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FREQUENCY SHAPING COMPENSATION FOR BACKSTEPPING SLIDING MODE CONTROL

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Abstract: In this paper, dynamic sliding surface design combined with recursive backstepping algorithm is introduced. The high frequency, high amplitude chattering effects associated with high gain properties of the backstepping algorithm are eliminated by using compensator dynamics introduced in sliding mode through a class of switching surfaces which has the interpretation of linear operators.

Keywords: Lyapunov stability, uncertainty, sliding mode, chattering, robustness

1. INTRODUCTION

The robust stabilization of uncertain systems with both matched and unmatched modeling uncertainties and unknown disturbances has been an important research area in control. Sliding mode control theory has been extensively applied to stabilize the uncertain systems satisfying matched uncertainties, (Utkin, 1992). In the presence of unmatched uncertainty, recursive backstepping based on Lyapunov design (M. Krsti´c and Kokotović, 1995) has been applied for certain classes of systems. Recursive backstepping combined with robust sliding mode performance has been studied widely in recent years (Haskara and Özgüner, 1999), (Koshkouei and Zinober, 2000). The Lyapunov function used to stabilize the uncertain systems may lead to high gain and undesirable high frequency, high amplitude chattering. A new Lyapunov function has been proposed to design softer control laws to eliminate the chattering effects (Freeman and Kokotović, 1993). Basically they decomposed the disturbances and designed the Linear Quadratic (LQ) smooth controller making principle minors dominant for positive definiteness.

This paper introduces recursive backstepping combined with dynamic sliding surface design. Lyapunov based recursive backstepping design is applied to robustly stabilize the linear system in the presence of both matched and unmatched uncertainties and the recursive smooth state feedback control laws are generated by forcing the sliding manifold through compensator dynamics. The compensators are designed to attenuate the frequency contents of the sliding mode dynamics such that high frequency, high amplitude chattering effects are eliminated (Acarman and Özgüner, 2001), (Young and Özgüner, 1993). Two design methods are presented, the first method is based on pole placement techniques satisfying desired transient performance specifications during sliding mode by using the free parameters of the dynamic compensator and the second one is based on frequency-shaped LQ design techniques, which have been used to minimize high frequency chattering effects associated with the high gain sliding mode and backstepping control algorithms. This paper is organized as follows: In Section 2, dynamic switching surface has been designed for the error dynamics derived based on recursive backstepping control. In Section 3, frequency-shaped optimal sliding mode has been designed to eliminate chattering effects. In Section 4, simulation results and comparisons are presented. Section 5 gives some conclusions of this work.

2. DYNAMIC SLIDING MODE CONTROL COMBINED WITH RECURSIVE BACKSTEPPING DESIGN

Let the plant be given in the regular form,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \mathrm{B}_2 \end{bmatrix} u + \begin{bmatrix} \mathrm{D}_1(x_1, t) \\ \mathrm{D}_2(x_1, x_2, t) \end{bmatrix} \quad (1)$$

where $x_1 \in \mathcal{R}^n$, $x_2 \in \mathcal{R}^m$, $u \in \mathcal{R}^m$, the matrices are real, of compatible dimensions, B_2 is of full rank and the functions $D_1(x_1, t)$, $D_2(x_1, x_2, t)$ represent the system nonlinearities and uncertainties. The control goal is to make asymptotically globally stable the state x_1 using sliding mode control design methodology. The switching surface is

$$\sigma \;=\; Cx_1 + \mathcal{L}(x_2) \quad (2)$$

where C is a $\mathcal{R}^{m\times n}$ constant matrix to define the desired dynamics of the sliding mode and $\mathcal{L}(\bullet)$ is a linear operator which has a realization as a transfer function

$$(sI + F)z \;=\; [K_1 + K_2 s]\, x_2 \quad (3)$$

where K_1, $K_2 \in \mathcal{R}^{m\times m}$, $F \in \mathcal{R}^{m\times m}$. And the linear operator is defined $\mathcal{L}(\bullet)$ as a dynamic system

$$\begin{aligned} \dot{z} &= -Fz + K_2A_{21}x_1 + (K_1 + K_2A_{22})\,x_2 + K_2B_2u \\ y &= Hz + x_2 \end{aligned} \quad (4)$$

