15th Triennial World Congress, Barcelona, Spain

DISCRETE-TIME SLIDING MODE CONTROL OF AN INDUCTION MOTOR [1]

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Abstract: A discrete-time sliding mode with block control aided design is applied to a nonlinear discrete-time induction motor model where the load torque is considered as unknown perturbation. With full state measurements, both rotor speed and rotor flux amplitude tracking objectives are satisfied. Then, a reduced order observer is implemented where speed and current measurements provide the observation for the unreachable fluxes and load torque. The simulations predict the system to be robust with respect to external load torques. *Copyright © 2002 IFAC*

Keywords: Inductions motors, discrete-time systems, sliding mode control, observer.

1. INTRODUCTION

Induction motor is one of the most used actuator for industrial applications due to its reliability, ruggedness and relatively low cost. The control of induction motor is challenging, since the dynamical system is multivariable, coupled, and highly nonlinear. A classical technique for induction motor control is field oriented control (Blaschke, 1972), which involves nonlinear state transformation and feedback for asymptotic decoupling of the rotor speed and rotor flux, and applying linear control methods such as PID. More recently, various nonlinear control design approaches have been applied to the induction motor control problem for better performance, like backstepping (Tan, and Chang; 1999), passivity (Ortega, *et al*., 1996), adaptive input-output linearization (Marino, and Tomei, 1995), and sliding modes (Utkin, *et al*., 1999; Doods, 1999). All of these approaches are based on the continuous-time model of the plant, and for practical implementation in a digital device, is necessary to design the controller for a discrete-time a discrete-time model of the plant.

This research work is based on a digital sliding mode (Utkin, et al., 1999) with block control aided design approach to achieve rotor speed and rotor flux amplitude tracking objectives for the fixed reference frame model. The uncertainty accounted for is an unknown load torque

The paper is organized as follows. Section 2 briefly reviews the continuous-time induction motor model and using the solution of the mechanical and rotor flux dynamics systems, this model is discretized. The main results are presented in Section 3, where the discrete-time sliding mode block control and the rotor flux and load torque observer, are designed. Section 4 deals with the proposed control law and observer simulations. Finally, in Section 5 are some concluding remarks drawn from simulations and control technique.

2. DISCRETIZATION OF THE CONTINUOS-TIME INDUCTION MOTOR MODEL

In this section, it is developed another representation of the induction motor model, called *discrete-time*

[1] *Work supported by Conacyt, Mexico, under grant 36960A*

induction motor model. Under the assumptions of equal mutual inductance and a linear magnetic circuit, a fifth-order induction motor model is given as

$$\frac{d\omega}{dt} = \mu \mathbf{I}^{\mathbf{T}} \Im \mathbf{\Psi} - \frac{T_L}{J}$$

$$\frac{d\mathbf{\Psi}}{dt} = -\alpha \mathbf{\Psi} + n_p \omega \Im \mathbf{\Psi} + \alpha M \mathbf{I}$$

$$\frac{d\mathbf{I}}{dt} = \varphi + \frac{1}{\sigma}\mathbf{u}$$

where $\Im = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is the skew matrix, and has the following property: $\mathbf{z}^T \Im \mathbf{z} = \mathbf{0}$; the other variables and parameters have the following definitions: $\mathbf{\Psi} \in \mathbf{R}^2$ is the rotor flux vector, $\mathbf{I} \in \mathbf{R}^2$ is the stator current vector, which in current-fed motors is the control input, $\mathbf{u} \in \mathbf{R}^2$ is the control input voltage vector, ω is the rotor angular velocity, T_L is the load torque, J is the rotor moment of inertia, and

$$\varphi = \begin{bmatrix} \alpha\beta\psi_\alpha + n_p\beta\omega\psi_\beta - \gamma i_\alpha \\ \alpha\beta\psi_\beta - n_p\beta\omega\psi_\alpha - \gamma i_\beta \end{bmatrix}$$

$\alpha = \frac{R_r}{L_r} = \frac{1}{T_r}$, $\beta = \frac{M}{\sigma L_r}$, $\gamma = \frac{M^2 R_r}{\sigma L_r^2} + \frac{Rs}{\sigma}$,

$\sigma = L_s - \frac{M^2}{L_r}$, $\mu = \frac{3}{2}\frac{Mn_p}{JL_r}$, L_s, L_r and M are the stator, rotor and mutual inductance respectively, R_s, R_r are the stator and rotor resistances respectively, and n_p is the number of pole pairs.

