15th Triennial World Congress, Barcelona, Spain

ON THE CHOICE OF SAMPLING PERIOD AND ROBUST POLE PLACEMENT

Xiao-long Peng *, **Bengt Lennartson** ** and **Carl-Magnus Fransson** **

* *Ericsson Mobile Data Design AB, Göteborg, Sweden*
email: pengxl@s2.chalmers.se
** *Chalmers University of Technology, Göteborg, Sweden*
email: bl@s2.chalmers.se

Abstract: In this paper a systematic procedure for the choice sampling period in sampled-data control is suggested. The procedure is based on an optimization scheme, where both low frequency performance, mid frequency stability margins and high frequency control activity are taken into account. The proposed evaluation method is applied to a pole placement controller for an integral plant with time-delay and a third order stable plant model. The results show that, especially when an antialiasing filter is included, a significantly shorter sampling period can be motivated compared to existing rules of thumb. As a second result an optimal pole location for the integral plant with time delay is achieved.

Keywords: sampling period, pole placement, time-delay, optimization, robustness

1. INTRODUCTION

Since the introduction of sampled-data control the question of a proper sampling period has been discussed. Very few systematic approaches have been introduced. Typical rules of thumb are related to the desired closed loop bandwidth, see e.g. (Åström and Wittenmark, 1990; Powell and Katz, 1975). In (Lennartson, 1990) closed loop performance (compensation of load disturbances) was investigated for different sampling periods keeping the control activity at a fixed level independent of the sampling rate. The results showed that control strategies having a continuous-time counterpart can be recommended. Hence, control design methods where poles are forced at the origin cannot be recommended (often suggested in the control literature), since shorter sampling periods implies increased control signal activity. This investigation was based an a stochastic framework.

More recently an evaluation procedure based on frequency response and $\mathcal{H}_\infty$-criteria has been used for synthesis and evaluation of PI and PID controllers, see (Lennartson and Kristiansson, 1997; Kristiansson and Lennartson, 2000). This approach includes minimization of performance with constraints on control activity but also stability margins. This general and systematic evaluation procedure is now generalized to sampled-data systems and applied to the choice of sampling period. The resulting constrained nonlinear optimization problem is solved by MATLABs Optimization Toolbox and TOMLAB (Holmström, 1999), an alternative optimization tool including global optimization routines. Note that the minimization is non-convex and ill-conditioned, and hence the choice of initial values in the optimization routine is a tricky task.

Buffer systems occur in many different application areas, such as transportation, chemical plants and data communication. A reasonable model for a buffer system is often just an integrator with time-delay. One example is the Internet TCP/IP protocol implementation, where communication buffers are modeled in that way and controlled by a Smith-predictor structure, see (Mascolo, 1999).

In this paper we examine a standard digital pole placement design as an alternative control strategy for integral plants with time-delay. As a result of our evaluation procedure both a suitable sampling period and an optimal pole placement location is suggested. When an antialiasing filter is included in the design a suitable sampling frequency is found to be around 50 times the closed loop bandwidth ω_b, which can be compared to recommendations like 10–20 times ω_b in the literature, see e.g. (Åström and Wittenmark, 1990; Lennartson, 1990). This choice of sampling period is also verified for a third order plant model.

2. EVALUATION METHOD

It is a well known fact that improvement of a controller design in one respect will very often bring deterioration in another. Different system qualities are not independent of each other. Especially we note that changes of some character in one frequency region usually will have influences in other frequency ranges. Therefore a method for comparison of two controllers must, if it claims to be fair, guarantee that all aspects that are not immediately compared are equally restricted during the comparison. The method proposed here will fulfill this demand. It is based on three criteria, each of them related to essential performance and robustness qualities of the actual system and also roughly related to different frequency ranges.

Consider the sampled-data SISO system in Figure 1, where a continuous-time plant with transfer function $G(s)$ together with an antialiasing filter $G_f(s)$ is controlled by a digital controller $K_d(z)$. The sampling period is h and the hold function is assumed to be of zero order. A corresponding discrete-time model for the plant including the antialiasing filter is denoted $G_d(z)$.

