15th Triennial World Congress, Barcelona, Spain

ROBUST H-INFINITY REDUCED ORDER FILTERING F OR UNCER TAIN BILINEAR SYSTEMS

H. Souley Ali, M. Zasadzinski, H. Rafaralah y and M. Darouac h

CRAN, IUT de Longwy, Université Henri Poincaré – Nancy I
186, rue de Lorraine, 54400 Cosnes et Romain, FRANCE
e-mail : {souley,mzasad}@iut-longwy.uhp-nancy.fr

Abstract: This paper investigates $\mathcal{H}_\infty$ reduced order unbiased filtering problems for a nominal bilinear system and a bilinear system affected b y norm-bounded structured uncertainties in all the system matrices. First, the un biasedness condition is derived. Second, a change of variable is introduced on the inputs of the system to reduce the conservatism inherent to the filter stability requirement and to treat the product of the inputs by the disturbances. Then the solution is expressed in terms of LMI b y transforming the problem in to a robust state feedback in the nominal case and a robust static output feedback in the uncertain case. *Copyright ©2002 IFAC*

Keywords: Bilinear systems, Robust estimation, H-infinit y optimization, Uncertain ty.

1. INTRODUCTION

The functional filtering purpose is to estimate a linear com bination of the states of a system using the measuremen ts. In this paper, a reduced order $\mathcal{H}_\infty$ filtering method is proposed to reconstruct a linear combination of the states of bilinear systems by exploiting the nonlinearities in the nominal and the robust cases. This is achieved through the design of an observer which dynamics has the same dimension than this linear com bination. In addition to the stability and $\mathcal{L}_2$ gain attenuation requirements, the filter must also be unbiased, i.e. the estimation error does not depend on the states of the system. This condition is expressed in terms of as many Sylvester equations as there are inputs, with an additional Sylvester equation due to the linear part of the bilinear system. Our approach is based on the resolution of a system of equations to find conditions for the existence of the unbiased reduced order filter; we then solve the exponential convergence and $\mathcal{L}_2$ gain attenuation problems which are reduced to a robust state feedback in the nominal case. It is shown that the robust functional unbiased filtering problem for uncertain bilinear systems subjected to time-varying norm-bounded uncertainties can be seen as a particular case of a static output feedback one under some conditions. This problem requires to solve **L**inear **M**atrix **I**nequalities (LMI) with an additional non convex **B**ilinear **M**atrix **I**nequality (BMI) constraint.

The paper is organized as follows. The conditions for the unbiasedness, exponential convergence and $\mathcal{L}_2$ gain attenuation of a reduced order $\mathcal{H}_\infty$ functional filter for continuous-time nominal bilinear system are studied in section 2. It is shown through section 3 that the concerned robust filtering problem for bilinear system affected by structured norm-bounded time-v arying uncertainties can be solved as a static output feedback problem. Then, section 4 concludes the paper.

2. REDUCED ORDER UNBIASED $\mathcal{H}_\infty$ FILTERING IN THE NOMINAL CASE

2.1 *Problem Formulation*

The nominal bilinear system considered in this section is given by

$$\dot{x} = A^0 x + \sum_{i=1}^{m} A^i u^i x + Bw \qquad (1a)$$

$$y = Cx + Dw \qquad (1b)$$

$$z = Lx \qquad (1c)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the measured output and $z(t) \in \mathbb{R}^r$ is the vector of variables to be estimated where $r \leqslant n$. The vector $w(t) \in \mathbb{R}^q$ represents the disturbance vector. Without loss of generality, it is assumed that $\operatorname{rank} L = r$. The problem is to estimate the vector $z(t)$ from the measurements $y(t)$. As in the most cases for physical processes, the bilinear system (1) has known bounded control inputs, i.e. $u(t) \in \Omega \subset \mathbb{R}^m$, where $(i = 1, \dots, m)$

$$\Omega := \left\{ u(t) \in \mathbb{R}^m \mid u^i_{\min} \leqslant u^i(t) \leqslant u^i_{\max} \right\}. \quad (2)$$

The ROUF (**R**educed **O**rder **U**nbiased **F**ilter) is described by

$$\dot{\eta} = H^0\eta + \sum_{i=1}^m H^i u^i \eta + J^0 y + \sum_{i=1}^m J^i u^i y \quad (3a)$$

$$\hat{z} = \eta + Ey \quad (3b)$$

where $\hat{z}(t) \in \mathbb{R}^r$ is the estimate of $z(t)$. In order to avoid that some linear combinations of components of vector z be directly estimated from the measurement y without using the filter state vector η, this assumption is made in the sequel.

Assumption 1. Matrices C and L verify

$$\operatorname{rank}\begin{bmatrix} C^T & L^T \end{bmatrix} = \operatorname{rank} C + \operatorname{rank} L. \quad (4)$$

The estimation error is given by

$$e = z - \hat{z} = Lx - \hat{z} = \overline{e} - EDw \quad (5)$$

where

$$\overline{e} = \Psi x - \eta \quad (6a)$$

$$\Psi = L - EC. \quad (6b)$$

The problem of the ROUF design is to determine H^0, H^i, J^0, J^i and E such that

- **(i)** the filter (3) is unbiased if $w = 0$, i.e. the estimation error is independent of x,
- **(ii)** the ROUF (3) is exponentially convergent for $u(t) \in \Omega$,
- **(iii)** the mapping from w to e has $\mathcal{L}_2$ gain less than a given scalar γ for $u(t) \in \Omega$ (van der Schaft, 1992).

2.2 *Unbiasedness condition fulfillment*

From (5), notice that the time derivative of the error e is function of the time derivative of the disturbances w. To avoid the use of $\dot{w}$ in the dynamics of the error e, consider $\overline{e}$ as a new "state vector". Setting $u^0 = 1$, then the $\mathcal{L}_2$ gain from w to e has the following state space realization

$$\begin{cases} \dot{\overline{e}} = \displaystyle\sum_{i=0}^m u^i H^i \overline{e} + (\Psi B - \sum_{i=0}^m J^i D u^i) w \\ + \displaystyle\sum_{i=0}^m (\Psi A^i - H^i \Psi - J^i C) u^i x \\ e = \overline{e} - EDw \end{cases} \quad (7)$$

and the unbiasedness of the filter is achieved iff the following Sylvester equations

$$\Psi A^i - H^i \Psi - J^i C = 0 \qquad i = 0, \dots, m \quad (8)$$

hold. As matrix L is of full row rank, the relations in (8) are equivalent to $(i = 0, \dots, m)$

$$(\Psi A^i - H^i \Psi - J^i C) \begin{bmatrix} L^\dagger & I_n - L^\dagger L \end{bmatrix} = 0 \quad (9)$$

where $L^\dagger$ is a generalized inverse of matrix L satisfying $L = LL^\dagger L$ (since $\operatorname{rank} L = r$, we have $LL^\dagger = I_r$). Using the definition of Ψ, (9) is equivalent to $(i = 0, \dots, m)$

$$0 = \Psi A^i L^\dagger - H^i \Psi L^\dagger - J^i C L^\dagger \quad (10a)$$

$$0 = \Psi \overline{A}^i + H^i E \overline{C} - J^i \overline{C} \quad (10b)$$

where $\overline{A}^i = A^i(I_n - L^\dagger L)$ and $\overline{C} = C(I_n - L^\dagger L)$.

