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CHECKING REDUNDANCIES IN SUPERVISORY CONTROL. A COMPLEXITY RESULT. [1]

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Abstract: The iterative nature of state feedback control appoaches based on the addition of control places (in Petri net models), can give rise to redundancies in the added control. These redundancies are characterised as implicit places. In this paper we prove that deciding the implicitness of a place added to a live marked graph is of polynomial complexity.

Keywords: Petri nets, supervisory control, linear programming.

1. INTRODUCTION

Petri net models have been extensively used in the synthesis of controllers for discrete event systems arising from the application domain of flexible manufacturing systems. The specification of the controller, in the case of the *state feedback control* approach (Holloway *et al.*, 1997), can be given as a set of forbidden states. The goal of the control is that of constraining the system behaviour so that a set of forbidden states cannot be reached. The corresponding control policy is called *state feedback*. An important class of these state specifications is related to the liveness enforcing of a system or the avoidance of (total or partial) deadlock states. The controller specification for deadlock avoidance is the set of deadlock states.

One of the fundamental problems in the computation of the control is the characterization of the deadlock states, because the synthesis of the controller is strongly dependent on it. For example, some approaches (Ezpeleta *et al.*, 1995; Xie and Jeng, 1999) consider a specification of deadlock states based on the emptyness of the siphons of the net system. That is, an empty siphon represents a region of the reachability set with the property that all places belonging to the siphon are empty. This means that the output transitions of an empty siphon are dead forever. In other cases (Banaszak and Krogh, 1990), deadlock states are specified by other particular substructures. The common factor to all these approaches (even no related to the deadlock avoindance control policy) is that the specification of the forbidden states can be expressed by means of linear inequalities, which set of integer solutions are the states to forbid in the behaviour of the net system. The controller to be sinthesized from each linear inequality is a place that can be obtained as a linear combination of a set of places derived from the places whose marking variables appear in the inequality, i.e. they are *structurally implicit places*. Therefore, the non-negativity of the marking of the control place cuts all forbidden states, or, in other words, the region composed by the markings satisfying the specification of the forbidden states is characterised by a negative marking of the control place. This approach has been generalised in (Park and Reveliotis, 2000) defining the *algebraic livenes enforcing supervisors*.

This approach has been shown as a fruitful way to obtain the controllers, but in general it requires an iterative method, because new deadlock states can appear in the system.

The iterative nature of the method means also that after an iteration we can add a set of control places that can become redundant with respect to the set of control places added in other iterations. Therefore, the detection of these redundancies is interesting in order to compute a controller as simple as possible. The

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concept that captures the idea of redundant control place is the concept of implicit place.

In this paper we investigate the complexity of testing if a place is implicit. This test requires the computation of the minimum initial marking making implicit a structural implicit place. After this computation, if the initial marking of the control place is greater than or equal to the computed marking making it implicit, then the control place is redundant and it can be removed to obtain a simpler controller.

The paper is organised as follows. In sections 2 and 3 the basic definitions on implicit places are given. The Minimum Initial Marking Problem (MIMP) is presented in Section 4, as well as its complexity for the case of free choice net systems. Section 5 presents new results concerning the complexity of the MIMP for marked graphs and proving that in this case the complexity is polynomial. Because of the lack of space, no basic definitions and notations about Petri nets are given. The used notation can be found in the related paper (García-Vallés and Colom, 1999).

2. IMPLICIT PLACES

An implicit place is a place whose removal does not change the behaviour of the net system. Two notions of behaviour equivalence are used to define implicit places. The first one considers that two net systems have the same behaviour if they present the same occurrence sequences of transitions. That is, these places can be removed without changing the *sequential observation* of the behaviour of the net system. Implicit places under this equivalence notion are called *sequential implicit places* (SIP). The second notion of equivalence imposes that the two net systems must have the same occurrence sequences of steps. In this case, implicit places are called *concurrent implicit places* (CIP) and its removal does not change the possibilities of simultaneous occurrences of transitions in the original net system.