The *loop transfer function* then becomes $L_d(z) = G_d(z)K_d(z)$. Hence, the discrete-time *sensitivity* and *complementary sensitivity* functions are

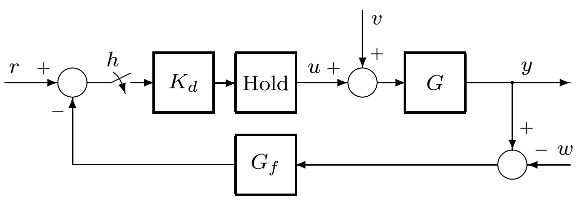

Fig. 1. Closed loop SISO system with continuous-time plant $G(s)$, antialiasing filter $G_f(s)$ and discrete-time controller $K_d(z)$.

$$S_d(z) = \frac{1}{1 + L_d(z)} \qquad T_d(z) = \frac{L_d(z)}{1 + L_d(z)}$$

2.1 *Stability margins*

Two classical measures are common to characterize the mid frequency (MF) robustness, the phase margin φ_m and the gain margin G_m. However, in recent years a restriction of the *maximum sensitivity function*

$$\|S_d\|_\infty = \max_\omega |S_d(e^{j\omega h})| \le M_S \qquad (1)$$

has been more and more accepted as an exclusive robustness measure, (Åström and Hägglund, 1995; Langer and Landau, 1999). The reason is that $\|S_d\|_\infty$ is equal to the inverse of the minimal distance from the loop transfer function $L_d(e^{j\omega h})$ to the critical point $(-1, 0)$ in the Nyquist plot. In many situations it is also a fully sufficient MF robustness measure.

When further damping of the step response or increased phase margin is required, without slowing down the system response too much, a restriction on the maximum complementary sensitivity function

$$\|T_d\|_\infty = \max_\omega |T_d(e^{j\omega h})| \le M_T \qquad (2)$$

should be added, especially for plants with integral action, see (Kristiansson, 2000). Hence, the proposed mid frequency robustness criterion, the *Generalized Maximum Sensitivity* GM_S is defined as

$$GM_S = \max(\|S_d\|_\infty, \alpha\|T_d\|_\infty) \qquad (3)$$

where $\alpha = M_S/M_T$. The two restrictions in this criterion correspond to two circles in the complex plane, inside which the Nyquist plot of $L_d(e^{j\omega h})$ is not allowed to come. In Figure 2 these circles are shown for the default values throughout this article, $M_S = 1.7$ and $M_T = 1.3$. When there is equality in at least one of the restrictions (1)-(2), as for the loop in the figure, this means that $GM_S = M_S$. Hence the GM_S criterion converts the restriction (2) to a corresponding M_S level.

2.2 *Sampled-data frequency response*

Before we introduce relevant performance measures we note that the frequency response for a sampled-data system from a continuous-time input signal can be defined in different ways, see (Lindgärde and Lennartson, 1997). One approach,

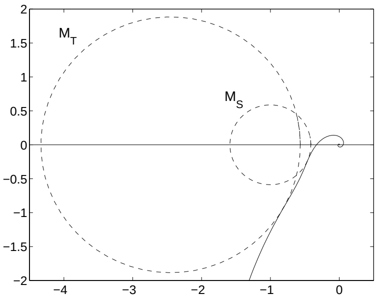

Fig. 2. The M_S-circle ($M_S = 1.7$) and the M_T-circle ($M_T = 1.3$) which together define GM_S.

suitable for performance analysis, is to assume that the input is a continuous-time periodic signal

$$v(t) = v_0 e^{j\omega t} \qquad |v_0| < \infty$$

Then introduce the square root of the signal power averaged over time, i.e.

$$\|v\|_{\mathcal{P}} = \lim_{\tau \to \infty} \sqrt{\frac{1}{2\tau} \int_{-\tau}^{\tau} \|v(t)\|^2 \, dt}$$

Assume that the closed-loop operator Ψ_{zv} for a sampled-data system from the continuous-time input v to the continuous-time output z is internally stable, piecewise continuous and h-periodic. The performance frequency gain (PFG) for Ψ_{zv} is then defined as

$$\overline{\gamma}_{zv}(\omega) = \max_{\|v_0\|=1} \frac{\|z\|_{\mathcal{P}}}{\|v\|_{\mathcal{P}}}$$

2.3 *Disturbance rejection*

Introduce the integrated output $Z(s) = Y(s)/s$. The systems ability to handle low frequency load disturbances v is then measured by the criterion

$$J_v = \max_{\omega} \overline{\gamma}_{zv}(\omega) \tag{4}$$

For J_v to be finite the controller must include integral action. In practice this criterion also happens to be related to the closed loop bandwidth ω_b for Ψ_{yr}, cf. (Kristiansson, 2000), and hence it can be regarded as a general performance measure.