Using (6a), relation (10) can be rewritten as

$$H^i = \overline{\overline{A}}^i - \mathcal{K}^i \overline{\overline{C}}^i \qquad i = 0, \dots, m \quad (11)$$

where $\overline{\overline{A}}^i = LA^iL^\dagger$, $\overline{\overline{C}}^i = \begin{bmatrix} CA^iL^\dagger \\ CL^\dagger \end{bmatrix}$ and

$$\mathcal{K}^i = \begin{bmatrix} E & K^i \end{bmatrix} \text{ with } K^i = J^i - H^i E. \quad (12)$$

Then relation (10a) can be expressed in the following compact form

$$\mathcal{K}\Sigma = L\overline{A} \quad (13)$$

where $\overline{A} = \begin{bmatrix} \overline{A}^0 & \dots & \overline{A}^m \end{bmatrix}$ and

$$\Sigma = \begin{bmatrix} C\overline{A}^0 & C\overline{A}^1 & \dots & C\overline{A}^m \\ \overline{C} & 0 & \dots & 0 \\ 0 & \overline{C} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & \overline{C} \end{bmatrix} \quad (14a)$$

$$\mathcal{K} = \begin{bmatrix} E & K^0 & \dots & K^m \end{bmatrix}, \quad (14b)$$

and a general solution to equation (13) if it exists, is given by

$$\mathcal{K} = L\overline{A}\Sigma^\dagger + Z(I_{(m+2)p} - \Sigma\Sigma^\dagger) \quad (15)$$

where $Z = \begin{bmatrix} Z_E & Z^0 & \dots & Z^m \end{bmatrix}$ is arbitrary.

Lemma 2. The unbiasedness of the filter (3) is achieved if and only if the following rank condition

$$\operatorname{rank}\Sigma = \operatorname{rank}\begin{bmatrix} (L\overline{A})^T & \Sigma^T \end{bmatrix} \quad (16)$$

holds. ■

Proof. Using the previous developments, filter (3) is unbiased, i.e. relation (8) holds, iff there exists a solution $\mathcal{K}$ to (13), then iff the rank condition (16) is satisfied. ❑

2.3 *Unbiasedness condition under $ED = 0$.*

Using the previous steps, the mapping from w to e, given by (7), becomes

$$\begin{cases} \dot{\overline{e}} = \left(\displaystyle\sum_{i=0}^m u^i \overline{\overline{A}}^i - L\overline{A}\Sigma^\dagger \Lambda_1(u) - Z_\Sigma \Lambda_1(u) \right) \overline{e} \\ + \left(LB - L\overline{A}\Sigma^\dagger \Lambda_2(u) - Z_\Sigma \Lambda_2(u) \right) w \\ + \left(\displaystyle\sum_{i=0}^m u^i \overline{\overline{A}}^i - L\overline{A}\Sigma^\dagger \Lambda_1(u) - Z_\Sigma \Lambda_1(u) \right) EDw \\ e = \overline{e} - EDw \end{cases}$$

(17)

where $Z_\Sigma = Z(I_{(m+2)p} - \Sigma\Sigma^\dagger)$ and

$$\Lambda_1(u)=\begin{bmatrix} \sum_{i=0}^{m} u^i CA^i L^\dagger \\ CL^\dagger \\ u^1 CL^\dagger \\ \vdots \\ u^m CL^\dagger \end{bmatrix}, \Lambda_2(u)=\begin{bmatrix} CB \\ D \\ u^1 D \\ \vdots \\ u^m D \end{bmatrix}. \quad (18)$$

Due to the product $Z(I_{(m+2)p} - \Sigma\Sigma^\dagger)\Lambda_1(u)E$, the error is bilinear in the gain parameter Z in system (17). This bilinearity is intrinsically linked to the unbiasedness condition (8). Indeed, the "bilinearity" $H^i\Psi$ in (8) yields a gain K^i (see (12)) containing the product H^iE. In order to avoid this bilinearity, we consider $ED = 0$ in the sequel; this allows us to have LMI tractable formulation for the problem instead of intractable BMI one. Adding the constraint $ED = 0$, relations (13), (14) and (15) become

$$\mathcal{K}\overline{\Sigma} = \begin{bmatrix} 0 & L\overline{A} \end{bmatrix} \text{ where } \overline{\Sigma} = \left[\begin{array}{c|c} D & \Sigma \\ 0 & \end{array}\right] \quad (19)$$

and a solution to (19), if it exists, is given by

$$\mathcal{K} = \begin{bmatrix} 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger + Z(I_{(m+2)p} - \overline{\Sigma}\,\overline{\Sigma}^\dagger). \quad (20)$$

So, we give the following lemma which is derived from lemma 2.

Lemma 3. The unbiasedness of the filter (3) is achieved under $ED = 0$ iff the following rank condition

$$\operatorname{rank} \overline{\Sigma} = \operatorname{rank} \left[\begin{array}{c} 0 \;\; L\overline{A} \\ \hline \overline{\Sigma} \end{array}\right] \quad (21)$$

holds. ■

Now, assume that the condition (21) in lemma 3 holds. Then relation (8) is verified with $\mathcal{K}$ given by (20) and we have $\overline{e}(t) = e(t)$ in (17), i.e.

$$\dot{e} = \underbrace{\left(\sum_{i=0}^{m} u^i \overline{\overline{A}}^i - \begin{bmatrix} 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger \Lambda_1(u) - Z_{\overline{\Sigma}}\Lambda_1(u)\right)}_{\sum_{i=0}^{m} u^i H^i} e$$
$$+ \left(LB - \begin{bmatrix} 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger \Lambda_2(u) - Z_{\overline{\Sigma}}\Lambda_2(u)\right) w \quad (22)$$

where $Z_{\overline{\Sigma}} = Z(I_{(m+2)p} - \overline{\Sigma}\,\overline{\Sigma}^\dagger)$. Now, as the item (i) of the design objectives has been solved, it remains to treat the points (ii) and (iii) of these objectives.