To face the problem of discretization it is necessary to found the solution of the system, but this system has no analytic solution at all. To overcome this problem, the model is divided in a current-fed induction motor third-order model, where the current inputs are considered as pseudo-inputs, and a second-order subsystem that only models the currents of the stator with voltages as inputs. The current-fed model will be exactly discretized by solving the set of differential equations and the other subsystem will be discretized by a first-order Taylor series (Kazantzis and Kravaris; 1999). Making use of the following globally defined change of coordinate:

$$\mathbf{Y} = e^{-np\theta\Im}\mathbf{\Psi}, \quad \mathbf{X} = e^{-np\theta\Im}\mathbf{I} \tag{1}$$

where $\dot{\theta} = \omega$, yields the following bilinear model

$$\frac{d\omega}{dt} = \mu \mathbf{X}^{\mathbf{T}} \Im \mathbf{Y} - \frac{T_L}{J}$$
$$\frac{d\mathbf{\Psi}}{dt} = -\alpha \mathbf{Y} + \alpha M \mathbf{X} \tag{2}$$

Founding a solution to (2) involves integral operations, where it is assumed that control is applied in a piecewise constant fashion. So, the control is constant over the integration time interval $[kT,(k+1)T]$, $k = 0,1,2,...,$ where $T > 0$ is the *sampling time*. The solution to (2) in this time interval is

$$\omega((k+1)T) = \omega(kT) + \frac{\mu}{\alpha}\left(1 - e^{-\alpha T}\right)\mathbf{X}^T(kT)\Im\mathbf{Y}(kT) - \frac{T_{Lk}}{J}T$$

$$\mathbf{Y}((k+1)T) = e^{-\alpha T}\mathbf{Y}(kT) + M\left[1 - e^{-\alpha T}\right]\mathbf{X}(kT).$$

Defining a common notation $\mathbf{x}_k = \mathbf{x}(kT)$, yields

$$\omega_{k+1} = \omega_k + \frac{\mu}{\alpha}\left(1 - e^{-\alpha T}\right)\mathbf{X}_k^T \Im \mathbf{Y}_k - \frac{T_{Lk}}{J}T \tag{3}$$
$$\mathbf{Y}_{k+1} = e^{-\alpha T}\mathbf{Y}_k + M\left[1 - e^{-\alpha T}\right]\mathbf{X}_k$$

where $a = e^{-\alpha T}$. Taking (3) to the original states with a inverse transformation of (1), finally yields

$$\omega_{k+1} = \omega_k + \frac{\mu}{\alpha}(1-a)M\left(i_k^\beta \psi_k^\alpha - i_k^\alpha \psi_k^\beta\right) - \left(\frac{T}{J}\right)T_{Lk}$$

$$\psi_{k+1}^\alpha = \cos\left(n_p\theta_{k+1}\right)\rho_1 - sin\left(n_p\theta_{k+1}\right)\rho_2$$

$$\psi_{k+1}^\beta = sin\left(n_p\theta_{k+1}\right)\rho_1 + \cos\left(n_p\theta_{k+1}\right)\rho_2$$

where

$$\rho_1 = a\left(\cos(n_p\theta_k)\psi_k^\alpha + \sin(n_p\theta_k)\psi_k^\beta\right) + (1-a)M\left(\cos(n_p\theta_k)i_k^\alpha + \sin(n_p\theta_k)i_k^\beta\right)$$

$$\rho_2 = a\left(\cos(n_p\theta_k)\psi_k^\beta - \sin(n_p\theta_k)\psi_k^\alpha\right) + (1-a)M\left(\cos(n_p\theta_k)i_k^\beta - \sin(n_p\theta_k)i_k^\alpha\right)$$

The rotor position is calculated from $\dot{\theta} = \omega$, in the same way, yielding

$$\theta_{k+1} = \theta_k + \omega_k T + \frac{\mu}{\alpha}\left[T - \frac{1}{\alpha}(1-a)\right]M(i_k^\beta\psi_k^\alpha - i_k^\alpha\psi_k^\beta) - \frac{T_{Lk}}{2J}T^2$$

There are left two differential current equations to discretize, by a first order Taylor series

$$i_{k+1}^\alpha = \varphi_k^\alpha + \frac{T}{\sigma}u_k^\alpha, \quad i_{k+1}^\beta = \varphi_k^\beta + \frac{T}{\sigma}u_k^\beta$$

where

$$\varphi_k^\alpha = i_k^\alpha + \alpha\beta T\psi_k^\alpha + n_p\beta T\omega_k\psi_k^\beta - \gamma T i_k^\alpha$$

$$\varphi_k^\beta = i_k^\beta + \alpha\beta T\psi_k^\beta - n_p\beta T\omega_k\psi_k^\alpha - \gamma T i_k^\beta$$