2.4 *Control activity*

When reasonable stability margins are fulfilled, design of a control system is typically a question of trade off between performance and control activity. The sampled-data systems sensitivity to sensor noise w in the control signal u is measured by the control activity criterion

$$J_u = \max_{\omega} \overline{\gamma}_{uw}(\omega) \tag{5}$$

This PFG from w to u typically has its maximum around or slightly above the closed loop bandwidth. The control criterion J_u is therefore a mid to high frequency measure.

In (Lindgärde and Lennartson, 1997) it is shown how PFG can be computed. For the kind of plant models considered in this paper it is also shown by Lindgärde that the corresponding ordinary discrete-time frequency gain is an appropriate approximation. Hence, this simplification is utilized in this paper. For a general procedure, however, we recommend to compute the more complex PFG instead.

2.5 *Evaluation procedure*

To evaluate a sampled-data controller for different sampling periods we assume that there are one or more free parameters available for tuning. These parameters are represented by the vector ρ. Based on the proposed criteria we then suggest the following evaluation method. Solve the optimization problem

$$\min_{\rho} J_v(\rho) \quad GM_S \le C_1 \quad J_u \le C_2 \tag{6}$$

for different values of h. The default value of C_1 in this paper is 1.7, while C_2 is related the specific model. To get a fair comparison between different sampling periods it is important to have the same control activity J_u for sampling periods of interest.

3. OPTIMAL POLE PLACEMENT AND ANTIALIASING FILTER

The optimal controller design strategy described above is general and independent of any specific controller design principle. In this paper we choose the closed-loop pole configuration as tuning parameters. To make the closed-loop pole configuration as independent as possible of the chosen sampling period h, it is considered in continuous time. We have chosen a pair of well-damped complex conjugate poles and the remaining poles spread out on the negative real axis. The characteristic equation in the continuous transform domain then becomes

$$P(s) = (s^2 + 2\zeta\omega_n s + {\omega_n}^2) \\ (s+a)(s+a/\beta)\ldots(s+a/\beta^{\nu-1}) \tag{7}$$

where $\zeta > 0$, $\omega_n > 0$, $a > 0$, $0 < \beta \le 1$. Hence these tuning parameters, included in the vector $\rho = [\zeta \ \ \omega_n \ \ a \ \ \beta]$, are tuned in the minimization of J_v subject to the constraints on GM_S and J_u, see (6).

3.1 *Pole placement*

The discrete-time model $G_d(z)$ takes the form $G_d(z) = B(z)z^{-d}/A(z)$ where

$$A(z) = 1 + a_1 z^{-1} + \ldots + a_{n_a} z^{-n_a}$$

$$B(z) = b_1 z^{-1} + \ldots + b_{n_b} z^{-n_b}$$

Integral action in the controller means that $K_d(z) = D(z)/((1 - z^{-1})C(z))$ where

$$C(z) = 1 + c_1 z^{-1} + \ldots + c_{n_c} z^{-n_b-d+1}$$

$$D(z) = d_0 + d_1 z^{-1} + \ldots + d_{n_d} z^{-n_a}$$

Discretization of $P(s)$ yields

$$P_d(z) = 1 + p_1 z^{-1} + p_2 z^{-2} + \ldots + p_{n_p} z^{-n_p}$$

The poles at the origin in $G_d(z)$ due to time delay shall not be moved. On the other hand the rest of the poles shall not be forced at the origin. Taking the integral action into account this means that $np = na + nb$ and $\nu = np - 2$ in (7). The parameters in $C(z)$ and $D(z)$ are derived analytically by the Diophantine equation

$$A(z)(1 - z^{-1})C(z) + B(z)z^{-d}D(z) = P_d(z)$$

With the plant including the antialiasing filter and the controller $K_d(z)$, the performance criterion J_v, the control activity J_u and the stability margin GM_S can be computed in the optimization procedure described above. The optimal closed-loop pole configuration is then achieved as a result of the optimization procedure (6). Note that both the controller $K_d(z)$ and hence the criteria depends on the tuning parameter vector ρ.