The composite system is given,

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + D_1(x_1, t) && (5) \\ \dot{x}_2 &= A_{21}x_1 + A_{22}x_2 + B_2z + D_2(x_1, x_2, t) && (6) \\ \dot{z} &= -Fz + K_2A_{21}x_1 + (K_1 + K_2A_{22})\,x_2 \\ &\quad + K_2B_2u + K_2D_2(x_1, x_2, t) && (7) \end{aligned}$$

We have assumed that $\mathcal{L}(\bullet)$ has an equal number poles and zeros, here the number of poles and zeros is equal to one to be able to introduce the first derivative of the state x_2 and the discontinuous control signal to the augmented state z. And z is the continuous input to the plant given in Eqn 1. The disturbances are assumed to satisfy,

$$\begin{aligned} \|D_1(x_1, t)\|_2 &\le d_1\|x_1\|_2 \\ \|D_2(x_1, x_2, t)\|_2 &\le \|D_1(x_1, t)\|_2 + \|D_1(x_2, t)\|_2 \\ &\le d_1\|x_1\|_2 + d_2\|x_2\|_2 \end{aligned} \quad (8)$$

where $\|x\|_2 = (x^Tx)^{1/2}$ for all $x \in \mathcal{R}^n$.

Step 1: Define the error variable $y_1 \equiv x_1$ and $y_2 \equiv x_2 - \alpha_1(x_1)$ where $x_2 = \alpha_1(x_1)$ with $\alpha_1(0) = 0$ is a smooth stabilizing state feedback control of the system given in Eqn 5 included to compensate for the uncertainty $D_1(x_1, t)$. A Lyapunov function, $\mathrm{V}_1 = \frac{1}{2}y_1^Ty_1$ and its first-order time derivative along the trajectory of Eqn 5

$$\begin{aligned} \dot{V}_1(y_1) &= \frac{\partial V_1}{\partial y_1}\dot{y}_1 = y_1^T(A_{11}y_1 + A_{12}x_2 + D_1(x_1, t)) \\ &\le \lambda_{min}(A_{11} - (\mathcal{K} - d_1))\|y_1\|_2^2 + d_1\|y_1\|_2^2 \\ &\le -k_1\|y_1\|_2^2 \end{aligned} \quad (9)$$

for all $x_1 \in \mathcal{R}^n$, $x_2 \in \mathcal{R}^m$ and $k_1 > 0$ is satisfied by using the state feedback,

$$x_2 = -A_{12}^{+}(\mathcal{K} - d_1)y_1 \quad (10)$$

where $\lambda_{min}(A_{11} - (\mathcal{K} - d_1)) < -d_1 < 0$ and A^+ is the pseudo-inverse of the matrix A. Then the time derivative of the error variable y_1 and the error variable y_2 are given by

$$\begin{aligned} \dot{y}_1 &= A_{11}y_1 + A_{12}(y_2 - \alpha_1(x_1)) + D_1(y_1, t) \quad (11) \\ &= (A_{11} - (\mathcal{K} - d_1))y_1 + A_{12}y_2 + D_1(y_1, t) \end{aligned}$$

$$\mathrm{y}_2 = x_2 + A_{12}^{+}(\mathcal{K} - d_1)x_1 = x_2 + \alpha_1(x_1)$$

Step 2: Define $y_3 \equiv z - \alpha_2(x_1, x_2)$ where $z = \alpha_2(x_1, x_2)$ with $\alpha_2(0, 0) = 0$ is a smooth stabilizing state feedback of the system given by Eqn 5 and Eqn 6 included to compensate for the uncertainties $D_1(x_1, t)$ and $D_2(x_1, x_2, t)$. The Lyapunov function $\mathrm{V}_2(y_1, y_2) = V_1(y_1) + \frac{1}{2}y_2^Ty_2$ and its time derivative along the trajectories of Eqn 5, Eqn 6 is given by

$$\begin{aligned} \dot{V}_2 &= y_1^T\,[(A_{11} - (\mathcal{K} - d_1))y_1 + A_{12}y_2 + D_1(y_1, t)] \\ &\quad + y_2^T\,[A_{21}x_1 + A_{22}x_2 + B_2z + D_2(x_1, x_2, t) \\ &\quad + A_{12}^{+}(\mathcal{K} - d_1)(A_{11}x_1 + A_{12}x_2 + D_1(x_1, t))] \end{aligned} \quad (12)$$