Definition 1. (Colom, 1989) Let $\mathscr{S} = \langle \mathscr{N}, \mathbf{m_0} \rangle$ be a net system and $\mathscr{S}' = \langle \mathscr{N}', \mathbf{m_0}' \rangle$ the net system $\mathscr{S}$ with an additional place p. p is a *Sequential [Concurrent] Implicit Place* (SIP) [CIP] iff $\mathrm{L}(\mathscr{S}) = \mathrm{L}(\mathscr{S}')$ $[\mathrm{LS}(\mathscr{S}) = \mathrm{LS}(\mathscr{S}')]$, i.e., the addition of p preserves all occurrence sequences [of steps] of $\mathscr{S}$.

Note that if p is a SIP (CIP), $\mathrm{RS}(\mathscr{S}) = \mathrm{RS}(\mathscr{S}'|_P)$, because the occurrence sequences are preserved, and therefore, if $\mathbf{m}$ is reachable in $\mathscr{S}$ by the occurrence sequence σ, a marking $\mathbf{m}'$ such that $\mathbf{m}'[q] = \mathbf{m}[q], \forall q \neq p$, and $\mathbf{m}'[p] = \mathbf{m_0}'[p] + \mathbf{C}'[p,T] \cdot \sigma$ will be reachable in $\mathscr{S}'$.

In the rest of the paper, whenever appropiated, we will use primed variables to denote objects of $\mathscr{S}'$ (the net system with the added place), and non-primed variables for objects of $\mathscr{S}$ (the “original” net system).

The relation between sequential and concurrent implicit places is shown in the following result.

Corollary 2. (García-Vallés and Colom, 1999) 1) If p is a CIP, then p is a SIP. 2) If p is self-loop free, then p is a CIP if and only if p is a SIP.

The removal of tokens from an implicit place can make it non-implicit. The following result guarantees that an implicit place remains implicit if its initial marking is increased. Therefore, given an implicit place, there exists a minimum initial marking for this place that makes it implicit.

Theorem 3. (García-Vallés and Colom, 1999) Let $\mathscr{S} = \langle \mathscr{N}, \mathbf{m_0} \rangle$ be a net system and p a SIP (CIP) in $\mathscr{S}$. The place p is a SIP (CIP) in all net systems $\mathscr{S}^+ = \langle \mathscr{N}, \mathbf{m_0}^+ \rangle$, such that $\mathbf{m_0}^+[p] \geq \mathbf{m_0}[p]$ and $\mathbf{m_0}^+[q] = \mathbf{m_0}[q]$ for all $q \neq p$. In other words, the SIP (CIP) p remains implicit if its initial marking is increased.

Finally, we define the implicitness property in terms of reachable markings instead of sequences. It means that a SIP (CIP) is never the only place that avoids the occurrence of (steps concerning) its output transitions. This characterisation is very useful because it can be manipulated using algebraic techniques.

Proposition 4. (García-Vallés and Colom, 1999) 1) p is a SIP of $\mathscr{S}'$ if and only if for all $\mathbf{m}' \in \mathrm{RS}(\mathscr{N}', \mathbf{m_0}')$, and for all $t \in p^\bullet$, if $\mathbf{m}'|_P \geq \mathbf{Pre}[P,t]$, then $\mathbf{m}'[p] \geq \mathbf{Pre}'[p,t]$. 2) p is a CIP of $\mathscr{S}'$ if and only if for all $\mathbf{m}' \in \mathrm{RS}(\mathscr{N}', \mathbf{m_0}')$, and for all $s_{p^\bullet} \in \mathbb{Z}_+^{|p^\bullet|}$, if $\mathbf{m}'|_P \geq \mathbf{Pre}[P, p^\bullet] \cdot s_{p^\bullet}$, then $\mathbf{m}'[p] \geq \mathbf{Pre}'[p, p^\bullet] \cdot s_{p^\bullet}$

3. STRUCTURALLY IMPLICIT PLACES

The implicitness of a place depends, in general, on the initial marking of the net system. If the initial marking of the net without the place changes, in some cases is always possible make the place implicit again by changing its initial marking acordingly. Places that fulfil this property are called *structurally implicit places*. The word “structurally” highlights the fact that this property depends exclusively on the net structure. For structurally implicit places, their implicitness is determined by its initial marking only.

