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SWING-UP CONTROL OF INVERTED PENDULUM BY PERIODIC INPUT

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Abstract: In this paper, the swing-up control of a 1-link pendulum with the sinusoidally excited pivot is studied from a viewpoint of energy. The amplitude of the periodic signal is used as the control input, and is manipulated so that the energy of the pendulum is equal to the potential energy at the up-right position. Numerical simulation confirms the validity of the proposed method.

Keywords: Sinusoidal signals, Energy control, Global stability, Nonlinear models, Inversion, Lyapunov function

1. INTRODUCTION

The control of a pendulum has been one of fundamental problems in control field. As a control strategy to stabilize at the up-right position, it is well known that a linear quadratic technique is effective. Various nonlinear control methods have been proposed aiming not only to stabilize but also to swing up the pendulum. Åström and Furuta (2000) realized both swing up and stabilization from the viewpoint of energy. The pendulum is controlled so that the total energy is equal to that of the up-right position (Åström, 1999).

Contrary to controlling the pendulum for stabilization, it has been studied that the pendulum is kept up-right position by adding sinusoidal signal at the pivot in the feedforward way (Phelps and Hunter, 1964). This has the long history since the beginning of 20th century. Using such sinusoidal signal, the swing-up of the pendulum has not been studied until Michituji et al. (2000), who proposed the use of chaotic phenomena. In their study, two kinds of amplitudes are used for the periodic excitation and one of them is chosen depending on state variables.

This study proposes a method by combining energy control and periodic excitation. Swing-up control is hardly possible when the amplitude of the sinusoidal input is fixed. In this study, the swing-up is done by controlling the amplitude of the sinusoidal signal depending on the state. The proposed method achieves the global stabilization of a pendulum at the up-right position.

2. MODELING OF A 1-LINK PENDULUM

The paper considers the 1-link pendulum model illustrated in Fig. 1. In the figure, x and y are given in the moving frame that the pivot is taken as the origin.

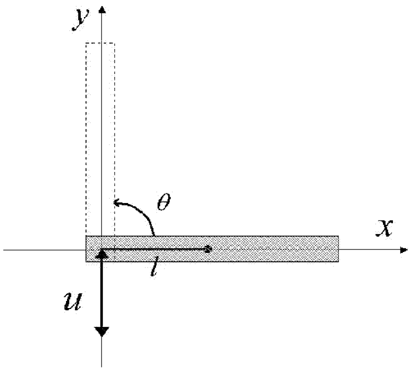

Fig. 1. The pendulum model. x and y are given in the moving frame that the pivot is taken as the origin. x and y denote the positions of center of gravity in the horizontal and the vertical axes, respectively. u is the periodic displacement at the pivot of the pendulum in the vertical direction.

$$x = l\cos\theta, \tag{1}$$

$$y = l\sin\theta, \tag{2}$$

where x and y denote the positions of center of gravity in the horizontal and the vertical axes, respectively. u is the periodic displacement at the pivot of the pendulum in the vertical direction,

$$u = r\sin(2\pi f t). \tag{3}$$

The angle θ is measured along counter-clockwise. The angle $\theta = \frac{\pi}{2}$ $[rad]$ is the up-right position (unstable equilibrium point), $\theta = 0$ is the horizontal position, and $\theta = -\frac{\pi}{2}$ $[rad]$ is the pendant position (stable equilibrium point). The parameters used in this paper are shown below.

Table 1. The Parameters

$m[kg]$	mass
$l[m]$	length from pivot to center of gravity
θ[rad]	angle of pendulum
$f[Hz]$	input frequency
$r[m]$	input amplitude
$g[m/s^2]$	acceleration of gravity
$C[kgm^2/s]$	viscose friction coefficient
$I[kgm^2]$	moment of inertia around center of gravity