using the state feedback,

$$\begin{aligned} z = -B_2^{+}[&A_{12}^Ty_1 + A_{21}x_1 + A_{22}x_2 \\ &+ A_{12}^{+}(\mathcal{K} - d_1)\,(A_{11}x_1 + A_{12}x_2) - k_2y_2] \end{aligned} \quad (13)$$

the time derivative of $V_2(y_1, y_2)$ is obtained,

$$\begin{aligned} \dot{V}_2 \le\; &-k_1\|y_1\|_2^2 - k_2\|y_2\|_2^2 + \|y_2\|_2\,(\|D_2(x_1, x_2, t)\|_2 \\ &+ \|A_{12}^{+}(\mathcal{K} - d_1)\|_2\|D_1(x_1, t)\|_2) \end{aligned} \quad (14)$$

and using disturbance inequalities Eqn 8

$$\begin{aligned} \dot{V}_2 \le\; &-k_1\|y_1\|_2^2 - k_2\|y_2\|_2^2 + \|y_2\|_2\,(d_1\|x_1\|_2 \\ &+ d_2\|x_2\|_2) + d_1\|A_{12}^{+}(\mathcal{K} - d_1)\|_2\|x_1\|_2) \end{aligned} \quad (15)$$

The change of coordinates $(y_1, y_2, y_3) \leftrightarrow (x_1, x_2, z)$ is well defined (Freeman and Kokotović, 1993), and using a change of coordinates $(x_2) \leftrightarrow (y_1, y_2)$ and the triangle inequality,

$$\|x_2\|_2 \le \|y_2\|_2 + \|A_{12}^{+}(\mathcal{K} - d_1)\|_2\|y_1\|_2$$

the time derivative of $V_2(y_1, y_2)$ is given

$$\dot{V}_2 \leq -k_1\|y_1\|_2^2 - k_2\|y_2\|_2^2 + d_2\|y_2\|_2^2 + \|y_1\|_2(d_1 + (d_1+d_2)\|A_{12}^{+}(\mathcal{K}-d_1)\|_2)\|y_2\|_2 \quad (16)$$

define

$$\|A_{12}^{+}(\mathcal{K}-d_1)\|_2 \equiv [\lambda_{max}(A_{12}^{+}(\mathcal{K}-d_1)^T (A_{12}^{+}\ (\mathcal{K}-d_1)]^{\frac{1}{2}} \equiv \delta \quad (17)$$

by inserting the above definition into Eqn 16,

$$\begin{aligned}\dot{V}_2 \leq& -k_1\|y_1\|_2^2 - (k_2-d_2)\|y_2\|_2^2 \\ &+ \|y_1\|_2(d_1 + (d_1+d_2)\delta)\|y_2\|_2 \quad (18)\\ =& -\begin{bmatrix}\|y_1\|_2\\ \|y_2\|_2\end{bmatrix}\begin{bmatrix}k_1 & -\mathcal{L}\\ -\mathcal{L} & k_2-d_2\end{bmatrix}\begin{bmatrix}\|y_1\|_2\\ \|y_2\|_2\end{bmatrix}\end{aligned}$$

where $\mathcal{L} = d_1 + (d_1+d_2)\delta$, the time derivative of $V_2(y_1, y_2)$ is negative when the state feedback gain is chosen

$$k_2 > d_2 + \frac{d_1 + (d_1+d_2)\delta}{4k_1} \quad (19)$$

yields,

$$\dot{V}_2 \leq -c\left[\|y_1\|_2^2 + \|y_2\|_2^2\right] \quad (20)$$

for some $c > 0$. Then the time derivative of the error variable y_2 and the error variable y_3 are given by

$$\dot{y}_2 = A_{21}x_1 + A_{22}x_2 + B_2 z + D_2(x_1,x_2,t) + A_{12}^{+}(\mathcal{K}-d_1)\left[A_{11}x_1 + A_{12}x_2 + D_1(x_1,t)\right] \quad (21)$$

$$\begin{aligned}y_3 =& z + B_2^{+}[A_{12}^T y_1 + A_{21}x_1 + A_{22}x_2 + k_2 y_2 \\ &+ A_{12}^{+}(\mathcal{K}-d_1)(A_{11}x_1 + A_{12}x_2)] \\ \equiv& z + \alpha_2(x_1,x_2) \quad (22)\end{aligned}$$