Definition 5. (Colom, 1989) A place p is a *sequential structurally implicit place (SSIP)* (*concurrent structurally implicit place (CSIP)*) of $\mathscr{N}'$ if and only if for each net system $\langle \mathscr{N}, \mathbf{m_0} \rangle$, there exists $a \in \mathbb{N}$ such that p is a SIP (CIP) in $\langle \mathscr{N}', \mathbf{m_0}' \rangle$, with $\mathbf{m_0}'[q] = \mathbf{m_0}[q] \, \forall q \neq p$, and $\mathbf{m_0}'[p] = a$.

According to the definition, a structurally implicit place can become implicit for any initial marking of the rest of places, if we have the freedom to select an adequate initial marking for it. However it should be noted that implicitness property does not implies its structural counterpart. That is, there can be implicit places that are not structurally implicit. Just like in the case of implicit places, some relations between SSIPs and CSIPs can be established.

Theorem 6. (García-Vallés and Colom, 1999) 1) If p is a self-loop free place, then p is a CSIP if and only if p is a SSIP. 2) p is a CSIP if and only if p is SSIP and

$$\begin{aligned} &\exists \mathbf{y}, k \text{ such that:} \\ &\quad \mathbf{y}^T \cdot \mathbf{C} \leq k \cdot \mathbf{C}'[p,T] \\ &\quad \mathbf{y}^T \cdot \mathbf{Pre} \geq \mathbf{Pre}'[p,T] \\ &\quad k \leq 1 \\ &\quad \mathbf{y} \geq 0 \end{aligned} \tag{1}$$

SSIPs can be efficiently characterised according to the following results. Obviously, the given structural conditions can be checked in polynomial time.

Theorem 7. (Colom, 1989) A place p is a SSIP if and only if there exists $\mathbf{y} \geq 0$, such that $\mathbf{y}^T \cdot \mathbf{C} \leq \mathbf{C}'[p,T]$.

Corollary 8. Let $\mathscr{N}$ be a structurally bounded net. p is a SSIP of $\mathscr{N}'$ if and only if there exists $\mathbf{y}$ such that $\mathbf{y}^T \cdot \mathbf{C} \leq \mathbf{C}'[p,T]$.

The algebraic characterization makes possible to prove that the equivalence between SSIPs and CSIPs can be extended to places with self loops, if the net without the place is structurally bounded, as the next corollary proves. Because live marked graphs are structurally bounded, only SSIPs have to be considered.

Corollary 9. Let $\mathscr{N}$ be a struturally bounded net. p is a CSIP of $\mathscr{N}'$ if and only if p is a SSIP of $\mathscr{N}'$.

Structural implicitness is not, in general, a necessary condition for implicitness. However, as the following result establishes, it turns to be necessary when the net without the place has some additional properties.

Theorem 10. (García-Vallés and Colom, 1999) Let p be a SIP of $\mathscr{S}'$. If $\mathscr{S}$ is structurally bounded and each minimal t-semiflow of $\mathscr{S}$ can occur in isolation from some reachable marking, then p is also a SSIP.

4. THE MINIMUM INITIAL MARKING PROBLEM (MIMP)

There exist several well-known subclasses of net systems, as for example live and safe free choice nets, and of course live marked graphs, fulfilling the conditions of Theorem 10, and therefore SSIPness is a necessary condition for SIPness for them. Moreover, SIPness is also a necessary condition for CIPness (Corollary 2). Taking into account the definition of structurally implicit place, in such cases the implicitness problem can be decomposed in two subproblems: 1) Determining if the place is a SSIP; 2) Determinig if the initial marking of the place under study is enough to make it implicit. Subproblem 1 is easily decided in polynomial time because it only requires to find a solution of a LPP (Theorem 7 or Corollary 8). Subproblem 2 is harder, and it will be drived in the rest of the paper. Note also that because the increasing of the initial marking of an implicit place does not affect its implicitness (Theorem 3), subproblem 2 can be enunciated as a minimum initial marking problem (making the place implicit). This problem is formally stated as follows:

Minimum Initial Marking (of a SSIP)
Given: A net system $\mathscr{S}'$, a SSIP p of $\mathscr{S}'$, and an initial marking $\mathbf{m_0}[p]$ such that p is a SIP in $\mathscr{S}'$.
To decide: is $\mathbf{m_0}'[p]$ the minimum initial marking that makes p a SIP (or CIP) in $\mathscr{S}$'?

