y, then for a.e. t, $y(t) \in S(x(t))$.

To simulate the whole system (2) we have to guarantee that when h is small the solutions are close to Krasowskii solutions to $\dot{x} = F(x) + G_1(x)s(x)$. From Theorem 2 we get:

Proposition 5. If s is bounded, as the L^1–norm of v tends to zero, Krasowskii solutions to (4) tend to solutions of $\dot{x} \in F(x) + (1 - h(x))G_1(x)S_K(x)$. The same conclusion holds for Filippov solutions.

4. SAFETY CONTROL FOR THE DUBINS' CAR

We take $h : \mathcal{X} \times [-\pi/2, \pi/2] \longrightarrow [0, 1]$ smooth such that

$$h(Z) = \begin{cases} 1 & \text{if } Z \in \mathcal{S} \\ 0 & \text{if } Z \in U(\mathcal{B}) \\ \delta(Z) & \text{if } Z \in \mathcal{L} \setminus U(\mathcal{B}) \end{cases}$$

where $\delta(Z) \in (0, 1)$, $U(\mathcal{B})$ is a neighborhood of $\mathcal{B}$ and $\mathrm{cl}(U(\mathcal{B})) \cap \mathcal{S} = \emptyset$.
We propose two choices for s. The £rst one comes from an optimal synthesis, while the second one is designed from the £rst one.
Consider the space (θ, y). Since the safe zone $\mathcal{S}$ surely contains the origin, we can consider the time optimal feedback to the origin. This feedback was described in (Bicchi *et al.* 2000) and the corresponding trajectories are represented in £gure 2. The optimal trajectories reach the origin with bang-bang controls, that is ± 1, and only one switching unless they reach the value $|\theta| = \pi/2$. In the latter case we have to use control 0, following a singular trajectory called turnpike, and then reach the origin with control ± 1. Two trajectories separate the zone where the feedback is $+1$ from the zone where it is -1, these are precisely the curves $y = 1 - \cos(\theta)$, $\theta \in [-\pi/2, 0]$ and $y = -1 + \cos(\theta)$, $\theta \in [0, \pi/2]$. Analyzing the one dimensional and zero dimensional singularities, according to the classi£cation of (Marigo and Piccoli 2002), we have the following

Proposition 6. The time optimal control to the origin is a stabilizing to $\mathcal{S}$ feedback.

Notice that if we are in the zone where $|\theta| \geq \pi/4$ and $\mathrm{sign}(\theta)y < -1+\cos(\theta)$, then we could let $s = 0$. This results in a minor cost of the safe control. Thus we propose a second strategy s_A whose trajectories are depicted in £gure 3. This strategy has the advantage of not leading the car to the line $|\theta| = \pi/2$ and thus being more robust for disturbances. Also in this case we have:

Proposition 7. The control s_A is a stabilizing to $\mathcal{S}$ feedback.

5. SIMULATION RESULTS

In this section we present some simulation results obtained for the suboptimal safety control s_A.
A £rst simulation tests the *good behavior* of the stabilizing feedback both in absence and in presence of disturbance (the random control v). In the £rst picture (£g.4) we depict two trajectories on the (θ, y) plane: one, labeled "Teacher", for the case of only the safe control acting ($h \equiv 0$) and one, labeled "Driver", when both the safe control and the free control are acting ($h \equiv 1/2$). Then we check that the control s_A is a safe control or stabilizing to $\mathcal{S}$ control, depending on the feedback h. For simplicity we choose the safe zone $\mathcal{S}$ (where $h \equiv 1$) to be $B(0, 0.3)$ the ball centered at the origin and radius 0.3 and the learning zone $\mathcal{L}$ to be $B(0, 1) \setminus B(0, 0.3)$, so $h \equiv 0$ outside $B(0, 1)$. In case of stabilizing control $h \equiv 0$ outside $B(0, 0.3)$.
In pictures 5 we show the behaviour, on the (y, θ)–plane of the whole system under safe control (with h satisfying condition (H1)) starting from point $(0, 1)$, which is on the boundary of the bad region $\mathcal{B}$.
Other simulations, not reported for space and £le length constraints, con£rm the theoretical results.

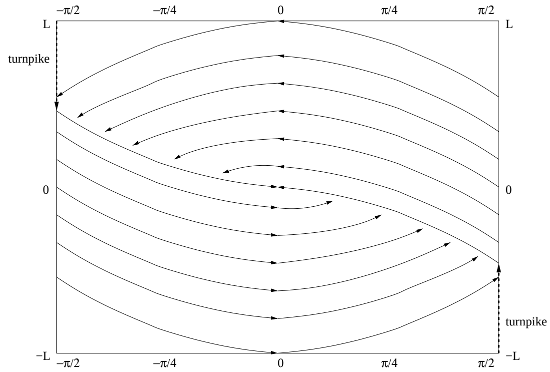

Fig. 2. The classical time optimal synthesis for the Dubins' Car

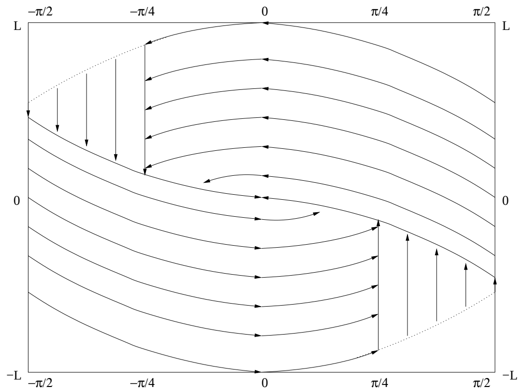

Fig. 3. The suboptimal synthesis for the Dubins' Car

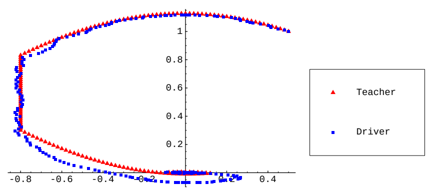

Fig. 4. Trajectory for s ($h \equiv 0$ "Teacher") and $s + v$ ($h \equiv 1/2$ "Driver")