inserting the state feedback $(z = y_3 - \alpha_2(x_1,x_2))$ Eqn 22, the error variable equation becomes,

$$\dot{y}_2 = -A_{12}^T y_1 - k_2 y_2 + B_2 y_3 + D_2(x_1,x_2,t) + A_{12}^{+}(\mathcal{K}-d_1)D_1(x_1,t) \quad (23)$$

and the third error variable equation is given by taking the first derivative of Eqn 22,

$$\begin{aligned}\dot{y}_3 =& -(F - \frac{\partial\alpha_2}{\partial x_2}B_2)y_3 + (K_1 + K_2A_{21} + \frac{\partial\alpha_2}{\partial x_1}A_{11} \\ &+ \frac{\partial\alpha_2}{\partial x_2}A_{21})(y_1 - \alpha_1(x_1)) + (K_2A_{22} + \frac{\partial\alpha_2}{\partial x_1}A_{12} \\ &+ \frac{\partial\alpha_2}{\partial x_2}A_{22})(y_2 - \alpha_2(x_1,x_2)) + \frac{\partial\alpha_2}{\partial x_1}D_1(x_1,t) \\ &+ \frac{\partial\alpha_2}{\partial x_2}D_2(x_1,x_2,t) + K_2B_2u \quad (24)\end{aligned}$$

2.1 *Sliding backstepping control*

Define sliding surface in terms of the error dynamics using

$$\sigma = Hy_3 + y_2 + Cy_1 \quad (25)$$

where H and C are defined in Eqn 2 and Eqn 4. The Lyapunov function of the composite error system (Eqn 11, Eqn 23, Eqn 24) is given

$$V_3 = \frac{1}{2}y_1^T y_1 + \frac{1}{2}y_2^T y_2 + \frac{1}{2}\sigma^T\sigma = V_2 + \frac{1}{2}\sigma^T\sigma$$

the time derivative of the Lyapunov function

$$\begin{aligned}\dot{V}_3 =& y_1^T\left[(A_{11} - (\mathcal{K}-d_1))y_1 + A_{12}y_2 + D_1(y_1,t)\right] \\ &+ y_2^T[-A_{12}^T y_1 - k_2y_2 + B_2y_3 + D_2(x_1,x_2,t) \\ &+ A_{12}^{+}(\mathcal{K}-d_1)D_1(x_1,t)] + \sigma^T(-H(F \\ &- \frac{\partial\alpha_2}{\partial x_2}B_2)y_3 + H(K_1 + K_2A_{21} + \frac{\partial\alpha_2}{\partial x_1}A_{11} + \frac{\partial\alpha_2}{\partial x_2}A_{21}) \\ &(y_1 - \alpha_1(x_1)) + H((K_2A_{22} + \frac{\partial\alpha_2}{\partial x_1}A_{12} + \frac{\partial\alpha_2}{\partial x_2}A_{22}) \\ &(y_2 - \alpha_2(x_1,x_2)) + K_2A_{22} + \frac{\partial\alpha_2}{\partial x_1}A_{12} + (\frac{\partial\alpha_2}{\partial x_2}A_{22}) \\ &(y_2 - \alpha_2(x_1,x_2)) + \frac{\partial\alpha_2}{\partial x_1}D_1(x_1,t) + \frac{\partial\alpha_2}{\partial x_2} \\ &D_2(x_1,x_2,t) + K_2B_2u)) - A_{12}^T y_1 - k_2y_2 + B_2y_3 \\ &+ D_2(x_1,x_2,t) + A_{12}^T(\mathcal{K}-d_1)D_1(x_1,t) \\ &+ C((A_{11} - (\mathcal{K}-d_1))y_1 + A_{12}y_2 + D_1(y_1,t))) \quad (26)\end{aligned}$$