In a previous work (García-Vallés and Colom, 1999) this problem was proved NP-Complete (for SIPs) even when the net system without p is a live and safe free-choice net system, a very simple and well-behaved subclass. In the following section, and for the case of a structurally implicit place added to a live marked graph, we characterise the MIMP by means of a linear programming problem, and thus the implicitness of the place can be decided in polynomial time.

5. THE MIMP IN LIVE MARKED GRAPHS

In this section the subclass of live marked graphs is addressed. Recall that the place under study is added to live marked graph $\mathscr{S} = \langle \mathscr{N}, \mathbf{m_0} \rangle$, and possibly the net with the added place, $\mathscr{S}' = \langle \mathscr{N}', \mathbf{m_0}' \rangle$, is not a marked graph any longer. Both sequential and concurrent versions of implicitness are studied. As it was noted in Section 4, SSIPness is a necessary condition for both kinds of SIPness, and then only the MIMP remains open. Making use of the important property that the net state equation of a live marked graph has not spurious solutions, and with the help of totally unimodular matrices theory, polynomial-time characterizations for both implicitness properties will be obtained.

Notice that determining liveness for marked graphs is of polynomial complexity: A marked graph is live if and only if it is conservative, consistent, and all circuits contain at least one token.

5.1 *Totally Unimodular Matrices*

An integer programming problem defined over the constraint set $S = \{\mathbf{x} \in \mathbb{Z}_+^n ; \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\}$ can be sometimes solved as a linear programming problem (making $\mathbf{x} \in \mathbb{R}_+^n$) if the matrix $\mathbf{A}$ has certain properties. *Totally Unimodular Matrices (TU)* are a kind of matrices fulfilling the needed properties. As it will be exposed in the following subsections, and for the case of live marked graphs, the matrices that define the sets of constraints associated with the MIMP are totally unimodular. This fact will allow to relax the domain of the variables from integer to real. All the results in this subsection have been taken from (Nemhauser and Wolsey, 1988).

Definition 11. An $m \times n$ integral matrix $\mathbf{A}$ is *totally unimodular* (TU) if the determinant of each square submatrix of $\mathbf{A}$ is equal to 0, 1, or -1.

The usefulness of TU matrices is that an IPP can be solved as a LPP, if the matrix that defines the constraints is TU, as the following propositions state.

Proposition 12. If $\mathbf{A}$ is TU, then $P(\mathbf{b}) = \{\mathbf{x} \in \mathbb{R}_+^n : \mathbf{A} \cdot \mathbf{x} \leq \mathbf{b}\}$ is integral for all $\mathbf{b} \in \mathbb{Z}^m$ for which it is not empty.

Proposition 13. Consider the linear programming problem LP over the polyhedron P given by $z_{LP} = \max\{\mathbf{c}^T \cdot \mathbf{x} : \mathbf{x} \in P\}$. Then, the following statements are equivalent: 1) P is integral. 2) LP has an integral optimal solution for all $\mathbf{c} \in \mathbb{R}^n$ for which it has an optimal solution.

The following characterization of TU will be very fruitful in the proofs of the main results for live marked graphs.

Theorem 14. $\mathbf{A}$ is TU if and only if for every $J \subseteq \{1, \ldots, n\}$, there exists a partition J_1, J_2 of J such that, for $i = 1, \ldots, m$:

$$\left| \sum_{j \in J_1} \mathbf{A}[i,j] - \sum_{j \in J_2} \mathbf{A}[i,j] \right| \leq 1$$

5.2 *Sequential Implicit Places*

In order to apply the theory of TU matrices to the minimum initial marking problem, an IPP characterization of SIPness is needed. Taking as starting point the characterization given in Proposition 4, the first step is to substitute such a set by the marking solutions of the net state equation. This substitution is possible, basically, because the net state equation of live marked graphs have not solutions that are not reachable (Murata, 1977).

Lemma 15. p is a SIP of $\mathscr{S}'$ if and only if for all $t \in p^\bullet$, and for all $\sigma \in \mathbb{Z}_+^{|T|}$, such that $\mathbf{m_0} + \mathbf{C} \cdot \sigma \geq \mathbf{Pre}[P,t]$, $\mathbf{m_0}'[p] \geq -\mathbf{C}'[p,T] \cdot \sigma + \mathbf{Pre}'[p,t]$.