2.4 *Exponential stability and $\mathcal{L}_2$ gain attenuation*

Now, we introduce a change of variables by considering each $u^i(t)$ in (22) as a "structured uncertainty". Notice that the definition of the "uncertainty set" Ω in (2) can leads to some conservatism since, in the general case, $|u^i_{\min}| \neq |u^i_{\max}|$ with $|u^i_{\min}| \neq 1$ and $|u^i_{\max}| \neq 1$. To reduce this conservatism, each $u^i(t)$ can be rewritten as follows

$$u^i(t) = \alpha^i + \sigma^i \varepsilon^i(t) \quad (23)$$

where $(i = 1, \dots, m)$

$$\alpha^i = \frac{1}{2}(u^i_{\min} + u^i_{\max}),\; \sigma^i = \frac{1}{2}(u^i_{\max} - u^i_{\min}) \quad (24)$$

and $\alpha^0 = 1$, $\sigma^0 = 0$. The new "uncertain" variable is $\varepsilon(t) \in \overline{\Omega} \subset \mathbb{R}^m$ where $\overline{\Omega}$ is defined as

$$\overline{\Omega} := \left\{ \varepsilon(t) \in \mathbb{R}^m \mid \varepsilon^i_{\min} = -1 \leqslant \varepsilon^i(t) \leqslant \varepsilon^i_{\max} = 1 \right\}. \quad (25)$$

By using relations (23)-(25), the dynamics of the error $e(t)$ in (22) can be rewritten as follows

$$\dot{e} = \left(\mathbb{A} - Z\mathbb{C} + \left(\widetilde{\mathbb{A}} - Z\widetilde{\mathbb{C}}\right) \Delta_e(\varepsilon)\overline{H}_e\right) e$$
$$+ \left(\mathbb{B} - Z\mathbb{G} + \left(\widetilde{\mathbb{B}} - Z\widetilde{\mathbb{G}}\right) \Delta_w(\varepsilon)\overline{H}_w\right) w \quad (26)$$

with

$$\mathbb{A} = \sum_{i=0}^{m} \alpha^i \overline{\overline{A}}^i - \begin{bmatrix} 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger \Phi, \quad (27a)$$
$$\mathbb{C} = (I_{(m+2)p} - \overline{\Sigma}\,\overline{\Sigma}^\dagger)\Phi, \quad (27b)$$
$$\widetilde{\mathbb{A}} = \begin{bmatrix} \sigma^1\overline{\overline{A}}^1 & \dots & \sigma^m\overline{\overline{A}}^m \end{bmatrix} - \begin{bmatrix} 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger \Gamma, \quad (27c)$$
$$\widetilde{\mathbb{C}} = (I_{(m+2)p} - \overline{\Sigma}\,\overline{\Sigma}^\dagger)\Gamma, \quad (27d)$$
$$\mathbb{B} = LB - \begin{bmatrix} 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger \Upsilon, \mathbb{G} = (I_{(m+2)p} - \overline{\Sigma}\,\overline{\Sigma}^\dagger)\Upsilon, \quad (27e)$$
$$\widetilde{\mathbb{B}} = -\begin{bmatrix} 0 & L\overline{A} \end{bmatrix} \overline{\Sigma}^\dagger \mathbb{D}, \widetilde{\mathbb{G}} = (I_{(m+2)p} - \overline{\Sigma}\,\overline{\Sigma}^\dagger)\mathbb{D}, \quad (27f)$$

and

$$\Gamma = \begin{bmatrix} \sigma^1 CA^1L^\dagger & \dots & \sigma^m CA^mL^\dagger \\ 0 & \dots & 0 \\ \sigma^1 CL^\dagger & \ddots & \vdots \\ \ddots & \ddots & 0 \\ \dots & 0 & \sigma^m CL^\dagger \end{bmatrix}, \Upsilon = \begin{bmatrix} \alpha^0 CB \\ \alpha^0 D \\ \vdots \\ \alpha^m D \end{bmatrix} \quad (28a)$$

$$\mathbb{D} = \begin{bmatrix} 0 & \dots & 0 \\ 0 & \dots & 0 \\ \sigma^1 D & \ddots & \vdots \\ \ddots & \ddots & 0 \\ \dots & 0 & \sigma^m D \end{bmatrix}, \Phi = \begin{bmatrix} \sum_{i=0}^{m} \alpha^i CA^iL^\dagger \\ \alpha^0 CL^\dagger \\ \vdots \\ \alpha^m CL^\dagger \end{bmatrix}. \quad (28b)$$

More, $\Delta_e(\varepsilon) \in \mathbb{R}^{mr\times mr}$, $\Delta_w(\varepsilon) \in \mathbb{R}^{mq\times mq}$, $\overline{H}_e \in \mathbb{R}^{mr\times r}$ and $\overline{H}_w \in \mathbb{R}^{mq\times q}$ are defined by

$$\Delta_e(\varepsilon) = \operatorname{bdiag}(\varepsilon^1 I_r, \dots, \varepsilon^m I_r), \quad (29a)$$
$$\Delta_w(\varepsilon) = \operatorname{bdiag}(\varepsilon^1 I_q, \dots, \varepsilon^m I_q) \quad (29b)$$
$$\overline{H}_e = \begin{bmatrix} I_r & \dots & I_r \end{bmatrix}^T,\; \overline{H}_w = \begin{bmatrix} I_q & \dots & I_q \end{bmatrix}^T, \quad (29c)$$

where bdiag(.) denotes a block-diagonal matrix. From (25), the "uncertain" matrices $\Delta_e(\varepsilon)$ and $\Delta_w(\varepsilon)$ are bounded as

$$\|\Delta_e(\varepsilon)\| \leqslant 1 \text{ and } \|\Delta_w(\varepsilon)\| \leqslant 1. \quad (30)$$

According to the previous developments, (26) can be rewritten as the following system

$$\begin{cases} \dot{e} = (\mathbb{A} - Z\mathbb{C})\, e + \left(\left[\begin{array}{cc|c} \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} & \mathbb{B} \end{array}\right] - Z \left[\begin{array}{cc|c} \widetilde{\mathbb{C}} & \widetilde{\mathbb{G}} & \mathbb{G} \end{array}\right]\right) \left[\begin{array}{c} p_e \\ p_w \\ \hline w \end{array}\right] \\ \left[\begin{array}{c} q_e \\ q_w \\ \hline e \end{array}\right] = \left[\begin{array}{c} \overline{H}_e \\ 0 \\ \hline I_r \end{array}\right] e + \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & \overline{H}_w \\ \hline 0 & 0 & 0 \end{array}\right] \left[\begin{array}{c} p_e \\ p_w \\ \hline w \end{array}\right] \end{cases} \quad (31)$$

connected with

$$\begin{bmatrix} p_e \\ p_w \end{bmatrix} = \begin{bmatrix} \Delta_e(\varepsilon) & 0 \\ 0 & \Delta_w(\varepsilon) \end{bmatrix} \begin{bmatrix} q_e \\ q_w \end{bmatrix}. \quad (32)$$

At this step, we can see that the $\mathcal{H}_\infty$ ROUF can be solved as a particular case of a dual robust state feedback problem with structured uncertainties.