A discontinuous control input can then be formulated as:

$$u = -(HK_2B_2)^{-1}\left[M\|y\|_2 + \Delta\right] sign(\sigma) \quad (27)$$

where $y = [y_1\ y_2\ y_3]^T$ and $\Delta > 0, M > 0$ are fairly high gains such that a sliding mode on the sliding surface $\sigma = 0$ is guaranteed.
If the sliding mode exists: $\sigma = 0$, $y_3 = -H^{-1}y_2 - H^{-1}Cy_1$ and the time derivative of the Lyapunov function given in Eqn 26 can be derived (through straight-forward algebraic manipulations)

$$\dot{V}_3 \leq -(k_1 - d_1 - \gamma_1)\|y_1\|_2^2 - (k_2 - d_2 - \gamma_2)\|y_2\|_2^2 + \|y_1\|_2\gamma_3\|y_2\|_2 \quad (28)$$

where γ_i, $i = 1,2,3$ denotes the Euclidean norm of the derived terms in Eqn 26 when $\sigma = 0$.
If the sliding mode exists: $\sigma = 0$, the state feedback gains k_1 and k_2 can be chosen such that Eqn 28 yields

$$\dot{V}_3 < -c\left[\|y_1\|_2^2 + \|y_2\|_2^2\right] \leq -W(y_1,y_2) \quad (29)$$

where $W(y_1,y_2)$ is a continuos positive semidefinite function and $c > 0$. Then the error system (Eqn 11, Eqn 23) is exponentially stable. From Barbalat Lemma, $W(y_1,y_2) \to 0$ *as* $t \to \infty$ This implies $y_i = 0$, $i = 1,2,3$ *as* $t \to \infty$ and $\sigma = 0$ *as* $t \to \infty$ Therefore, the stability of the composite system along the dynamic sliding surface $\sigma = 0$ is guaranteed.

Now consider the error dynamics, the switching surface is defined as $\sigma = Hy_3 + y_2 + Cy_1$ where $\dot{y}_3$ is given in Eqn 22. If sliding mode exists on $\sigma = 0$, then the equation of sliding mode is

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} A_{11} - (\mathcal{K} - d_1)I & A_{12} \\ A_{12}^T - B_2 H^{-1} L & -k_1 I - B_2 H^{-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{30}$$

and the poles of the error dynamics can be placed by the selection of {K, H, L} if (A_{11}, A_{12}) is a controllable pair. The above system may be written as,

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix} \times \begin{bmatrix} I & 0 \\ \text{-}A_{12}^T - B_2 H^{-1} L & -k_1 I - B_2 H^{-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} A_{12} & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & I \\ (\mathcal{K} - d_1) & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \tag{31}$$

and its poles can be placed, if the pair

$$\left(\begin{bmatrix} A_{11} & 0 \\ 0 & I \end{bmatrix}, \quad \begin{bmatrix} A_{12} & I \\ 0 & 0 \end{bmatrix} \right)$$

is controllable. The controllability of the above pair is evidently the same as of (A_{11}, A_{12}) as claimed.

Remark : The constant matrices K_1, and K_2 do not affect the dynamics of the sliding mode. They may be considered as a Lead, Lag controller design parameters to improve transient performance response of the dynamic compensator.

3. FREQUENCY-SHAPED OPTIMAL SLIDING MODE

The idea of frequency shaping, which was proposed by (Gupta, 1980), is to introduce frequency dependent weighting matrices in a Linear Quadratic optimal regulator design formulation. The performance index,

$$J = \int_{-\infty}^{\infty} \left[x^*(jw) Q(w) x(jw) + u^*(jw) R(w) u(jw) \right] dt \tag{32}$$

where $Q(w) \geq 0$ and $R(w) > 0$ for all frequencies w and x^*, u^* are the complex conjugate transposes of x and u respectively. The frequency-shaping matrices are chosen based on the argument that if $R(w)$ is chosen to be large over a certain frequency band, and small outside this band, the control action whose frequencies lie in this band would be penalized more. Effectively, a reduction of the loop gain of the closed loop system at high frequencies is achieved over this frequency band. If high frequency in the control action is undesirable, by selecting high-pass characteristics for the elements of $R(w)$, high frequency control action is minimized. Equally, choosing low pass characteristics for $Q(w)$ to penalize the low frequency motion of the system produces a similar effect on the optimal feedback control.