Proof: $\Longrightarrow$ Because p is a SIP, $\mathrm{RS}(\mathscr{S}) = \mathrm{RS}(\mathscr{S}'|_P)$. $\mathrm{RS}(\mathscr{S})$ can be characterized by the marking solutions of the net state equation, because $\mathscr{S}$ is a live marked graph (Murata, 1977), that is $\mathrm{RS}(\mathscr{S}) = \{\mathbf{m} \mid \mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \sigma,\ \mathbf{m} \geq 0,\ \sigma \in \mathbb{Z}_+^{|T|}\}$. On the other hand, an because p is a SIP, $\mathrm{L}(\mathscr{S}) = \mathrm{L}(\mathscr{S}')$ and, if $\mathbf{m_0} \xrightarrow{\sigma} \mathbf{m}$, then $\mathbf{m_0}'[p] \xrightarrow{\sigma} \mathbf{m}[p]$. Therefore $\mathrm{RS}(\mathscr{S}')$ can be characterized as $\mathrm{RS}(\mathscr{S}') = \{\mathbf{m},\ \mathbf{m}'[p] \mid \mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \sigma \geq 0,\ \mathbf{m}'[p] = \mathbf{m_0}[p] + \mathbf{C}[p,T] \cdot \sigma,\ \sigma \in \mathbb{Z}_+^{|T|}\}$. Taking into account this fact, the right-hand side of Poposition 4.1 can be directly rewritten as stated in this lemma.
$\Longleftarrow$ The condition can be rewritten as: for all $t \in p^\bullet$, and for all $\sigma \in \mathbb{Z}_+^{|T|}$, such that $\mathbf{m} = \mathbf{m_0} + \mathbf{C} \cdot \sigma \geq \mathbf{Pre}[P,t]$ (≥ 0) and $\mathbf{m}'[p] = \mathbf{m_0}'[p] + \mathbf{C}'[p,T] \cdot \sigma$, then $\mathbf{m}'[p] \geq \mathbf{Pre}'[p,t]$. If this condition is met, then it is also true for the particular case when $\mathbf{m}'[p] \geq 0$. But in that case $\mathbf{m}, \mathbf{m}'[p]$ are the solutions of the net state equation of $\mathscr{S}'$. This set includes $\mathrm{RS}(\mathscr{S}')$, and therefore the condition is also true for $\mathrm{RS}(\mathscr{S}')$. By Proposition 4.1, p is a SIP. $\square$

The result of Lemma 15 as a family of IPPs (one for each output transition of p) can be rewriten, searching for the initial marking of p that fulfils the inequality for every reachable marking.

Lemma 16. Let the family of IPPs 2, defined for every $t \in p^\bullet$:

$$\begin{aligned} z_t^1 = \ & \max. \ -\mathbf{C}'[p,T] \cdot \sigma + \mathbf{Pre}'[p,t] \\ & \text{s.t.} \quad \mathbf{m_0} + \mathbf{C} \cdot \sigma \geq \mathbf{Pre}[P,t] \\ & \qquad\quad \sigma \in \mathbb{Z}_+^{|T|} \end{aligned} \tag{2}$$

p is a SIP of $\mathscr{S}'$ if and only if for all $t \in p^\bullet$, its corresponding IPP 2 is bounded, and $\mathbf{m_0}'[p] \geq \max\{z_t^1 \mid t \in p^\bullet\}$.

Proof: The result is easily deduced from Lemma 15, taking to account that each IPP 2 computes an initial marking that fulfils the condition of Lemma 15 in any case. Note that each IPP 2 always has a feasible solution, because in live net systems, and given any transition, there exists at least a reachable marking that enables it, that is, there exists $\mathbf{m} \in \mathrm{RS}(\mathscr{S})$ such that $\mathbf{m} \geq \mathbf{Pre}[P,t]$, for any transition t. $\square$

Finally the main result, the characterization of SIPness in terms of a set of linear programming problems, is obtained taken into account that the incidence matrix of a marked graph is totally unimodular. Moreover, the boundedness of the solutions of IPPs in Lemma 16 will be proved as equivalent to the condition of SSIPness.