So, we can give the theorem which ensures the exponential stability of the ROUF (3) and the $\mathcal{L}_2$ gain attenuation from w to e.

Theorem 4. Suppose that condition (21) is verified. If there exist matrices $P = P^T > 0$, $S_e > 0$, $S_w > 0$, Y and a scalar $\mu > 0$ such that ("$\bullet$" is the transpose of the off-diagonal part)

$$\begin{bmatrix} (a) & \bullet & \bullet & \bullet & \overline{H}_e^T S_e & 0 \\ \widetilde{\mathbb{A}}^T P - \widetilde{\mathbb{C}}^T Y & -S_e & 0 & 0 & 0 & 0 \\ \widetilde{\mathbb{B}}^T P - \widetilde{\mathbb{G}}^T Y & 0 & -S_w & 0 & 0 & 0 \\ \mathbb{B}^T P - \mathbb{G}^T Y & 0 & 0 & -\gamma^2 I_q & 0 & \overline{H}_w^T S_w \\ S_e \overline{H}_e & 0 & 0 & 0 & -S_e & 0 \\ 0 & 0 & 0 & S_w \overline{H}_w & 0 & -S_w \end{bmatrix} < 0. \quad (33)$$

with $(a) = \mathbb{A}^T P + P\mathbb{A} - \mathbb{C}^T Y - Y^T \mathbb{C} + (1+\mu) I_r$, $Y = Z^T P$ and where S_e and S_w are diagonal matrices with the same structure as $\Delta_e(\varepsilon)$ and $\Delta_w(t)$ respectively, then the ROUF (3) for the system (1) is exponentially stable and has a $\mathcal{L}_2$ gain from w to e less than or equal to γ. ∎

Proof. By considering system (31)-(32) as a diagonal norm-bound linear differential inclusion (Boyd *et al.*, 1994), the following auxiliary system derived from (31) (see (Li and Fu, 1997) for details, omitted because of lack of place)

$$\begin{cases} \dot{e} = (\mathbb{A} - Z\mathbb{C})\, e + \Big(\left[\widetilde{\mathbb{A}} S_e^{-1/2} \ \widetilde{\mathbb{B}} S_w^{-1/2} \,\vdots\, \gamma^{-1}\mathbb{B} \right] \\ \qquad - Z \left[\widetilde{\mathbb{C}} S_e^{-1/2} \ \widetilde{\mathbb{G}} S_w^{-1/2} \,\vdots\, \gamma^{-1}\mathbb{B} \right] \Big) \begin{bmatrix} p_e \\ p_w \\ \hline w \end{bmatrix} \\ \begin{bmatrix} q_e \\ q_w \\ \hline e \end{bmatrix} = \begin{bmatrix} S_e^{1/2} \overline{H}_e \\ 0 \\ \hline I_r \end{bmatrix} e + \left[\begin{array}{cc|c} 0 & 0 & 0 \\ 0 & 0 & \gamma^{-1} S_e^{1/2} \overline{H}_w \\ \hline 0 & 0 & 0 \end{array}\right] \begin{bmatrix} p_e \\ p_w \\ \hline w \end{bmatrix} \end{cases} \quad (34)$$

is introduced, where S_e and S_w have been defined in the theorem. Let $Y = Z^T P$. Then by using the bounded real lemma, system (31)-(32) is exponentially stable and has a $\mathcal{L}_2$ gain from w to e less than or equal to γ if there exist $P = P^T > 0$, $S_e > 0$, $S_w > 0$, Y and a scalar $\mu > 0$ such that matrices in system (34) satisfy this inequality

$$\begin{bmatrix} (a) & \bullet & \bullet & \bullet & \bullet & 0 & I_r \\ S_e^{-1/2} \widetilde{\mathbb{A}}^T P - S_e^{-1/2} \widetilde{\mathbb{C}}^T Y & -I_{mr} & 0 & 0 & 0 & 0 & 0 \\ S_w^{-1/2} \widetilde{\mathbb{B}}^T P - S_w^{-1/2} \widetilde{\mathbb{G}}^T Y & 0 & -I_{mq} & 0 & 0 & 0 & 0 \\ \gamma^{-1} \mathbb{B}^T P - \gamma^{-1} \mathbb{G}^T Y & 0 & 0 & -I_q & 0 & \bullet & 0 \\ S_e^{1/2} \overline{H}_e & 0 & 0 & 0 & -I_{mr} & 0 & 0 \\ 0 & 0 & 0 & (b) & 0 & -I_{mq} & 0 \\ I_r & 0 & 0 & 0 & 0 & 0 & -I_r \end{bmatrix} < 0$$

with $(b) = \gamma^{-1} S_w^{1/2} \overline{H}_w$. Pre- and post-multiply this inequality by $\mathrm{bdiag}(I_n, S_e^{1/2}, S_w^{1/2}, \gamma I_q, S_e^{1/2}, S_w^{1/2}, I_r)$, and use the Schur lemma to delete the last r rows and the last r columns, the LMI (33) is obtained. Assume that (21) is verified. Then using $Z = P^{-1} Y^T$ and (20), the matrices of ROUF (3) are given by (11) and (12). ❑

3. ROBUST REDUCED ORDER $\mathcal{H}_\infty$ FILTER

In this section, we will make reference to the following uncertain bilinear system

$$\dot{x} = \sum_{i=0}^{m} (A^i + \Delta_{A^i}(t)) u^i x + (B + \Delta_B(t)) w \quad (35a)$$
$$y = (C + \Delta_C(t)) x + (D + \Delta_D(t)) w \quad (35b)$$
$$z = Lx \quad (35c)$$

where $x(t)$, $y(t)$, $z(t)$, $w(t)$ and $u(t)$ have been defined in section 2. The matrices A^0, A^i, B, C, D and L are known constant matrices that describe the nominal system of (35) given by (1).