Finally, putting all together, the discrete-time version of the induction motor model, is feature

$$\omega_{k+1} = \omega_k + \frac{\mu}{\alpha}(1-a)M\left(i_k^\beta\psi_k^\alpha - i_k^\alpha\psi_k^\beta\right) - \left(\frac{T}{J}\right)T_{Lk}$$

$$\psi_{k+1}^\alpha = \cos\left(n_p\theta_{k+1}\right)\rho_1 - \sin\left(n_p\theta_{k+1}\right)\rho_2$$

$$\psi_{k+1}^\beta = \sin\left(n_p\theta_{k+1}\right)\rho_1 + \cos\left(n_p\theta_{k+1}\right)\rho_2$$

$$i_{k+1}^\alpha = \varphi_k^\alpha + \frac{T}{\sigma}u_k^\alpha \tag{4}$$

$$i_{k+1}^\beta = \varphi_k^\beta + \frac{T}{\sigma}u_k^\beta$$

$$\theta_{k+1} = \theta_k + \omega_k T + \frac{\mu}{\alpha}\left[T - \frac{1}{\alpha}(1-a)\right]M(i_k^\beta\psi_k^\alpha - i_k^\alpha\psi_k^\beta) - \frac{T_{Lk}}{2J}T^2$$

Fig. 1 compares the open-loop velocity simulation of both models.

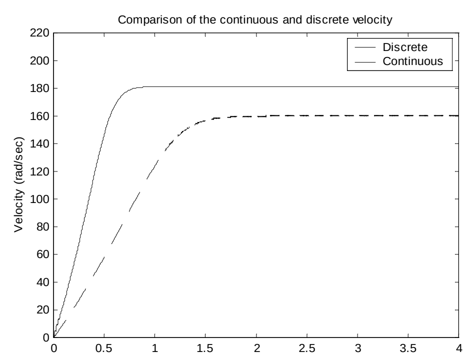


Fig. 1. Comparison of the continuous and discrete velocity.

There is a slight amount of error introduced by the current dynamical equations that were discretized by a first order Taylor series. Since the control input appears in these equations, the error can be eliminated.

3. DISCRETE-TIME SLIDING MODE CONTROL

Given full state measurements, the control objectives are to develop velocity and flux amplitude tracking for the electromechanical dynamics founded in the discrete-time induction motor model (4), using block control and discrete-time sliding mode.

3.1 Control design

Let us define the following states as

$$\mathbf{x}_k^1=\begin{bmatrix} x_k^1 \\ x_k^2 \end{bmatrix}=\begin{bmatrix} \omega_k-\omega_k^r \\ \psi_k-\psi_k^r \end{bmatrix},\quad \mathbf{x}_k^2=\begin{bmatrix} x_k^3 \\ x_k^4 \end{bmatrix}=\begin{bmatrix} i_k^\alpha \\ i_k^\beta \end{bmatrix} \tag{5}$$

where $\psi_k=\psi_k^{\alpha\,2}+\psi_k^{\beta\,2}$ is the rotor flux magnitude, ω_k^r and ψ_k^r are reference signals. If the resulting control, drives the state $\mathbf{x}_k^1$ toward zero, then ω_k and ψ_k will track exactly their respective reference signals, accomplishing in that way the control objectives. The system (4) involving (5), can be represented in the Block Controllable Form (BCF) consisting of two blocks

$$\begin{aligned}\mathbf{x}_{k+1}^1&=\mathbf{f}^1(\mathbf{x}_k^1)+\mathbf{B}_1(\mathbf{x}_k^1)\mathbf{x}_k^2 \\ \mathbf{x}_{k+1}^2&=\mathbf{f}^2(\mathbf{x}_k^1,\mathbf{x}_k^2)+\mathbf{B}_2\mathbf{u}_k\end{aligned} \tag{6}$$

where

$$\mathbf{f}^1=\begin{bmatrix} f^1 \\ f^2 \end{bmatrix}=\begin{bmatrix} \omega_k-\left(\frac{T}{J}\right)T_{Lk}-\omega_{k+1}^r \\ a^2(\psi_k)+(1-a)^2M^2\left(i_k^{\alpha 2}+i_k^{\beta 2}\right)-\psi_{k+1}^r \end{bmatrix}$$

$$\mathbf{B}_1=\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}\begin{bmatrix} -\psi_k^\beta & \psi_k^\alpha \\ \psi_k^\alpha & \psi_k^\beta \end{bmatrix},\quad \mathbf{f}^2=\begin{bmatrix} f^1 \\ f^2 \end{bmatrix}=\begin{bmatrix} \varphi_k^\alpha \\ \varphi_k^\beta \end{bmatrix}$$

$$\mathbf{B}_2=\begin{bmatrix} \frac{T}{\sigma} & 0 \\ 0 & \frac{T}{\sigma} \end{bmatrix},\quad \mathbf{u}_k=\begin{bmatrix} u_k^\alpha \\ u_k^\beta \end{bmatrix},\quad \begin{aligned} c_1&=\frac{\mu}{\alpha}(1-a)M, \\ c_2&=2a(1-a)M. \end{aligned}$$