Theorem 17. Let the family of IPPs 3, defined for every $t \in p^\bullet$:

$$\begin{aligned} z_t^2 = \ & \min. \ \mathbf{y}^T \cdot \left(\mathbf{m_0} - \mathbf{Pre}[P,t]\right) + \mathbf{Pre}'[p,t] \\ & \text{s.t.} \quad \mathbf{y}^T \cdot \mathbf{C} \leq \mathbf{C}'[p,T] \\ & \qquad\quad \mathbf{y} \geq 0 \end{aligned} \tag{3}$$

p is a SIP of $\mathscr{S}'$ if and only if p is a SSIP and $\mathbf{m_0}'[p] \geq \max\{z_t^2 \mid t \in p^\bullet\}$.

Proof: The family of IPPs 2 can be rewritten in standard form as: $z_t^1 = \max\{-\mathbf{C}'[p,T]\cdot\sigma : \sigma \in P_t\} + \mathbf{Pre}'[p,t]$, being $P_t = \{\sigma \in \mathbb{Z}_+^{|T|} : \mathbf{C}\cdot\sigma \leq -\mathbf{Pre}[P,t] + \mathbf{m_0}\}$. $\mathbf{C}$ is TU because it is the node-incidence matrix of a bipartite graph (the incidence matrix of a marked graph); additionally, the polyhedra $P_t' = \{\sigma \in \mathbb{R}_+^{|T|} : \mathbf{C}\cdot\sigma \leq -\mathbf{Pre}[P,t] + \mathbf{m_0}\}$ are not empty because the net system is live (see the proof of Lemma 16). These two conditions imply that for all $t \in p^\bullet$, P_t' is integral (Proposition 12). Therefore, for all $t \in p^\bullet$, $z_t' = \max\{-\mathbf{C}'[p,T]\cdot\sigma : \sigma \in P_t'\}$ has an integral optimal solution (Proposition 13), and $z_t' = z_t^1$. Applying the Alternatives Theorem (Nemhauser and Wolsey, 1988) to each z_t', we obtain the family of LPPs 3. Because z_t' has always a feasible solution, its corresponding LPP 3 is either bounded (if z_t' is bounded) or non-feasible (if z_t' is unbounded).

Therefore we can establish that $\forall t \in p^\bullet$, its corresponding IPP 2 is bounded, and $\mathbf{m_0}'[p] \geq \max\{z_t^1 \mid t \in p^\bullet\}$ if and only if $\forall t \in p^\bullet$, its corresponding LPP 3 has a feasible solution, and $\mathbf{m_0}'[p] \geq \max\{z_t^2 \mid t \in p^\bullet\}$. Finally, note that the existence of solutions of LPPs 3 is equivalent to the SSIPness of p (Theorem 7). $\square$

5.3 *Concurrent Implicit Places*

In the case of CIPs the procedure to obtain the desired result is similar to that followed for SIPs. However, in this case, total unimodularity of the matrix corresponding to the MIMP must be explicitly proved.

Lemma 18. p is a CIP of $\mathscr{S}'$ if and only if for all $\sigma \in \mathbb{Z}_+^{|T|}$, and for all $s_{p^\bullet} \in \mathbb{Z}_+^{|p^\bullet|}$, such that $\mathbf{m_0} + \mathbf{C}\cdot\sigma \geq \mathbf{Pre}[P,p^\bullet]\cdot s_{p^\bullet}$, $\mathbf{m_0}'[p] \geq -\mathbf{C}'[p,T]\cdot\sigma + \mathbf{Pre}'[p,p^\bullet]\cdot s_{p^\bullet}$.

Proof: The proof follows the same steps that the proof of Lemma 15, reasoning with ocurrence sequences of steps instead of transitions. $\square$

Just like for SIPness, the condition for CIPness in Lemma 18 is only sufficient for net systems in general, because of the existence of spurious solutions.