Associated energies considered in the moving frame with the pivot as the origin are described as

$$\begin{aligned} &\text{Rotation energy} &&: \mathcal{T}_{\mathcal{R}} = \frac{1}{2}I\dot{\theta}^2, \\ &\text{Translation energy} &&: \mathcal{T}_{\mathcal{T}} = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2), \\ &\text{Potential energy} &&: \mathcal{V} = m(g + \ddot{u})y, \\ &\text{Dissipation energy} &&: \mathcal{R} = \frac{1}{2}C\dot{\theta}^2. \end{aligned} \tag{4}$$

Substituting of the energies eq(4) into the Lagrange equation

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} - \frac{\partial \mathcal{L}}{\partial \theta} + \frac{\partial \mathcal{R}}{\partial \dot{\theta}} = 0, \tag{5}$$

the equations of the motion are given by

$$\begin{bmatrix} I + ml^2 & ml\cos\theta \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{u} \end{bmatrix} + \begin{bmatrix} C & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{u} \end{bmatrix} + \begin{bmatrix} mgl\cos\theta \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -r(2\pi f)^2\sin(2\pi f t) \end{bmatrix}, \tag{6}$$

where $\mathcal{L}$ denotes the Lagrangian $\mathcal{L} = (\mathcal{T}_{\mathcal{R}} + \mathcal{T}_{\mathcal{T}}) - \mathcal{V}$. The parameters $m = 0.1,\ l = 0.1,\ g = 9.8$, $C = 0.36 \times 10^{-5}$ and $I = ml^2/3$ are fixed in our study, and $f = f_0 \times 3 = 25.7$ is the frequency of the excitation of the pivot. $f_0(= 8.57)$ is the natural frequency of the pendulum system.

3. STABILIZATION BY PERIODIC INPUT WITH CONSTANT AMPLITUDE

First, the behavior of the periodically driven pendulum is analyzed for the amplitude of the periodic input, which drives the pivot of the pendulum in the vertical direction. Simulation results are shown from Fig.2 with respect to θ and $\dot{\theta}$. In the figure, the vertical and horizontal axes correspond to angular velocity $\dot{\theta}$ and angle θ, respectively. Both ends of the horizontal axis correspond to the position $\theta = -\frac{1}{2}\pi, \frac{3}{2}\pi$ $[rad]$, and the center of the horizontal axis corresponds to the up-right position $\frac{1}{2}\pi$ $[rad]$. In Fig.2, the simulation are done for f and r, namely $f = 25.7$ $[Hz]$ and $r = 0.011$ $[m]$. $dt = 1/(10 \times 2\pi f)[sec]$ is used for the sampling interval in the simulation.

In Fig.2, the region enclosed by a circle means the situation where the pendulum is stabilized in the neighborhood of the up-right position. Fig.2 shows that there is an attractor where the pendulum

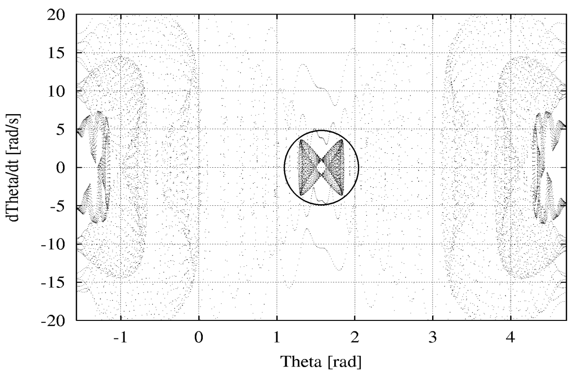

Fig. 2. The simulation result with $r = 0.011[m]$ and $f = 25.7[Hz]$. The vertical and horizontal axes denote angular velocity $\dot{\theta}$ and angle θ, respectively. Both ends of the horizontal axis correspond to the position $\theta = -\frac{1}{2}\pi, \frac{3}{2}\pi$ $[rad]$, and the center of the horizontal axis corresponds to the up-right position $\frac{1}{2}\pi$ $[rad]$.

is staying in the neighborhood of the up-right position. In this simulation, the stabilization is achieved if the initial state is carefully selected. On the other hands, outside the circle in Fig.2, there are cases that some trajectories are going around the pendant equilibrium position. If the amplitude r is properly chosen, the pendulum can be stabilized in the situation that its initial position is set around the up-right position.