3.2 *Antialiasing filter*

The antialiasing filter is chosen as a fourth-order Butterworth-filter, with two complex pole pair. The damping $\zeta = 0.38$ and 0.92, and the natural frequency ω_{bf} for both pole pair is

$$\omega_{bf} = \frac{0.56\pi}{h}$$

This means that the filter reduces the measurement noise by a factor 10 at the Nyquist frequency π/h. In the poleplacement design the antialiasing filter is simplified as a time-delay

$$G_f(s) = e^{-1.5h}$$

In this way the dimension of the resulting controller $K_d(z)$ is restricted. However, in the final evaluation of J_v, J_u and GM_S in the optimization procedure the complete fourth order filter is used.

4. CONTROL OF INTEGRAL PLANT INCLUDING TIME-DELAY

The evaluation procedure in (6) is now applied to the plant model

$$G(s) = \frac{e^{-s}}{s} \tag{8}$$

This model is normalized in the sense that the time scale is chosen as the length of the time delay, and the control signal is normalized such that the integral plant gain is set to unity.

When the time-delay is not a multiple of the sampling period, three non-zero poles have to be assigned ($nb = 2 = na + 1$). Solving the optimization procedure (6) for different sampling periods h then gives the optimal performance criterion J_v, the control activity J_u, the optimal closed-loop pole parameters ζ and ω_n as shown in Fig. 3-7. Four cases are considered, with and without antialiasing filter, free J_u and a constraint on $J_u \leq 1$. In the figures the curves including antialiasing filter are denoted $G_f = B_4$ (4th order Butterworth filter).

The following results can be noted

- An optimum is achieved when no constraint on J_u is introduced. The optimal J_u is some-

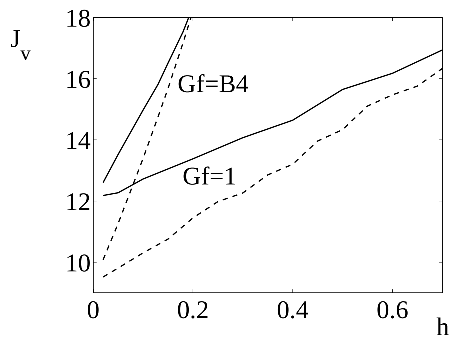

Fig. 3. Performance J_v as a function of the sampling interval h. Solid lines $J_u = 1$, dashed lines no restriction on J_u. $G_f = 1$ means no antialiasing filter, and $G_f = B_4$ means a fourth order Butterworth filter.

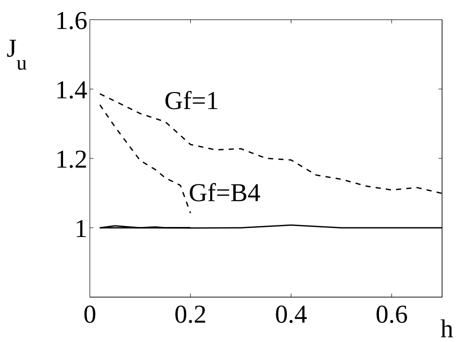

Fig. 4. Control activity J_u as a function of h.

what above 1, but decreases when h increases, cf Fig. 4.

- The performance criterion J_v increases with the sampling interval h as expected. Note however that a fair comparison between different sampling periods means that the control activity has to be fixed. The choice of sampling interval is therefore based on the constrained solution when $J_u = 1$.
- It can be seen that the introduction of the antialiasing filter deteriorates the performance J_v significantly, in the sense that J_v increases much more rapidly as a function of the sampling period. The reason is that the filter implies an extra phase shift, which increases with h (the filter is modeled as a delay equal $1.5h$).