Note that matrices $\mathbf{B}_1$ and $\mathbf{B}_2$ have full rank, that is: $rank(\mathbf{B}_1)=rank(\mathbf{B}_2)=2$, and

$$\|\mathbf{B}_1\|\le\beta_1,\qquad \|\mathbf{B}_2\|\le\beta_2. \tag{7}$$

Applying the block control technique, define an error vector, $\mathbf{z}_k^1$ is defined as $\mathbf{z}_k^1=(z_k^1,\ \ z_k^2)^T=\mathbf{x}_k^1$, then, the error dynamical equation is

$$\mathbf{z}_{k+1}^1=\mathbf{f}^1(\mathbf{x}_k^1)+\mathbf{B}_1(\mathbf{x}_k^1)\mathbf{x}_k^2 \tag{8}$$

Handling $\mathbf{x}_k^2$ as a fictitious control for (8) and making the error $\mathbf{z}_k^1$ to tends to zero, with the anticipation of its dynamics as follows

$$\mathbf{z}_{k+1}^1=\mathbf{f}^1(\mathbf{x}_k^1)+\mathbf{B}_1(\mathbf{x}_k^1)\mathbf{x}_k^2=\mathbf{K}_1\mathbf{z}_k^1 \tag{9}$$

where $\mathbf{K}_1=diag\{k_1,k_2\}$, with $k_1>0$ and $k_2>0$.

Then, the desired value $\mathbf{x}_k^{2^d}$ of $\mathbf{x}_k^2$ is calculated from (9) as

$$\mathbf{x}_k^{2^d}=\mathbf{B}_1^{-1}(\mathbf{x}_k^1)\left[-\mathbf{f}^1(\mathbf{x}_k^1)+\mathbf{K}_1\mathbf{z}_k^1\right].$$

It is desired that $\mathbf{x}_k^2=\mathbf{x}_k^{2^d}$. In this way, it is defined a second new error vector, $\mathbf{z}_k^2$ as

$$\mathbf{z}_k^2=\begin{bmatrix} z_k^3 \\ z_k^4 \end{bmatrix}=\mathbf{x}_k^{2^d}-\mathbf{x}_k^2.$$

The error dynamical equation is

$$\mathbf{z}_{k+1}^2=\bar{\mathbf{f}}^2-\mathbf{B}_2\mathbf{u}_k$$

where

$$\bar{\mathbf{f}}^2=\bar{\mathbf{B}}_1\left[-\bar{\mathbf{f}}^1+\mathbf{K}_1\mathbf{z}_{k+1}^1\right]-\mathbf{f}^2$$

with

$$\bar{\mathbf{B}}_1=\left[\begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix}\begin{bmatrix} -\psi_{k+1}^\beta & \psi_{k+1}^\alpha \\ \psi_{k+1}^\alpha & \psi_{k+1}^\beta \end{bmatrix}\right]^{-1}$$

and

$$\bar{\mathbf{f}}^1=\begin{bmatrix} \omega_{k+1}-\left(\frac{T}{J}\right)T_{Lk+1}-\omega_{k+2}^r \\ a^2(\psi_{k+1})+(1-a)^2M^2\left(i_{k+1}^{\alpha\ 2}+i_{k+1}^{\beta\ 2}\right)-\psi_{k+2}^r \end{bmatrix}$$

Since all the states variable are measurable at time 'kT' , the states variables at time '$(k+1)T$' are calculated from (4). It is assumed that the load signal is constant, so

$$T_{Lk+1}=T_{Lk}\,.$$

The system (6) in the new coordinates is

$$\begin{aligned}\mathbf{z}_{k+1}^1&=\mathbf{K}_1\mathbf{z}_k^1-\mathbf{B}_1\mathbf{z}_k^2 \\ \mathbf{z}_{k+1}^2&=\bar{\mathbf{f}}^2-\mathbf{B}_2\mathbf{u}_k.\end{aligned} \tag{10}$$