Lemma 19. p is a CIP of $\mathscr{S}'$ if and only if IPP 4 is bounded, and $\mathbf{m_0}'[p] \geq z$.

$$\begin{aligned} z = \ \max.\ & -\mathbf{C}'[p,T]\cdot\sigma + \mathbf{Pre}'[p,p^\bullet]\cdot s_{p^\bullet} \\ \text{s.t.}\ & \mathbf{m_0} + \mathbf{C}\cdot\sigma \geq \mathbf{Pre}[P,p^\bullet]\cdot s_{p^\bullet} \\ & \sigma \in \mathbb{Z}_+^{|T|} \\ & s_{p^\bullet} \in \mathbb{Z}_+^{|p^\bullet|} \end{aligned} \tag{4}$$

Proof: The result is easily deduced from Lemma 18, taking to account that IPP 4 computes an initial marking that fulfils the condition of Lemma 18 in any case. Note that IPP 4 always has a feasible solution, because in live net systems, there exists at least a reachable marking that enables any transition, that is, there exists $\mathbf{m} \in \mathrm{RS}(\mathscr{S})$ such that $\mathbf{m} \geq \mathbf{Pre}[P,t]$, for any transition t. $\square$

The most dificult part in this subsection is to prove that the matrix that defines the polyhedron of IPP 4 is TU. Previously, let us rewrite IPP 4 in standard form: $z_{LP} = \max\{\mathbf{c}^T\cdot\mathbf{x} : \mathbf{x} \in P\}$, $P = \{\mathbf{x} \in \mathbb{Z}_+^{|T|+|p^\bullet|} : \mathbf{A}\cdot\mathbf{x} \leq \mathbf{m_0}\}$, where:

$$\mathbf{A} = \begin{bmatrix} -\mathbf{C} & \mathbf{Pre}[P,p^\bullet] \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} -\mathbf{C}'[p,T] \\ \mathbf{Pre}'[p,p^\bullet] \end{bmatrix}$$

$$\mathbf{x} = \begin{bmatrix} \sigma \\ s_{p^\bullet} \end{bmatrix}$$

Lemma 20. The matrix $\mathbf{A}$ is totally unimodular.

Proof: Let us assume, without loss of generality, that columns in $\mathbf{A}$ are ordered in the following way: $\mathbf{A} = [-\mathbf{C}[P,p^\bullet] \mid -\mathbf{C}[P,T\setminus p^\bullet] \mid \mathbf{Pre}'[p,p^\bullet]]$. Moreover, the order taken in the columns of $\mathbf{C}[P,T\setminus p^\bullet]$ is maintained in the columns of $\mathbf{Pre}[P,p^\bullet]$. That is, if the column with index j of $\mathbf{C}[P,p^\bullet]$ corresponds to the transition t, then the column with index $j+|T|$ of $\mathbf{A}$ (the column with index j in $\mathbf{Pre}[P,p^\bullet]$) corresponds also to transition t.

The result is proved with the help of Theorem 14. Let $J \subseteq \{1,\ldots,|T|+|p^\bullet|\}$, that is, J is a subset of indexes that identify a subset of columns of $\mathbf{A}$. Let J_1, J_2 be a partition of J defined in the following way: Let $j \in J$; i) If $j \in \{1,\ldots,|T|\}$, then $j \in J_1$; ii) In the case of $j \in \{|T|+1,\ldots,|T|+|p^\bullet|\}$, let $j' = j-|T|$; if $j' \in J$, then $j \in J_2$ else $j \in J_1$. Let $i \in \{1,\ldots,|P|\}$ be the index of a row of $\mathbf{A}$, and $a_i = \left|\sum_{j\in J_1}\mathbf{A}[i,j] - \sum_{j\in J_2}\mathbf{A}[i,j]\right|$. The goal is to prove that $a_i \leq 1$, $\forall i \in \{1,\ldots,|P|\}$. Because $\mathbf{C}$ is the incidence matrix of a live marked graph, there exists only two elements not equal to zero in $-\mathbf{C}[i,T]$, one of them with value 1 and the other -1. Let j_1 and j_2 be the indexes of the columns of $\mathbf{A}$ corresponding to such elements, respectively. Because $\mathbf{Pre}[P,p^\bullet]$ corresponds also to a live marked graph, there is at most one element not equal to zero in $\mathbf{Pre}[i,p^\bullet]$ (whose value will be 1). If such an element does not exist, then $a_i \leq 1$, because if j_1, j_2 or both belong to J, then they belong to J_1, being $\mathbf{A}[i,j_1] = 1$ and $\mathbf{A}[i,j_2] = -1$. On the contrary, if there exists an element not equal to zero in $\mathbf{Pre}[i,p^\bullet]$, let j_3 be the index of the column of such an element. Then $j_3 = j_1 + |T|$, because $\mathbf{C}$ and $\mathbf{Pre}[P,p^\bullet]$ correspons to the same net, and the sign of $\mathbf{C}$ is changed in $\mathbf{A}$. Three cases are distinguished: i) If $j_3 \notin J$, then $a_i \leq 1$, due to the same reasons than before; ii) If $j_3 \in J$ and $j_1 \in J$ then $j_3 \in J_2$. In this case, if $j_2 \in J$, $a_i = |\mathbf{A}[i,j_1] + \mathbf{A}[i,j_2] - \mathbf{A}[i,j_3]| = |1-1-1| \leq 1$. If