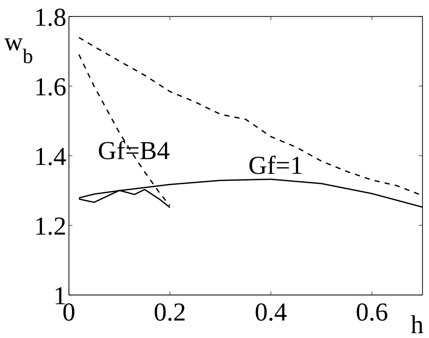

Fig. 5. Closed loop bandwidth ω_b as a function of h. Solid lines $J_u = 1$, dashed lines no restriction on J_u. $G_f = 1$ means no antialiasing filter, and $G_f = B_4$ means a fourth order Butterworth filter.

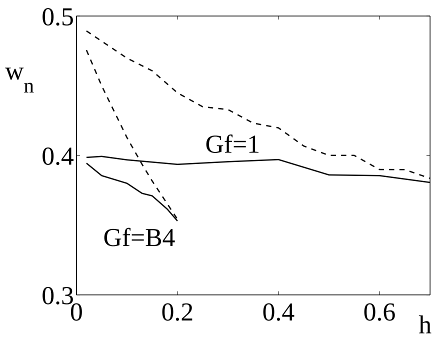

Fig. 6. Natural frequency ω_n in (7) as a function of h.

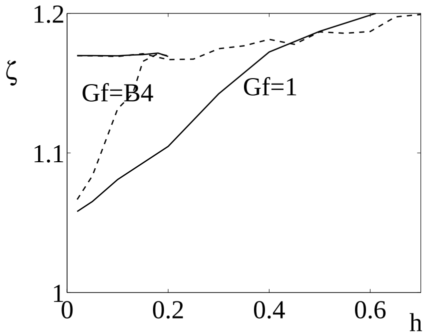

Fig. 7. Damping ζ in (7) as a function of h.

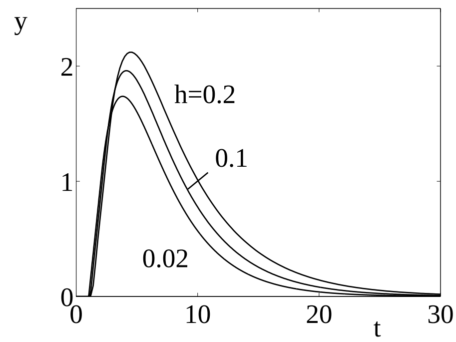

Fig. 8. Load disturbance step responses for the controlled time-delayed plant.

- A reasonable strategy for the choice of sampling period could be to choose a value when J_v has increased 10%—20% compared to the almost continuous-time solution. From Fig. 3 20% increase, based on $J_v = 12.1$ for $h \approx 0$, gives the following result

	J_v	h	ω_b	ω_s/ω_b
$G_f = 1$	14.5	0.19	1.3	25
$G_f = B_4$	14.5	0.08	1.3	58

- The table shows that introduction of an antialiasing filter motivates more than doubling of the sampling frequency. Typical rules of thumb in the literature suggest $\omega_s/\omega_b = 10 - 20$.
- When no limitation on the control activity is required Fig. 6 and 7 illustrate optimal closed-loop pole parameters ζ and ω_n (dashed lines). The damping ζ is quite invariant; an average value could be $\zeta = 1.15$. For the suggested sampling interval in the table a possible choice of the natural frequency ω_n is slightly above 0.4.
- The third free pole is put at the origin in the discrete-time domain, where it cancels a zero. Hence, only ω_n and ζ are necessary to include in the optimization parameter vector ρ.
- Figure 8 shows the step response from a load disturbance for three different sampling periods, when the antialiasing filter is included. Compared to the short interval $h = 0.02$, the choice $h = 0.1$ gives a reasonable degradation, while $h = 0.2$ implies an unnecessary large deterioration compared to the result with ten times higher sampling frequency ($h = 0.02$). These step responses confirm the suggested choice of sampling period $h = 0.08$ according to the 20% degradation rule.