The next step is to design the control law from the last results. The first step in sliding mode control is to choose the surface $\mathbf{S}_k=\mathbf{0}$, and, a smart selection is

$$\mathbf{S}_k=\mathbf{z}_k^2=0\cdot$$

This surface will be zeroing as the state trajectories reach the surface, and then the control objectives will be accomplished. The transformed system (10) is redefined as

$$\begin{aligned}\mathbf{z}_{k+1}^1&=\mathbf{K}_1\mathbf{z}_k^1-\mathbf{B}_1\mathbf{z}_k^2 \\ \mathbf{S}_{k+1}&=\bar{\mathbf{f}}^2-\mathbf{B}_2\mathbf{u}_k\end{aligned} \tag{11}$$

In order to design a control law, a discrete-time sliding mode version (Utkin, *et al.*, 1999), is implemented as

$$\mathbf{u}_k=\begin{cases} \mathbf{u}_{keq} & \text{for}\ \ \|\mathbf{u}_{keq}\|\le u_0 \\ u_0\dfrac{\mathbf{u}_{keq}}{\|\mathbf{u}_{keq}\|} & \text{for}\ \ \|\mathbf{u}_{keq}\|>u_0 \end{cases}$$

where $\mathbf{u}_{keq}$ is calculated from $\mathbf{S}_{k+1}=\mathbf{0}$ of the form

$$\mathbf{u}_{keq}=\mathbf{B}_2^{-1}\left[\bar{\mathbf{f}}^2\right]$$

and u_0 is the control resources that bound the control. Proceeding with a stability analysis, where the case $\|\mathbf{u}_{keq}\|\le u_0$ is first analyzed. To reveal the structure of $\mathbf{u}_{keq}$ and $\mathbf{S}_{k+1}$, let us represent them as the following functions:

$$\mathbf{u}_{keq}=\mathbf{B}_2^{-1}\left[\bar{\mathbf{f}}^2+\mathbf{S}_k-\mathbf{x}_k^{2^d}+\mathbf{x}_k^2\right]$$

and

$$\mathbf{S}_{k+1}=\mathbf{S}_k+\bar{\mathbf{f}}^2-\mathbf{x}_k^{2^d}+\mathbf{x}_k^2-\mathbf{B}_2\mathbf{u}_k\,. \tag{12}$$

In order to decrease $\|\mathbf{S}_k\|$ monotonically to zero, it is necessary to satisfy $\mathbf{S}_{k+1} - \mathbf{S}_k \le 0$, and using the fact that control can vary within $\left\| \mathbf{u}_{keq} \right\| \le u_0$ **,** then, the condition that guarantees sliding mode stability, is calculated as

$$\left\| \mathbf{B}_2^{-1} \left(\bar{\mathbf{f}}^2 - \mathbf{x}_k^{2^d} + \mathbf{x}_k^2 \right) \right\| \le u_0 . \qquad (13)$$

Note that otherwise, the control resources are insufficient to stabilize the system. Let us turn to the case when $\left\| \mathbf{u}_{keq} \right\| \le u_0$. Replacing $\mathbf{u}_k = u_0 \dfrac{\mathbf{u}_{keq}}{\left\| \mathbf{u}_{keq} \right\|}$ in (12) yields

$$\mathbf{S}_{k+1} = \left(\mathbf{S}_k + \bar{\mathbf{f}}^2 - \mathbf{x}_k^{2^d} + \mathbf{x}_k^2 \right) \left(1 - \frac{u_0}{\left\| \mathbf{u}_{keq} \right\|} \right)$$

$$\left\| \mathbf{S}_{k+1} \right\| \le \left\| \mathbf{S}_k \right\| + \left\| \bar{\mathbf{f}}^2 - \mathbf{x}_k^{2^d} + \mathbf{x}_k^2 \right\| - \frac{u_0}{\left\| B_2^{-1} \right\|}$$

$$\left\| \mathbf{S}_{k+1} \right\| < \left\| \mathbf{S}_k \right\|$$

due to (13). Hence $\left\| \mathbf{S}_k \right\|$ decreases monotonically to zero, and, after a finite number of steps, $\left\| \mathbf{u}_k \right\| \le u_0$ is achieved, i.e.

$$\mathbf{S}_k = \mathbf{z}_k^2 = \mathbf{0} \Rightarrow \mathbf{x}_k^2 = \mathbf{x}_k^{2^d} .$$

Discrete-time sliding mode will take place from the following sampling point onwards. Under the condition (7), the transformed system (11) of order 4, reduces its order to 2, and it is modeled by

$$\mathbf{z}_{k+1}^1 = \mathbf{K}_1 \mathbf{z}_k^1 .$$

This system represents the sliding mode dynamics which achieves the control objectives.