Following the work (Young and Özgüner, 1993), which converted the frequency-dependent performance index (Eqn 32) into a standard constant weighting matrix through an augmentation to the original state-space with additional compensator states and dynamics which are defined by the frequency shaping matrices (Gupta, 1980), the quadratic cost is given for the error dynamics derived to stabilize the system given in Eqn 1,

$$\begin{aligned} \dot{y}_1 &= [A_{11} - (\mathcal{K} - d_1)]\, y_1 + A_{12} y_2 + D_1(x_1, t) \\ \dot{y}_2 &= \left[A_{21} + A_{12}^{+}(\mathcal{K} - d_1) A_{11}\right] y_1 \qquad (33) \\ &\quad + \left[A_{22} + A_{12}^{+}(\mathcal{K} - d_1) A_{12}\right] [y_2 - \alpha_1(y_1)] \\ &\quad + B_2 u + A_{12}^{+}(\mathcal{K} - d_1)\, [D_1(x_1, t)] \\ &\quad + D_2(x_1, x_2, t) \end{aligned}$$

$$\mathrm{J} = \int_{-\infty}^{\infty} \left[y_1^*(jw) Q_{11} y_1(jw) \;+\; y_2^*(jw) Q_{22}(w) y_2(jw) \right] dw$$

in which, without of loss generality, the cross state and control term has been removed. The frequency-shaped optimal switching surface is given by,

$$\sigma(y_1, y_2, z) = y_2 + R_e^{-1} \left(B_e^T P_e + N_e^T \right) \begin{bmatrix} \mathrm{z} \\ \mathrm{y}_1 \end{bmatrix} = 0 \tag{34}$$

where z is the state of the dynamic compensator realizing the transfer function $\tilde{Q}_{22}(s)$,

$$\begin{aligned} \dot{z} &= Fz + Gy_2 \qquad (35) \\ \eta &= Hz + Dy_2 \end{aligned}$$

$$\tilde{Q}_{22}(s) = D + H\,(sI - F)^{-1}\, G \tag{36}$$

where P_e is the solution of the Riccati equation

$$\begin{aligned} &A_e^T P_e + P_e A_e \qquad (37) \\ &- (P_e B_e + N_e)\, R_e^{-1}\, (P_e B_e + N_e)^T + Q_e = 0 \end{aligned}$$

with $\mathrm{A}_e = diag(F, A_{11} - (\mathcal{K} - d_1))$,

$$\mathrm{B}_e = \begin{bmatrix} \mathrm{G} \\ \mathrm{A}_{12} \end{bmatrix}, \quad N_e = \begin{bmatrix} \mathrm{H}^T D \\ 0 \end{bmatrix},$$

$$Q_e = diag(H^T H, Q_{11}), R_e = D^T D, \tag{38}$$

The frequency-shaped optimal sliding surface (Eqn 34) is a linear operator on the states and depending on the weighting matrices, certain frequency band is penalized. Define sliding surface in terms of the error dynamics using

$$\sigma = y_2 + c_1 y_1 + c_2 z \tag{39}$$

where c_1 and c_2 are determined by the transfer function $\tilde{Q}_{22}(s)$, by the error dynamics, Eqn 33, and the solution of the Riccati Equation given in Eqn 37 The Lyapunov function of the system given in Eqn 33

$$\mathrm{V}_2 = \tfrac{1}{2}y_1^T y_1 + \tfrac{1}{2}z^T z + \tfrac{1}{2}\sigma^T \sigma$$

the time derivative of the Lyapunov function

$$\begin{aligned}\dot{V}_2 \leq\ & y_1^T \left[(A_{11} - (\mathcal{K} - d_1))y_1 + A_{12}y_2 + D_1(y_1,t)\right] \\ & + z^T \left[Fz + Gy_2\right] + \sigma^T (c_1((A_{11} - (\mathcal{K} - d_1))y_1 + A_{12}y_2 \\ & + D_1(x_1,t)) + [A_{21} + A_{12}^{+}(\mathcal{K} - d_1)A_{11}]y_1 \\ & + \left[A_{22} + A_{12}^{+}(\mathcal{K} - d_1)A_{12}\right][y_2 - \alpha_1(y_1)] \\ & + B_2 u + A_{12}^{+}(\mathcal{K} - d_1)\left[D_1(x_1,t)\right] + D_2(x_1,x_2,t) \\ & + c_2(Fz + Gy_2)) \qquad (40)\end{aligned}$$

a discontinuous control input to

$$u = -B_2^{-1}\left[\tilde{M}\|y\|_2 + \tilde{\Delta}\right] sign(\sigma) \qquad (41)$$

where $y = [y_1\ y_2]^T$ and $\tilde{\Delta} > 0, \tilde{M} > 0$ are fairly high constants such that a sliding mode on the sliding surface $\sigma = 0$ is guaranteed. If the sliding mode exists: $\sigma = 0$, $y_2 = -c_1 y_1 - c_2 z$ and the time derivative of the Lyapunov function given in Eqn 40 can be derived (through straight-forward algebraic manipulations)