The uncertain matrices $\Delta_{A^0}(t)$, $\Delta_B(t)$, $\Delta_C(t)$, $\Delta_D(t)$ and $\Delta_{A^i}(t)$ can be written as

$$\begin{bmatrix} \Delta_{A^0}(t) & \Delta_B(t) \\ \Delta_C(t) & \Delta_D(t) \end{bmatrix} = \begin{bmatrix} M_x^0 \\ M_y^0 \end{bmatrix} \Delta^0(t) \begin{bmatrix} E_x^0 & E_w^0 \end{bmatrix} \quad (36a)$$

$$\begin{bmatrix} \Delta_{A^1}(t) & \dots & \Delta_{A^m}(t) \end{bmatrix} = \begin{bmatrix} M_x^1 & \dots & M_x^m \end{bmatrix} \times \underbrace{\mathrm{bdiag}(\Delta^i(t))}_{\Delta(t)} \begin{bmatrix} E_x^{1\,T} & \dots & E_x^{m\,T} \end{bmatrix}^T \quad (36b)$$

where $M_x^i \in \mathbb{R}^{n \times \ell_i}$, $M_y^0 \in \mathbb{R}^{p \times \ell_0}$, $E_x^i \in \mathbb{R}^{\ell_i \times n}$ and $E_w^0 \in \mathbb{R}^{\ell_0 \times q}$ $(i = 0, \dots, m)$ are known constant matrices which specify how the elements of the matrices of the nominal system are affected by the uncertain parameters in $\Delta^i(t)$, $(i = 0, \dots, m)$. The time-varying uncertainties in (36a) and (36b) are assumed to be structured and bounded, i.e. $\Delta^i(t)$ are diagonal matrices satisfying $\|\Delta^i(t)\| \leqslant I_{\ell_i}$.

For the same reasons explained in section 2.3, the constraint

$$E \begin{bmatrix} M_y^0 & D \end{bmatrix} = 0 \quad (37)$$

is used instead of $ED = 0$ in the sequel of section 3. Then (19)-(21) must be replaced by

$$\mathcal{K}\widehat{\Sigma} = \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix} \text{ with } \widehat{\Sigma} = \left[\begin{array}{c|c} M_y^0 & \\ 0 & \overline{\Sigma} \end{array}\right] \quad (38a)$$

$$\mathcal{K} = \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix} \widehat{\Sigma}^\dagger + Z(I_{(m+2)p} - \widehat{\Sigma}\widehat{\Sigma}^\dagger) \quad (38b)$$

$$\mathrm{rank}\, \widehat{\Sigma} = \mathrm{rank} \begin{bmatrix} 0 \ 0 \ L\overline{A} \\ \hline \widehat{\Sigma} \end{bmatrix}. \quad (38c)$$

Introducing the augmented state vector ξ given by

$$\xi^T = \begin{bmatrix} x^T & e^T \end{bmatrix}, \quad (39)$$

using (22) and the change of variables in (23)-(25), the system obtained by the concatenation of (3) and (35)-(36b) is given by

$$\begin{cases} \dot{\xi}(t) = \Bigg(\sum_{i=0}^{m} \Big(\begin{bmatrix} \alpha^i A^i & 0 \\ 0 & \alpha^i H^i \end{bmatrix} + \begin{bmatrix} \alpha^i M_x^i \\ \alpha^i \Psi M_x^i \end{bmatrix} \Delta^i(t) \begin{bmatrix} E_x^i & 0 \end{bmatrix} \Big) \\ \qquad + \begin{bmatrix} M_x^0 \\ \Psi M_x^0 - \sum_{i=0}^{m} \alpha^i J^i M_y^0 \end{bmatrix} \Delta^0(t) \begin{bmatrix} E_x^0 & 0 \end{bmatrix} \\ \quad + \sum_{i=1}^{m} \varepsilon^i \Big(\begin{bmatrix} \sigma^i A^i & 0 \\ 0 & \sigma^i H^i \end{bmatrix} + \begin{bmatrix} \sigma^i M_x^i \\ \sigma^i \Psi M_x^i \end{bmatrix} \Delta^i(t) \begin{bmatrix} E_x^i & 0 \end{bmatrix} \\ \quad - \begin{bmatrix} 0 \\ \sigma^i J^i M_y^0 \end{bmatrix} \Delta^0(t) \begin{bmatrix} E_x^0 & 0 \end{bmatrix} \Big) \Bigg) \xi + \Bigg(\begin{bmatrix} B \\ \Psi B - \sum_{i=0}^{m} \alpha^i J^i D \end{bmatrix} \\ \qquad + \begin{bmatrix} M_x^0 \\ \Psi M_x^0 - \sum_{i=0}^{m} \alpha^i J^i M_y^0 \end{bmatrix} \Delta^0(t) E_w^0 \\ \quad - \sum_{i=1}^{m} \Big(\varepsilon^i \begin{bmatrix} 0 \\ \sigma^i J^i D \end{bmatrix} + \begin{bmatrix} 0 \\ \sigma^i J^i M_y^0 \end{bmatrix} \varepsilon^i \Delta^0(t) E_w^0 \Big) \Bigg) w \\ e = \begin{bmatrix} 0 & I_r \end{bmatrix} \xi \end{cases} \quad (40)$$

where matrix Ψ satisfies the unbiasedness relation (8). Using (11), (12), (27)-(29) and (38a), the uncertain system (40) is equivalent to

$$
\left\{
\begin{aligned}
\dot{\xi}(t) &= \begin{bmatrix} \sum_{i=0}^{m} \alpha^i A^i & 0 \\ 0 & \mathbb{A}-Z\mathbb{C} \end{bmatrix} \xi + \begin{bmatrix} \overline{A}_\sigma & 0 & 0 \\ 0 & \widetilde{\mathbb{A}}-Z\widetilde{\mathbb{C}} & \widetilde{\mathbb{B}}-Z\widetilde{\mathbb{G}} \end{bmatrix} \begin{bmatrix} p_x \\ p_e \\ p_w \end{bmatrix} \\
&+ \begin{bmatrix} M_x^0 & \overline{M}_\alpha \\ \mathbb{B}_M - Z\mathbb{G}_M & \mathbb{B}_{\overline{M}_\alpha} - Z\mathbb{G}_{\overline{M}_\alpha} \end{bmatrix} \begin{bmatrix} p_\xi^0 \\ p_\xi \end{bmatrix} \\
&+ \begin{bmatrix} 0 & \overline{M}_\sigma \\ \widetilde{\mathbb{B}}_{\overline{M}_\sigma} - Z\widetilde{\mathbb{G}}_{\overline{M}_\sigma} & \mathbb{B}_{\overline{M}_\sigma} - Z\mathbb{G}_{\overline{M}_\sigma} \end{bmatrix} \begin{bmatrix} \overline{p}_\xi^0 \\ \overline{p}_\xi \end{bmatrix} + \begin{bmatrix} M_x^0 \\ \mathbb{B}_M - Z\mathbb{G}_M \end{bmatrix} p_w^0 \\
&+ \begin{bmatrix} 0 \\ \widetilde{\mathbb{B}}_{\overline{M}_\sigma} - Z\widetilde{\mathbb{G}}_{\overline{M}_\sigma} \end{bmatrix} \overline{p}_w^0 + \begin{bmatrix} B \\ \mathbb{B}-Z\mathbb{G} \end{bmatrix} w \\
\begin{bmatrix} q_x \\ q_e \\ q_w \\ \hline q_\xi^0 \\ q_\xi \\ q_w^0 \\ \hline \overline{q}_\xi^0 \\ \overline{q}_\xi \\ \overline{q}_w^0 \end{bmatrix} &= \begin{bmatrix} \overline{H}_x & 0 \\ 0 & \overline{H}_e \\ 0 & 0 \\ \hline E_x^0 & 0 \\ \overline{E}_x & 0 \\ 0 & 0 \\ \hline \overline{E}_x^0 & 0 \\ \overline{E}_x & 0 \\ 0 & 0 \end{bmatrix} \xi + \begin{bmatrix} 0 \\ 0 \\ \overline{H}_w \\ \hline 0 \\ 0 \\ E_w^0 \\ \hline 0 \\ 0 \\ \overline{E}_w^0 \end{bmatrix} w \\
e &= \begin{bmatrix} 0 & I_r \end{bmatrix} \xi
\end{aligned}
\right.
\tag{41}
$$