3.1 *Stability conditions by Mathieu equation*

In order to find stabilizing parameters (f, r, θ), the Mathieu equation is known to provide sufficient conditions for systems to be stabilized at the up-right position. The equations of the motion for the 1-link pendulum can be written from eq(6);

$$\ddot{\theta} + \frac{ml(g+\ddot{u})\cos\theta}{I+ml^2} + C\dot{\theta} = 0. \tag{7}$$

Let $C = 0$ and $X = \theta - \frac{\pi}{2}$, eq(7) is written as

$$\ddot{X} + \frac{ml(g+\ddot{u})\cos(X+\frac{\pi}{2})}{I+ml^2} = 0,$$

$$\ddot{X} - \frac{ml(g+\ddot{u})\sin X}{I+ml^2} = 0. \tag{8}$$

In the neighborhood of the up-right position $X = 0$, eq(8) is written as

$$\ddot{X} - \frac{ml(g+\ddot{u})}{I+ml^2}X = 0. \tag{9}$$

By exciting the sinusoidal input

$$u = r\sin(\omega t) = -r\cos(\omega t + \frac{\pi}{2}) \tag{10}$$

where

$$\omega = 2\pi f. \tag{11}$$

Eq(9) yields

$$\ddot{X} - \frac{ml(g + r\omega^2\cos(\omega t + \frac{\pi}{2}))}{I+ml^2}X = 0,$$

$$\ddot{X} - \omega_n^2\left\{1 + \frac{r\omega^2}{g}\cos\left(\omega t + \frac{\pi}{2}\right)\right\}X = 0, \tag{12}$$

where

$$\omega_n^2 = \frac{mgl}{I+ml^2}. \tag{13}$$

Eq(12) is called the Mathieu equation. The Mathieu equation is conventionally written as

$$d^2X/dT^2 + (\alpha + \beta\cos T)X = 0, \tag{14}$$

where T denotes the generalized time defined by $T = \omega t + \pi/2$. Comparing eq(12) and eq(14), one obtains

$$\alpha = -\omega_n^2/\omega^2 \tag{15}$$

and

$$\beta = -\omega_n^2 r/g. \tag{16}$$

So α should be negative. The stability boundary of the Mathieu equation is given by Acheson (1993),

$$|\alpha| > \left|\frac{1}{2}\beta^2\right|.$$

Moreover when $|\alpha|$ is small, Acheson (1993) and Blackburn, et al. (1992), give the upper boundary as $|\beta| = 0.450$:

$$(2\omega_n^2/\omega^2)^{\frac{1}{2}} < \omega_n^2 r/g < 0.450, \tag{17}$$

$$\sqrt{2}g/\omega_n\omega < r < 0.450g/\omega_n^2. \tag{18}$$

f and r should be chosen so that the pendulum is staying in the neighborhood of the up-right position, i.e., the above equations are satisfied. In eq(18), for vibration with $f = 25.7[Hz]$, the amplitude r should be inside a region as follows, regardless of the mass,

$$0.01 < r < 0.06 . \tag{19}$$

Since the amplitudes are chosen as $r = 0.011$ $[m]$ (Fig.2), the figure shows that the pendulum has a stable attractor near the up-right position. So the stability criterion of it given by the Mathieu equation (12) is found to give the satisfied result.

4. SWING-UP AND STABILIZATION OF PENDULUM BY USING AMPLITUDE CONTROL

Stabilization of the pendulum in the neighborhood of the up-right position has been discussed in the previous section with the constant amplitude excitation of the pivot, where the initial condition is carefully chosen. The constant amplitude of the periodic input for the stabilization should be chosen to satisfy (19). It is emphasized that the swing-up control is impossible with constant amplitude excitation. Our control strategy is to operate the amplitude of the periodic input such a way that the energy of the pendulum is equal to mgl, which is the potential energy at the stabilized position. The designing procedure is based on the energy of the controller. The control action is derived in the followings.