$$\begin{aligned}\dot{V}_2 &\leq -(k_1 - d_1 - \phi_1)\|y_1\|_2^2 + \phi_2\|z\|_2^2 + \|y_1\|_2\phi_3\|z\|_2 \\ &= -\begin{bmatrix}\|y_1\|_2 \\ \|z\|_2\end{bmatrix}\begin{bmatrix}k_1 - d_1 - \phi_1 & -\phi_3/2 \\ -\phi_3/2 & \phi_2\end{bmatrix}\begin{bmatrix}\|y_1\|_2 \\ \|z\|_2\end{bmatrix}\end{aligned}$$

Choosing $\mathrm{k}_1 > d_1 + \phi_1 + \frac{\phi_3^2}{4\phi_2}$ yields

$$\dot{V}_2 < -c\left[\|y_1\|_2^2 + \|z\|_2^2\right] \leq -W(y_1, z) \qquad (42)$$

where ϕ_i, $i = 1, 2, 3$ denotes the Euclidean norm of the derived terms in Eqn 40 when $\sigma = 0$, $c > 0$ and $W(y_1, z)$ is a continuos positive semidefinite function.
Following Theorem 2.2, the state feedback gains k_1 is chosen such that Eqn 42 is satisfied then the error system (Eqn 11) is exponentially stable. From Barbalat Lemma, $W(y_1, z) \to 0\ \ as\ \ t \to \infty$ This implies $\mathrm{y}_i = 0,\ i = 1, 2\ \ as\ \ t \to \infty$ and $\sigma = 0\ \ as\ t \to \infty$ Therefore, the stability of the composite system along the dynamic sliding surface $\sigma = 0$ is guaranteed.

4. SIMULATION RESULTS

For simulation purposes, the following plant is chosen.

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}1 & 2 \\ 2 & 1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ \mathrm{B}_2\end{bmatrix}u + \begin{bmatrix}x_1 \sin(10\mathrm{t}) \\ x_1\cos(10\mathrm{t}) + x_2\sin(20\mathrm{t})\end{bmatrix} \qquad (43)$$

The eigenvalues of the system are $\lambda = \{-1, 3\}$. The discontinuous input to the error dynamics is

$$u = -(M\|x\| + \Delta)sign(s) \qquad (44)$$

where M and Δ are positive constants.

4.1 *Dynamic switching surface design*

For dynamic switching surface design, the linear operator is chosen,

$$(s + 1000)z = (s + 1)x_2 \qquad (45)$$

Following the recursive (Eqn 1 thru Eqn 24), the error variables $y_1 \equiv x_1$, $\mathrm{y}_2 \equiv x_2 + 2x_1$ and $y_3 \equiv z + 6x_1 + 5x_2 + 8(x_2 + 2x_1)$, and the error dynamics of the plant given Eqn 43 are derived,

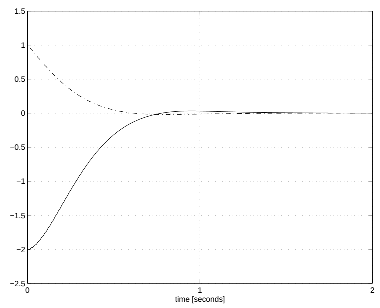
$$\begin{aligned}\dot{y}_1 &= -3y_1 + 2y_2 + y_1 sin(10t) \qquad (46) \\ \dot{y}_2 &= -6y_1 + 5y_2 + y_3 + y_1 cos(10t) + y_2 sin(10t) \\ \dot{y}_3 &= -4016y_1 + 12890y_2 - 987y_3 + u \\ &\quad + 14y_1 cos(10t) + 14y_2 sin(10t) - 6y_1 sin(20t)\end{aligned}$$

The Lyapunov function of the error system (Eqn 46) $\mathrm{V} = \frac{1}{2}(\mathrm{y}_1^2 + y_2^2 + \sigma^2)$ and the sliding manifold $\sigma = 1.995y_1 + 5y_2 + y_3$ are chosen and using the discontinuous input

$$u = -(625|z| + 3750|y_2| + 250|y_1| + 10)sign(\sigma)$$

finite time convergence to the sliding manifold is guaranteed. During the sliding mode, dynamics of the closed loop system are at $\{-1.5 \pm j3.7\}$. The time responses of the states, the continuous plant control input (Fig.1, Fig.2) have been simulated.