It is an obvious fact that the proposed control $\mathbf{u}_k$ depends on $\bar{\mathbf{f}}^2$ in order to eliminate old dynamics, but this function depends of control $\mathbf{u}_k$ squared, due to term ${i_{k+1}^{\alpha}}^2 + {i_{k+1}^{\beta}}^2$, that appears in $\bar{\mathbf{f}}^1$, making the system in that way, unsolvable. To overcome this problem it is designed an observer only with current measurements, for the new variable Im_k, defined as follows

$$\mathrm{Im}_k = \sqrt{{i_k^{\alpha}}^2 + {i_k^{\beta}}^2} .$$

It is assumed that Im_k is constant, i.e.

$$\mathrm{Im}_{k+1} = \mathrm{Im}_k .$$

Then the observer is presented as the original plant plus a tracking error

$$\hat{\mathrm{Im}}_{k+1} = \hat{\mathrm{Im}}_k + g e_k^I$$

where $e_k^I = \mathrm{Im}_k - \hat{\mathrm{Im}}_k$ is the tracking error. Taking one step ahead

$$e_{k+1}^I = (1-g) e_k^I$$

it is easy to see that with the following condition:

$$2 > g > 0$$

the observer error will tends asymptotically to zero, and the estimation $\hat{\mathrm{Im}}_k$ will track the real value Im_k. Avoiding the control dependency of $\mathbf{u}_k$ squared. Fig. 2 shows a simulation of the observer.

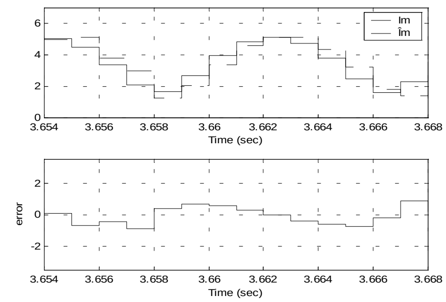


Fig. 2. (1) Comparison of Im_k with $\hat{\mathrm{Im}}_k$.
(2) The tracking error.

Again, there is an error that can be eliminated by the control action.

3.2 Reduced order nonlinear observer

The last control algorithm works with the full state and parameters measurement assumption. But in reality, the rotor fluxes and torque measurement is a difficult task. Here, it is design a reduced order nonlinear observer for fluxes and load, with the rotor speed and currents measurements only. System (4) is written as

$$\begin{aligned} \omega_{k+1} &= \omega_k + \mathbf{I}_k \Im \boldsymbol{\psi}_k - (T/J) T_{Lk} \\ \mathbf{I}_{k+1} &= \varphi_k + (T/\sigma) \mathbf{u}_k \qquad (14) \\ \boldsymbol{\psi}_{k+1} &= a \mathbf{G}_k \boldsymbol{\Psi}_k + (1-a) M \mathbf{G}_k \mathbf{I}_k \end{aligned}$$

where $\mathbf{G}_k$ is defined as

$$\mathbf{G}_k = \begin{bmatrix} \cos(npT\omega_k) & -\sin(npT\omega_k) \\ \sin(npT\omega_k) & \cos(npT\omega_k) \end{bmatrix}.$$

The proposed observer for the system (14), assumes the speed and current measurements, and an unknown constant load

$$\begin{aligned} \hat{\omega}_{k+1} &= \omega_k + \mathbf{I} \Im \hat{\boldsymbol{\psi}}_k - \left(\frac{T}{J} \right) \hat{T}_{Lk} + l_1 (\omega_k - \hat{\omega}_k) \\ \hat{T}_{Lk+1} &= \hat{T}_{Lk} + l_2 (\omega_k - \hat{\omega}_k) \qquad (15) \\ \hat{\boldsymbol{\psi}}_{k+1} &= a \mathbf{G}_k \hat{\boldsymbol{\Psi}}_k + (1-a) M \mathbf{G}_k \mathbf{I}_k . \end{aligned}$$

Let e_k^{ω}, be the difference between the measured rotor speed and the estimated one, i.e.