connected with

$$
\underbrace{\left[p_x^T \; p_e^T \; p_w^T \,\vdots\, p_\xi^{0\,T} \; p_\xi^T \; p_w^{0\,T} \,\vdots\, \overline{p}_\xi^{0\,T} \; \overline{p}_\xi^T \; \overline{p}_w^{0\,T} \right]^T}_{\widehat{p}} = \widehat{\Delta}(\varepsilon,t)
$$

$$
\times \underbrace{\left[q_x^T \; q_e^T \; q_w^T \,\vdots\, q_\xi^{0\,T} \; q_\xi^T \; q_w^{0\,T} \,\vdots\, \overline{q}_\xi^{0\,T} \; \overline{q}_\xi^T \; \overline{q}_w^{0\,T} \right]^T}_{\widehat{q}} \tag{42}
$$

with $\widehat{\Delta}(\varepsilon,t) = \mathrm{bdiag}(\Delta_x(\varepsilon), \Delta_e(\varepsilon), \Delta_w(\varepsilon), \Delta^0(t), \Delta(t), \Delta^0(t), \overline{\Delta}^0(\varepsilon,t), \overline{\Delta}(\varepsilon,t), \overline{\Delta}^0(\varepsilon,t))$ and where the matrices which have not yet been defined are given by (with $\overline{\ell} = \ell_1 + \ldots + \ell_m$)

$$
\begin{aligned}
\overline{A}_\sigma &= \begin{bmatrix} \sigma^1 A^1 & \cdots & \sigma^m A^m \end{bmatrix}, && (43a) \\
\overline{M}_\alpha &= \begin{bmatrix} \alpha^1 M_x^1 & \cdots & \alpha^m M_x^m \end{bmatrix}, && (43b) \\
\overline{M}_\sigma &= \begin{bmatrix} \sigma^1 M_x^1 & \cdots & \sigma^m M_x^m \end{bmatrix}, && (43c) \\
\mathbb{B}_M &= L M_x^0 - \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix} \widehat{\Sigma}^\dagger \Upsilon_M, && (43d) \\
\mathbb{G}_M &= (I_{(m+2)p} - \widehat{\Sigma}\widehat{\Sigma}^\dagger)\Upsilon_M, && (43e) \\
\mathbb{B}_{\overline{M}_\alpha} &= L\overline{M}_\alpha - \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix} \widehat{\Sigma}^\dagger \Upsilon_{\overline{M}_\alpha}, && (43f) \\
\mathbb{G}_{\overline{M}_\alpha} &= (I_{(m+2)p} - \widehat{\Sigma}\widehat{\Sigma}^\dagger)\Upsilon_{\overline{M}_\alpha}, && (43g)
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{B}_{\overline{M}_\sigma} &= L\overline{M}_\sigma - \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix} \widehat{\Sigma}^\dagger \Upsilon_{\overline{M}_\sigma}, && (44a) \\
\mathbb{G}_{\overline{M}_\sigma} &= (I_{(m+2)p} - \widehat{\Sigma}\widehat{\Sigma}^\dagger)\Upsilon_{\overline{M}_\sigma}, && (44b) \\
\widetilde{\mathbb{B}}_{\overline{M}_\sigma} &= - \begin{bmatrix} 0 & 0 & L\overline{A} \end{bmatrix} \widehat{\Sigma}^\dagger \mathbb{M}, && (44c) \\
\widetilde{\mathbb{G}}_{\overline{M}_\sigma} &= (I_{(m+2)p} - \widehat{\Sigma}\widehat{\Sigma}^\dagger)\mathbb{M}. && (44d) \\
\overline{H}_x &= \begin{bmatrix} I_n & \ldots & I_n \end{bmatrix}^T \in \mathbb{R}^{mn\times n} && (44e) \\
\overline{E}_x^0 &= \begin{bmatrix} E_x^{0\,T} & \ldots & E_x^{0\,T} \end{bmatrix}^T \in \mathbb{R}^{m\ell_0\times n} && (44f) \\
\overline{E}_x &= \begin{bmatrix} E_x^{1\,T} & \ldots & E_x^{m\,T} \end{bmatrix}^T \in \mathbb{R}^{\overline{\ell}\times n} && (44g) \\
\overline{E}_w^0 &= \begin{bmatrix} E_w^{0\,T} & \ldots & E_w^{0\,T} \end{bmatrix}^T \in \mathbb{R}^{m\ell_0\times q} && (44h) \\
\Delta_x(\varepsilon) &= \mathrm{bdiag}(\varepsilon^1 I_n, \ldots, \varepsilon^m I_n) && (44i) \\
\overline{\Delta}^0(\varepsilon,t) &= \mathrm{bdiag}(\varepsilon^1 \Delta^0(t), \ldots, \varepsilon^m \Delta^0(t)) && (44j) \\
\overline{\Delta}(\varepsilon,t) &= \mathrm{bdiag}(\varepsilon^1 \Delta^1(t), \ldots, \varepsilon^m \Delta^m(t)) && (44k)
\end{aligned}
$$

with

$$
\Upsilon_M = \begin{bmatrix} CM_x^0 \\ \alpha^0 M_y^0 \\ \vdots \\ \alpha^m M_y^0 \end{bmatrix}, \Upsilon_{\overline{M}_\alpha} = \begin{bmatrix} C\overline{M}_\alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{45a}
$$

$$
\Upsilon_{\overline{M}_\sigma} = \begin{bmatrix} C\overline{M}_\sigma \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbb{M} = \begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \sigma^1 M_y^0 & \ddots & \vdots \\ \ddots & \ddots & 0 \\ \cdots & 0 & \sigma^m M_y^0 \end{bmatrix} \tag{45b}
$$

Notice that, from (25) and the definition of $\Delta^i(t)$, we have the following bounds

$$
\|\Delta_x(\varepsilon)\| \leqslant 1, \; \left\|\overline{\Delta}^0(\varepsilon,t)\right\| \leqslant 1, \; \left\|\overline{\Delta}(\varepsilon,t)\right\| \leqslant 1. \tag{46}
$$

Note that the unbiasedness condition (8) for the filter (3) is verified for the nominal case, i.e. for $M_x^i = 0$, $(i = 0,\ldots,m)$ and $M_y^0 = 0$.