For $r = 0$, the energy of the pendulum is

$$E = \frac{1}{2}(I+ml^2)\dot{\theta}^2 + mgl\sin\theta. \tag{20}$$

It is assumed that there is no friction. The time derivative of the energy E is calculated as

$$\frac{dE}{dt} = (I + ml^2)\dot{\theta}\ddot{\theta} + mgl\dot{\theta}\cos\theta \qquad (21)$$

Substituting eq(6) into eq(21) yields

$$\begin{aligned}\frac{dE}{dt} &= -m(g + \ddot{u})l\dot{\theta}\cos\theta + mgl\dot{\theta}\cos\theta \\ &= -m\ddot{u}\dot{\theta}l\cos\theta. \qquad (22)\end{aligned}$$

It can be seen from eq(22) that the derivative of E can be controlled by $\ddot{u}$.

Let consider a candidate of Lyapunov function as

$$V = \frac{1}{2}(E - E_0)^2,$$

where $E_0(= mgl)$ denotes the energy when the pendulum is at the up-right position. The time derivative of V is calculated as follows:

$$\begin{aligned}\frac{dV}{dt} &= (E - E_0)\frac{dE}{dt} \\ &= -(E - E_0)m\ddot{u}l\dot{\theta}\cos\theta \qquad (23)\end{aligned}$$

If the authors control the amplitude K of the sinusoidal input as follows, then the derivative of V yields negative, so it will vanish as $t \to \infty$.

$$\begin{aligned}\ddot{u} &= K(2\pi f)^2 \sin(2\pi f t) \qquad (24) \\ K &= \mathrm{sgn}((E - E_0)\sin(2\pi f t)\dot{\theta}\cos\theta)r,\end{aligned}$$

where $\mathrm{sgn}(x) = 1\ (x \geq 0)$, $\mathrm{sgn}(x) = -1\ (x < 0)$. The control method (24) leads to $dV/dt < 0$ $(V \to 0)$. Hence $E \to E_0$, the pendulum can be swung up.

4.1 *Motion of the pendulum using amplitude control*

Table 2. Simulation parameters

$f[Hz]$	$r[m]$	initial condition $\theta[rad]$	$dt[sec]$
25.7	0.1	$-\pi/2$	0.00389

Our method shall be evaluated by extensive numerical simulation. Simulation parameters are listed in Table 2. $dt = 1/(10f)$ denotes the sampling interval in the simulation. Fig.3 shows the simulation result. This figure shows the angle.

Fig.3 shows that the angle is converged $\theta \to -\frac{3}{2}\pi(= \frac{1}{2}\pi)[rad]$ after 6 [sec]. Using this control method, the amplitude by eq(24) is not satisfying eq(19).

Numerical simulation confirms that this control method can swing up and stabilize the pendulum.

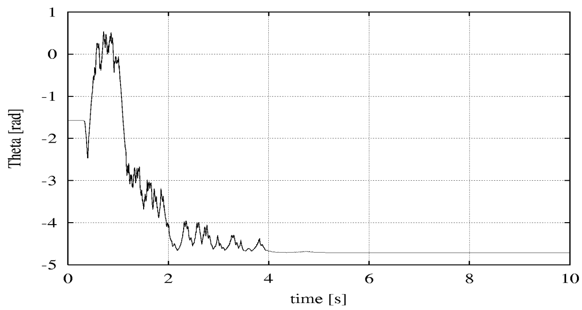

Fig. 3. The simulation result with the condition in Table 2. This figure shows that the angle is converged $\theta \to -\frac{3}{2}\pi(= \frac{1}{2}\pi)[rad]$ after 6 [sec].

5. CONCLUSION

It is well known that a 1-link pendulum can be kept around the up-right position by a periodic excitation of the pivot, but its initial state needs to set around the up-right position.

This report presents that the pendulum can swing up from the pendant position to the up-right position by controlling the amplitude of the periodic excitation so that the derivative of the criterion, which is given by the squared difference of energy of the pendulum from that of the up-right position, is negative.

In the future, the authors should consider the swing-up of the 2-link pendulum and apply to the real systems. They are currently under experiment.

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