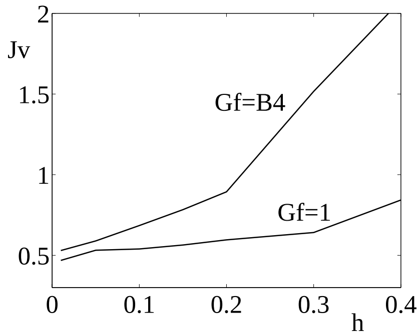

Fig. 9. Performance J_v as a function of the sampling interval h for the plant (9) when $J_u \leq 20$.

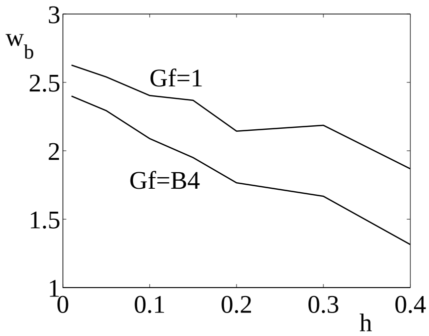

Fig. 10. Closed loop bandwidth ω_b as a function of h for the plant (9).

5. CONTROL OF A LAG PROCESS

To show that the results for the integral plant with time-delay was not just a special result for that plant, we end up this paper by a brief investigation of the following third order lag process

$$G(s) = \frac{1}{(s+1)^3} \tag{9}$$

Now there are six poles to assign, implying that $\nu = 4$ in (7). Optimal J_v is shown in Fig. 9 when the control activity is restricted to $J_u \leq 20$. Corresponding bandwidth ω_b is shown in Fig. 10. The 20% degradation rule for the choice of sampling interval based on continuous-time $J_v = 0.5$ gives the following result

	J_v	h	ω_b	ω_s/ω_b
$G_f = 1$	0.6	0.21	2.15	14
$G_f = B_4$	0.6	0.06	2.27	50

6. CONCLUSIONS

A systematic procedure for evaluation of suitable sampling periods for sampled-data control has been suggested. The procedure results in a nonlinear performance optimization problem with constraints on stability margin and control activity.

The framework is applied to an integral plant with time-delay, where also an optimal pole placement strategy is achieved. When an antialiasing filter is included in the design a suitable choice of sampling frequency is

$$\omega_s = \frac{2\pi}{h} = N\omega_b \quad \text{where} \quad N = 50 - 60$$

This implies a degradation of about 20% compared to a corresponding continuous-time solution. This rule is confirmed also for a third order lag process. Hence, the introduction of antialiasing filters motivates a significantly higher sampling rate than what is normally recommended in the literature ($N = 10 - 20$).

7. REFERENCES

Åström, K.J. and B. Wittenmark (1990). *Computer-Controlled Systems, 2nd Ed.* Prentice Hall.

Åström, K.J. and T. Hägglund (1995). *PID Controllers: Theory, Design and Tuning.* Instrument Society of America.

Holmström, K. (1999). The TOMLAB optimization environment in MATLAB. *Advanced Modeling and Optimization* **1(1)**, 47–69.

Kristiansson, B. (2000). Evaluation and tuning of PID controllers. licentiate thesis. Dept of Signals and Systems, Chalmers University of Technology. Göteborg, Sweden.

Kristiansson, B. and B. Lennartson (2000). From PI to $\mathcal{H}_\infty$ control in a unified framework. In: *Proc 39th Conf on Decision and Control.* Sydney.

Langer, J. and I.D. Landau (1999). Combined pole placement/sensitivity function shaping method using convex optimization criteria. *Automatica* **35(6)**, 1111–1120.

Lennartson, B. (1990). On the choice of controller and sampling period for linear stochastic control. *Automatica* **26**, 573–578.

Lennartson, B. and B. Kristiansson (1997). Pass band and high frequency robustness for PID control. In: *Proc 36th Conf on Decision and Control.* San Diego.

Lindgärde, O. and B. Lennartson (1997). Performance and robust frequency response for multirate sampled-data systems. In: *Proc American Control Conference '97.* pp. 3877–3881.

Mascolo, S. (1999). Congestion control in high-speed communication networks using the smith principle. *Automatica* **Vol 35**, 1921–1935.

Powell, J.D. and P. Katz (1975). Sample rate selection for aircraft digital control. *AIAA Journal* **13**, 975–979.