Define a block-diagonal matrix $\mathbf{S} > 0$ with the same structure as the uncertain matrix $\widehat{\Delta}(\varepsilon,t)$ given in (42)

$$
\mathbf{S} = \mathrm{bdiag}(S_x, S_e, S_w, S_\Delta^0, S_\Delta, S_\Delta^0, \overline{S}_\Delta^0, \overline{S}_\Delta, \overline{S}_\Delta^0). \tag{47}
$$

where all the submatrices are diagonal. Then, in the system (41), the determination of gain Z can be transformed into this robust static output feedback control problem (see (Li and Fu, 1997) for details, omitted because of lack of place)

$$
\left\{
\begin{aligned}
\dot{\xi} &= \mathbf{A}\xi + \overline{\mathbf{B}}_w \begin{bmatrix} \widehat{p} \\ w \end{bmatrix} + \mathbf{B}_u \overline{u} \\
\begin{bmatrix} \widehat{q} \\ e \end{bmatrix} &= \overline{\mathbf{C}}_z \xi + \overline{\mathbf{D}}_{zw} \begin{bmatrix} \widehat{p} \\ w \end{bmatrix} \\
\overline{y} &= \mathbf{C}_y \xi + \overline{\mathbf{D}}_{yw} \begin{bmatrix} \widehat{p} \\ w \end{bmatrix}
\end{aligned}
\right. \tag{48a}
$$

$$
\overline{u} = -Z\,\overline{y} \tag{48b}
$$

where Z is the static output feedback controller to be designed in order to achieve stability and performances like attenuation from the "augmented perturbation" $[\widehat{p}^T \; w^T]^T$ to the "augmented controlled output" $[\widehat{q}^T \; e^T]^T$. The vectors $\overline{u}$ and $\overline{y}$ play the role of "control input" and "measured output", respectively. The matrices and the vectors introduced in (48) are given by

$$
\overline{\mathbf{B}}_w = \begin{bmatrix} \mathbf{B}_p \mathbf{S}^{-1/2} & \gamma^{-1}\mathbf{B}_w \end{bmatrix}, \overline{\mathbf{C}}_z = \begin{bmatrix} \mathbf{S}^{1/2}\mathbf{C}_q \\ \mathbf{C}_e \end{bmatrix} \tag{49a}
$$

$$
\overline{\mathbf{D}}_{zw} = \begin{bmatrix} \mathbf{S}^{1/2}\mathbf{D}_{qp}\mathbf{S}^{-1/2} & \gamma^{-1}\mathbf{S}^{1/2}\mathbf{D}_{qw} \\ \mathbf{D}_{ep}\mathbf{S}^{-1/2} & \gamma^{-1}\mathbf{D}_{ew} \end{bmatrix} \tag{49b}
$$

$$
\overline{\mathbf{D}}_{yw} = \begin{bmatrix} \mathbf{D}_{yp}\mathbf{S}^{-1/2} & \gamma^{-1}\mathbf{D}_{yw} \end{bmatrix}, \tag{49c}
$$

where

$$
\mathbf{A} = \begin{bmatrix} \sum_{i=0}^{m} \alpha^i A^i & 0 \\ 0 & \mathbb{A} \end{bmatrix}, \mathbf{B}_u = \begin{bmatrix} 0 \\ I_r \end{bmatrix}, \mathbf{C}_y = \begin{bmatrix} 0 & \mathbb{C} \end{bmatrix}
$$

$$
\begin{bmatrix} \mathbf{B}_p & \mathbf{B}_w \end{bmatrix} =
$$

$$
\left[\begin{array}{ccccccccc|c} \overline{A}_\sigma & 0 & 0 & M_x^0 & \overline{M}_\alpha & 0 & \overline{M}_\sigma & M_x^0 & 0 & B \\ 0 & \widetilde{\mathbb{A}} & \widetilde{\mathbb{B}} & \mathbb{B}_M & \mathbb{B}_{\overline{M}_\alpha} & \widetilde{\mathbb{B}}_{\overline{M}_\sigma} & \mathbb{B}_{\overline{M}_\sigma} & \mathbb{B}_M & \widetilde{\mathbb{B}}_{\overline{M}_\sigma} & \mathbb{B} \end{array}\right]
$$

$$
\begin{bmatrix} \mathbf{C}_q \\ \mathbf{C}_e \end{bmatrix}^T = \left[\begin{array}{ccccccccc|c} \overline{H}_x^T & 0 & 0 & E_x^{0\,T} & \overline{E}_x^T & 0 & \overline{E}_x^{0\,T} & \overline{E}_x^T & 0 & 0 \\ \hline 0 & \overline{H}_e^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_r \end{array}\right]
$$

$$
\begin{bmatrix} \mathbf{D}_{qp} & \mathbf{D}_{qw} \\ \mathbf{D}_{ep} & \mathbf{D}_{ew} \end{bmatrix} = \left[\begin{array}{ccccccccc:c}
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&\overline{H}_w \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&E_w^0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&0 \\
0&0&0&0&0&0&0&0&0&\overline{E}_w^0 \\ \hdashline
0&0&0&0&0&0&0&0&0&0
\end{array}\right]
$$

$$
\begin{bmatrix} \mathbf{D}_{yp} & \mathbf{D}_{yw} \end{bmatrix} =
$$

$$
\left[\begin{array}{cccccccccc:c} 0 & \widetilde{\mathbb{C}} & \widetilde{\mathbb{G}} & \mathbb{G}_M & \mathbb{G}_{\overline{M}_\alpha} & \widetilde{\mathbb{G}}_{\overline{M}_\sigma} & \mathbb{G}_{\overline{M}_\sigma} & \mathbb{G}_M & \widetilde{\mathbb{G}}_{\overline{M}_\sigma} & & \mathbb{G} \end{array}\right].
$$

Then the following theorem is dedicated to the design of the gain Z.