$$e_k^{\omega} = \omega_k - \hat{\omega}_k .$$

Then the following error definition is e_k^L, and represents the difference between the real and the estimated load

$$e_k^L = T_{Lk} - \hat{T}_{Lk}$$

and the difference between real flux vector and the estimated one is as follows

$$\mathbf{e}_k^{\psi} = \boldsymbol{\psi}_k - \hat{\boldsymbol{\psi}}_k .$$

Taking one step ahead of the three error equations, it yields to the dynamical error equations

$$\begin{bmatrix} e_{k+1}^{\omega} \\ e_{k+1}^{L} \end{bmatrix} = \begin{bmatrix} -l_1 & -\dfrac{T}{J} \\ -l_2 & 1 \end{bmatrix} \begin{bmatrix} e_k^{\omega} \\ e_k^{L} \end{bmatrix} + \mathbf{i}_k \Im \mathbf{e}_k^{\psi} \qquad (16)$$

$$\mathbf{e}_{k+1}^{\psi} = a \mathbf{G}_k \mathbf{e}_k^{\psi}$$

A Lyapunov function can be used to proof stability of $\mathbf{e}_k^{\psi}$

$$\mathbf{V}_k = \mathbf{e}_k^{\Psi^T} \mathbf{e}_k^{\Psi}.$$

Taking one step ahead of the Lyapunov function

$$\mathbf{V}_{k+1} = \mathbf{e}_k^{\Psi^T} a^2 \mathbf{G}_k^T \mathbf{G}_k \mathbf{e}_k^{\Psi}.$$

The increment of the Lyapunov function should be negative, and is expressed as

$$\Delta \mathbf{V}_k = \mathbf{e}_k^{\Psi^T} \left(a^2 \mathbf{G}_k^T \mathbf{G}_k - \mathbf{I}_{2x2} \right) \mathbf{e}_k^{\Psi} < 0$$

where

$$\left(a^2 \mathbf{G}_k^T \mathbf{G}_k - \mathbf{I}_{2x2} \right) < 0$$

or

$$a^2 \mathbf{G}_k^T \mathbf{G}_k < \mathbf{I}_{2x2} \,.$$

With some basic manipulations yields

$$\begin{bmatrix} a^2 & 0 \\ 0 & a^2 \end{bmatrix} < \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow a < 1 \qquad (17)$$

where $a = e^{-\alpha T}$. The condition (17) is satisfied due to the fact that T and α are always positive. So, the increment of the Lyapunov function is negative implying that the tracking error tends asymptotically to zero, i.e.

$$\lim_{k \to \infty} \hat{\mathbf{\Psi}}_k = \mathbf{\Psi}_k$$

Since $\mathbf{i}_k^T \Im$ is bounded, (16) is reduced to

$$\begin{bmatrix} e_{k+1}^{\omega} \\ e_{k+1}^{L} \end{bmatrix} = \begin{bmatrix} -l_1 & -\frac{T}{J} \\ -l_2 & 1 \end{bmatrix} \begin{bmatrix} e_k^{\omega} \\ e_k^{L} \end{bmatrix}. \qquad (18)$$

Finding suitable l_1 and l_2 constants, the system (18) will be asymptotically stable and the observer (15) will asymptotically track the plant. A well known Jury's stability test (Åström and Wittenmark; 1997) criterion for a second order system will help to find l_1 and l_2. The characteristic equation of (18) is

$$z^2 + (l_1 - 1)z + (-l_1 - \frac{T}{J} l_2) = 0 \,. \qquad (19)$$

Comparing (19) with an algebraic second order equation, yields

$$z^2 + a_1 z + a_2 = 0$$
$$a_1 = (l_1 - 1)$$
$$a_2 = (-l_1 - \frac{T}{J} l_2).$$

The Jury's stability test establishes for a second order system the following conditions

$$a_1 < 1$$
$$a_2 > -1 + a_1$$
$$a_2 > -1 - a_1$$

and with some computations the conditions that make the observer a stable system, are

$$1 < l_1 < 2$$
$$l_2 < 0$$

4. CONTROL LAW SIMULATIONS

Simulations are carried out to demonstrate the effectiveness of the above discrete-time sliding mode control and observers. The worst case scenario is simulated, i.e., the flux magnitude tracks an exponential signal and the speed tracks a sinusoidal shape signal. The unknown load torque is proposed as a noisy square shape signal that goes from minus nominal torque to positive nominal torque. Table 1 shows the induction motor parameters and Table 2 shows the control law parameters.

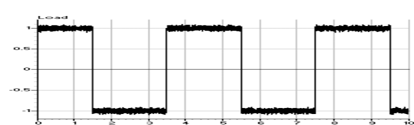
Table 1. Parameters of the induction motor. It is considered a three-phase, two-pole machine, with a stator-referred rotor.

Parameter	Value	Description
Rs	14 ohms	Stator Resistance
Ls	400 Mh	Stator Inductance
M	377 Mh	Mutual Inductance
Rr	10.1 ohms	Rotor Resistance
Lr	412.8 mH	Rotor Inductance
n_p	2	Number of Pole Pairs
J	0.01 Kgm^2	Moment of Inertia
ω_n	168.5 rad/sec	Nominal speed
T_{Ln}	1.1 Nm	Nominal Load

Table 2. Parameters used in the control law and the observer.