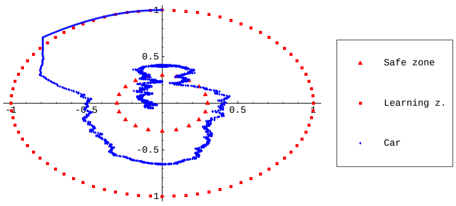

Fig. 5. (θ, y)–trajectory for safety control and initial conditions $(0, 1)$

APPENDIX

To de£ne a regular strati£ed feedback, for a system $\dot{x} = f(x, u)$, we £rst need to recall the de£nition of Withney strati£ed set.

De£nition 6. Let M be a subset of $\mathbb{R}^n$ and assume $M = \cup_{j \in J} M_j$, where $J \subset \mathbb{N}$ and M_j are disjoint nonemtpy connected embedded $\mathcal{C}^1$ submanifolds of $\mathbb{R}^n$. Then M is a Withney strati£ed set if the collection $\mathcal{P} := \{M_j\}_{j \in J}$, called the strati£cation of M, is locally £nite and the following holds.

- If $M_k \cap \mathrm{cl}(M_j) \neq \emptyset$ $(j \neq k)$ then $M_k \subset \partial M_j$ and $\dim(M_k) < \dim(M_j)$.
- Let $x_n, y_n \in M_j$, $n \in \mathbb{N}$, $x_n, y_n \to \bar{x} \in M_k \subset \mathrm{cl}(M_j)$ and denote by ℓ_n the straight line in $\mathbb{R}^n$ containing the segment joining x_n with y_n. If $T_{x_n} M_j \to T$(in the Grassmannian) and $\ell_n \to \ell$, then $\ell \subset T$ and $T_{\bar{x}} M_k \subset T$.

The dimension of M is $\dim(M) = \max_j \dim(M_j)$.

We restrict to the case of point target assumed to be the origin. For our system we can choose any $\bar{x} \in \mathcal{S}$ as target.

De£nition 7. Let Ω be an open set containing the origin. A regular strati£ed feedback on Ω is a 6–tuple $\Xi = (\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2, \Pi, \Sigma, s)$ such that

(RSF.1) Ω is a Withney strati£ed set with strati£cation $\mathcal{P}$. $\{0\} \in \mathcal{P}$. The elements of $\mathcal{P}$ are called "cells".

(RSF.2) $\mathcal{P} \backslash \{\{0\}\}$ is the disjoint union of $\mathcal{P}_1$ (the set of "type I cells") and $\mathcal{P}_2$ (the set of "type II cells"),

(RSF.3) the feedback $s : \{x : x \in P_1 \in \mathcal{P}_1\} \to U$ and $\Pi : \mathcal{P}_1 \to \mathcal{P}$ are maps, $\Sigma : \mathcal{P}_2 \to \mathcal{P}_1$ is a multifunction, with non empty values, such that the following properties are satis£ed:

(RSF3.A) The function s is of class $\mathcal{C}^1$ on each cell.

(RSF3.B) If $P_1 \in \mathcal{P}_1$ then $f(x, s(x)) \in T_x P_1$ (the tangent space to P_1 at x) for every $x \in P_1$. In addition, for each $x \in P_1$, if we let ξ_x be the maximally de£ned solution of the initial value problem

$$\dot{\xi} = f(\xi, s(\xi)), \quad \xi(0) = x, \quad \xi \in P_1, \qquad (6)$$

and de£ne $t_x = \sup Dom(\xi_x)$, then the limit $\xi_x(t_x-) \stackrel{\text{def}}{=} \lim_{t \uparrow t_x} \xi_x(t)$ exists and belongs to $\Pi(P_1)$.

(RSF3.C) If $P_2 \in \mathcal{P}_2$, then for each $P \in \Sigma(P_2)$ and $x \in P_2$ there exists a unique curve $\xi_x^P : [0, t_x^P[\to \Omega$ such that the restriction of ξ_x^P to $]0, t_x^P[$ is a maximally de£ned integral curve of the vector £eld $f(\cdot, s(\cdot))$ on P, and $\xi_x^P(0) = x$.

(RSF3.D) On every cell $P \in \mathcal{P}_1$, $x \to t_x$ is a continuously differentiable function, and $(t, x) \to \xi_x(t)$, $(t, x) \to u_x(t) \stackrel{\text{def}}{=} s(\xi_x(t))$ are continuously differentiable maps on the set

$$E(P) \stackrel{\text{def}}{=} \{(t, x) : x \in P\,,\ t \in [0, t_x]\}$$

in the sense that they can be prolonged to maps of class $\mathcal{C}^1$ on some open subset of $\mathbb{R} \times P$ containing $E(P)$. If $P_2 \in \mathcal{P}_2$ the same holds for every t_x^P, ξ_x^P, u_x^P, with $P \in \Sigma(P_2)$.

(RSF3.E) For every $x \in \Omega \backslash \{0\}$, if we let $\tilde{\xi}_x$ denote a curve, starting at x, obtained by piecing together the trajectories on every single cell, then $\tilde{\xi}_x$ ends at the origin in £nite time.

The trajectories $\tilde{\xi}_x$ of $(RSF3.E)$ are called strati£ed solutions to Ξ.

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