Fig. 1. The time response of the states – (Dynamic switching surface design)



4.2 *Frequency-shaped sliding mode design*

Defining the error variables $y_1 \equiv x_1$ and $y_2 \equiv x_2 + 2x_1$ and following the recursive backstepping steps, the error dynamics are obtained,

$$\begin{aligned}\dot{y}_1 &= -3y_1 + 2y_2 + y_1 sin(10t) \qquad (47) \\ \dot{y}_2 &= -6y_1 + 5y_2 + u + y_1 cos(10t) + y_2 sin(20t)\end{aligned}$$

A high-pass characteristic for $\tilde{Q}_{22}(w)$ with corner frequencies of 1 and 10 $rads^{-1}$, and a 40dB per decade slope is selected, whereas a unity weighting for Q_{11}, i.e.

$$Q_{11} = 1, \quad \tilde{Q}_{22}(w) = \frac{(jw + 2.25)^2}{(jw + 5)^2} \qquad (48)$$

Fig. 2. The time response of the input to the plant (Eqn 43) – (Dynamic switching surface design)

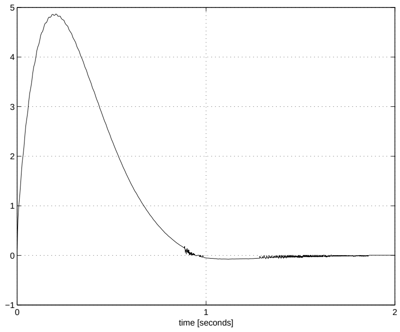
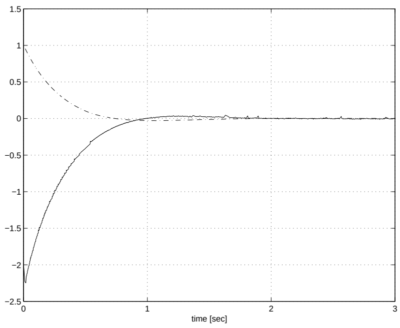


Fig. 3. The time response of the states – (Frequency-shaped sliding mode design)



The gain value is chosen such that a 40 dB weighting is applied to the high frequencies in x_2, and a 0 dB weighting for low frequencies to ensure that the optimal control action avoids the high frequency disturbances and eliminates high frequency, high gain chattering due to the high gain controller design. The system matrices for the Riccati equation Eqn 37 are:

$$A_e = \begin{bmatrix} 0 & 1 & 0 \\ -25 & -10 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

$$Q_e = \begin{bmatrix} 30.25 & 109.65 & 0 \\ 109.65 & 397.51 & \\ 0 & 0 & 1 \end{bmatrix}$$

$$B_e = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, \mathrm{Re} = 1 \times 10^4, \mathrm{N}_e = \begin{bmatrix} -5.5 \\ -19.93 \\ 0 \end{bmatrix}$$

The resulting shaped optimal switching surface is $\sigma = y_2 - 4.7221 \times z_1 - 0.0011 \times z_2 + 0.3224 \times y_1 = 0$

The states z_1 and z_2 are defined by a state-space realization of $\tilde{Q}_{22}(s)$,

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -25 & -10 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} y_2$$

The Lyapunov function of the error system (Eqn 47) $\mathrm{V} = \frac{1}{2}(\mathrm{y}_1^2 + \sigma^2)$ using the discontinuous input

$$u = -(10|y_1| + 5|y_2| + 100|z_1| + 34|z_2| + 10)sign(\sigma)$$

finite time convergence to the sliding manifold is guaranteed. During the sliding mode, dynamics of the closed loop system are at $\{-2.83, -3.64, -7.35\}$. The time responses of the states (Fig 3) have been simulated.

5. CONCLUSIONS

This paper contributes to the elimination of high frequency, high amplitude chattering caused by large state feedback gains derived by Lyapunov based recursive backstepping controller design. Transient performance improvement and finite time convergence to the switching surface have been assured by this proposed backstepping dynamic sliding mode methodology. The proposed controllers and conventional sliding mode controller combined with recursive backstepping design have been applied to linear systems in regular form with both matched and unmatched time-varying disturbances and simulation results have been presented.

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