$j_2 \notin J$, $a_i = |\mathbf{A}[i, j_1] - \mathbf{A}[i, j_3]| = |1-1| \leq 1$. iii) If $j_3 \in J$ and $j_1 \notin J$ then $j_3 \in J_1$. In this case, if $j_2 \in J$, $a_i = |\mathbf{A}[i, j_2] + \mathbf{A}[i, j_3]| = |-1+1| \leq 1$. If $j_2 \notin J$, $a_i = |\mathbf{A}[i, j_3]| = |1| \leq 1$. □

Total unimodularity of **A** allows to prove the main result concerning CIPs, analogous to the presented for SIPs.

Theorem 21. p is a CIP of $\mathscr{S}'$ if and only if p is a SSIP and $\mathbf{m_0}'[p] \geq z$.

$$
\begin{aligned}
z = \ \text{min.}\ & \mathbf{y}^T \cdot \mathbf{m_0} \\
\text{s.t.}\ & \mathbf{y}^T \cdot \mathbf{C} \leq \mathbf{C}'[p, T] \\
& \mathbf{y}^T \cdot \mathbf{Pre}[P, p^\bullet] \geq \mathbf{Pre}'[p, p^\bullet] \\
& \mathbf{y} \geq 0
\end{aligned}
\tag{5}
$$

Proof: Because **A** is TU, the proof follows the same steps than in Theorem 17, except for the last one, the equivalence between SSIPness and the existence of a solution of LPP 5. Let us prove this last point. Obviously, the existence of a solution of LPP 5 implies that p is a SSIP (Theorem 7). On the other hand, if p is a SSIP, there exists $\mathbf{y} \geq 0$ such that $\mathbf{y}^T \cdot \mathbf{C} \leq \mathbf{C}'[p, T]$ (Theorem 7). If $\mathbf{y}^T \cdot \mathbf{Pre}[P, p^\bullet] \geq \mathbf{Pre}'[p, p^\bullet]$, then $\mathbf{y}$ is a solution of LPP 5. On the contrary, let $k = \max\{\mathbf{Pre}'[p, q] - \mathbf{y}^T \cdot \mathbf{Pre}[P, q] \mid q \in p^\bullet\}$ $(k > 0)$. Because live marked graphs are structurally bounded, there exists $\mathbf{x} \geq \mathbf{1}$ such that $\mathbf{x}^T \cdot \mathbf{C} \leq 0$. Let $\mathbf{y_1} = \mathbf{y} + k \cdot \mathbf{x}$. Then $\mathbf{y_1}^T \cdot \mathbf{C} \leq \mathbf{C}'[p, T]$ (because $\mathbf{y}^T \cdot \mathbf{C} \leq \mathbf{C}'[p, T]$ and $k \cdot \mathbf{x}^T \cdot \mathbf{C} \leq 0$) and $\mathbf{y_1}^T \cdot \mathbf{Pre}[P, p^\bullet] \geq \mathbf{Pre}'[p, p^\bullet]$; that is, $\mathbf{y_1}$ is a solution of LPP 5. □

Finally, recall that Corollary 2 showed the equivalence between SIPs and CIPs for the case of self-loop free places. This fact makes that Theorem 21 can be considered as a alternative characterization of SIPness for self-loop free places.

6. CONCLUSIONS

In the state feedback control approach by the addition of control places, implicitness characterises redundancies in the control. In this paper we have proved that for the case of live marked graphs, checking if a control place is implicit is of polynomial complexity. This result complements a previous work on live and safe free-choice net systems (García-Vallés and Colom, 1999).

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