Parameter	Value	Description
T	0.001 sec	Sampling Period
u_o	330 Volts	Voltage bound
k_1	0.9	Control law gain
k_2	0.9	Control law gain
l_1	0.5	Observer gain
l_2	-0.5	Observer gain
G	1.9	Observer gain
$\hat{\psi}_k^{\alpha}(0)$	0.001 wb	Initial condition
$\hat{\psi}_k^{\beta}(0)$	0.001 wb	Initial condition
$z_k^3(0)$	-0.5	Initial condition
$z_k^4(0)$	0.5	Initial condition

The flux amplitude tracks an exponential signal at $0.2wb^2$. The rotor velocity tracks a sinusoidal signal with peak value of 70 *volts* and frequency of 3 *rad/sec*. The load torque is considered as a noisy square shape signal. Fig. 3 shows this load signal.



Time (sec)

Fig. 3. Square shape load. The load torque goes from –1.1Nm to 1.1Nm.

Fig. 4 illustrates the speed output signal and its references, and Fig. 5 shows the tracking error.

Time (sec)

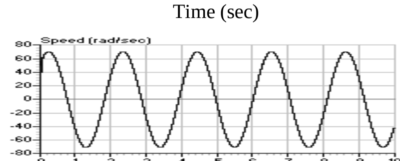
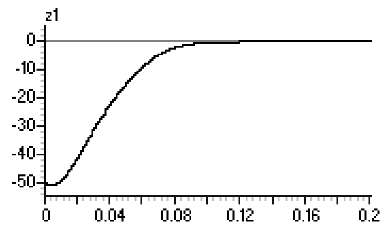


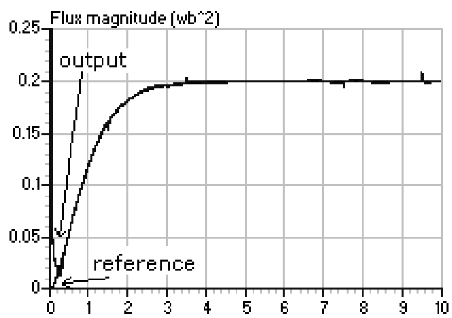
Fig. 4. Speed output signal and its reference. Note that the output exactly tracks its reference.



Time (sec)

Fig. 5. Tracking error. Note that the error tends asymptotically to zero.

Fig. 6 shows the flux amplitude output and its reference signal as well. And Fig. 7 shows the tracking error signal.



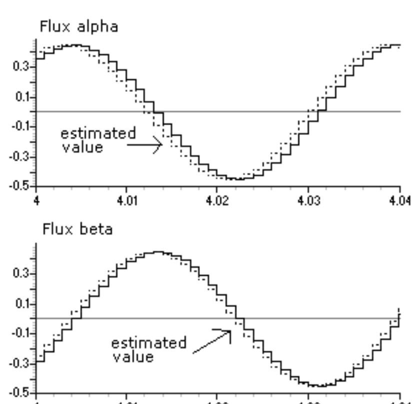
Time (sec)

Fig. 6. Flux amplitude output signal and its reference. Note that the output tracks its reference with a slight amount of error .

Time (sec)

Fig. 7. Tracking error. Note that the error oscillates around zero.

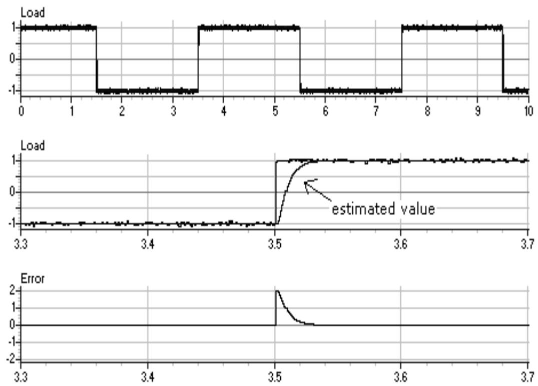
Fig. 8 shows the flux observer results



Time (sec)

Fig. 8. Flux observation graphs. Note that the amplitude is well tracked, but, the phase angle differs a little bit.

Fig. 9 illustrates the load observation results. Despite that the observer models the load as constant load , it tracks so fine a square shape signal.



Time (sec)

Fig. 9. Observed load and tracking error. Note when the load change its value, the observer response is fast.

5. CONCLUSIONS

The contributions of this paper can be stated as follows. The combination of sliding mode and block control results in a control law that achieves an excellent performance in the worst case scenario. With the flux observer it was demonstrated that its dynamics are stable. The load torque observer performs well.

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