Theorem 5. Assume that the rank condition in (38) holds, there exists a robust ROUF (3) for the uncertain system (35) if there exist matrices $\mathbf{P} = \mathbf{P}^T > 0$, $\mathbf{Q} = \mathbf{Q}^T > 0$ and diagonal matrices $\mathbf{S} > 0$, $\overline{\mathbf{S}} > 0$, such that (with $\overline{\ell} = \ell_1 + \ldots + \ell_m$, $\overline{s} = mn + mr + mq + 2\ell_0 + 2\overline{\ell} + 2m\ell_0$)

$$
\overline{\overline{\mathbf{K}}}_y^T \begin{bmatrix} \mathbf{A}^T\mathbf{P}+\mathbf{P}\mathbf{A} & \mathbf{P}\mathbf{B}_w & \mathbf{P}\mathbf{B}_p & \mathbf{C}_e^T & \mathbf{C}_q^T\mathbf{S} \\ \mathbf{B}_w^T\mathbf{P} & -\gamma^2 I_q & 0 & \mathbf{D}_{ew}^T & \mathbf{D}_{qw}^T\mathbf{S} \\ \mathbf{B}_p^T\mathbf{P} & 0 & -\mathbf{S} & \mathbf{D}_{ep}^T & \mathbf{D}_{qp}^T\mathbf{S} \\ \mathbf{C}_e & \mathbf{D}_{ew} & \mathbf{D}_{ep} & -I_r & 0 \\ \mathbf{S}\mathbf{C}_q & \mathbf{S}\mathbf{D}_{qw} & \mathbf{S}\mathbf{D}_{qp} & 0 & -\mathbf{S} \end{bmatrix} \overline{\overline{\mathbf{K}}}_y < 0 \qquad (50a)
$$

$$
\overline{\overline{\mathbf{K}}}_u^T \begin{bmatrix} \mathbf{Q}\mathbf{A}^T+\mathbf{A}\mathbf{Q} & \mathbf{Q}\mathbf{C}_e^T & \mathbf{Q}\mathbf{C}_q^T & \mathbf{B}_w & \mathbf{B}_p\overline{\mathbf{S}} \\ \mathbf{C}_e\mathbf{Q} & -I_r & 0 & \mathbf{D}_{ew} & \mathbf{D}_{ep}\overline{\mathbf{S}} \\ \mathbf{C}_q\mathbf{Q} & 0 & -\overline{\mathbf{S}} & \mathbf{D}_{qw} & \mathbf{D}_{qp}\overline{\mathbf{S}} \\ \mathbf{B}_w^T & \mathbf{D}_{ew}^T & \mathbf{D}_{qw}^T & -\gamma^2 I_q & 0 \\ \overline{\mathbf{S}}\mathbf{B}_p^T & \overline{\mathbf{S}}\mathbf{D}_{ep}^T & \overline{\mathbf{S}}\mathbf{D}_{qp}^T & 0 & -\overline{\mathbf{S}} \end{bmatrix} \overline{\overline{\mathbf{K}}}_u < 0 \qquad (50b)
$$

$$
I_{n+r} = \mathbf{P}\mathbf{Q} \qquad (50c)
$$

where $\overline{\mathbf{S}} = \mathbf{S}^{-1}$, $\overline{\overline{\mathbf{K}}}_y = \mathrm{bdiag}(\overline{\mathbf{K}}_y, I_{r+\overline{s}})$ and $\overline{\overline{\mathbf{K}}}_u = \mathrm{bdiag}(\overline{\mathbf{K}}_u, I_{q+\overline{s}})$ such that $\overline{\mathbf{K}}_y$ and $\overline{\mathbf{K}}_u$ are basis of the null spaces of $[\, \mathbf{C}_y \; \mathbf{D}_{yp} \; \mathbf{D}_{yw} \,]$ and $[\, \mathbf{B}_u^T \; 0 \,]$, respectively. All gains Z are given by

$$
Z = \overline{\mathbf{B}}_R^\dagger \mathbf{K} \overline{\mathbf{C}}_L^\dagger + \mathbf{Z} - \overline{\mathbf{B}}_R^\dagger \overline{\mathbf{B}}_R \mathbf{Z} \overline{\mathbf{C}}_L \overline{\mathbf{C}}_L^\dagger \qquad (51)
$$

with

$$
\mathbf{K} = \mathbf{R}_1^{-1}\mathbf{V}_2^{1/2}\mathbf{R}_2\left(\overline{\mathbf{C}}_R\mathbf{V}_1\overline{\mathbf{C}}_R^T\right)^{\frac{-1}{2}} - \mathbf{R}_1^{-1}\overline{\mathbf{B}}_L^T\mathbf{V}_1\overline{\mathbf{C}}_R^T\left(\overline{\mathbf{C}}_R\mathbf{V}_1\overline{\mathbf{C}}_R^T\right)^{-1}
$$

$$
\mathbf{V}_1 = \left(\overline{\mathbf{B}}_L\mathbf{R}_1^{-1}\overline{\mathbf{B}}_L^\dagger - \overline{\mathbf{Q}}\right)^{-1} > 0
$$

$$
\mathbf{V}_2 = \mathbf{R}_1 - \overline{\mathbf{B}}_L^T\left(\mathbf{V}_1 - \mathbf{V}_1\overline{\mathbf{C}}_R^T\left(\overline{\mathbf{C}}_R\mathbf{V}_1\overline{\mathbf{C}}_R^T\right)^{-1}\overline{\mathbf{C}}_R\mathbf{V}_1\right)\overline{\mathbf{B}}_L
$$

$$
\begin{bmatrix} \overline{\mathbf{B}} & \overline{\mathbf{Q}} \\ \bullet & \overline{\mathbf{C}} \end{bmatrix} = \left[\begin{array}{c:cccccc} -\mathbf{B}_u & \mathbf{Q}\mathbf{A}^T+\mathbf{A}\mathbf{Q} & \mathbf{Q}\mathbf{C}_e^T & \mathbf{Q}\mathbf{C}_q^T & \mathbf{B}_w & \mathbf{B}_p\overline{\mathbf{S}} \\ 0 & \mathbf{C}_e\mathbf{Q} & -I_r & 0 & \mathbf{D}_{ew} & \mathbf{D}_{ep}\overline{\mathbf{S}} \\ 0 & \mathbf{C}_q\mathbf{Q} & 0 & -\overline{\mathbf{S}} & \mathbf{D}_{qw} & \mathbf{D}_{qp}\overline{\mathbf{S}} \\ 0 & \mathbf{B}_w^T & \mathbf{D}_{ew}^T & \mathbf{D}_{qw}^T & -\gamma^2 I_q & 0 \\ 0 & \overline{\mathbf{S}}\mathbf{B}_p^T & \overline{\mathbf{S}}\mathbf{D}_{ep}^T & \overline{\mathbf{S}}\mathbf{D}_{qp}^T & 0 & -\overline{\mathbf{S}} \\ \hdashline \bullet & \mathbf{C}_y\mathbf{Q} & \overline{\mathbf{S}}\mathbf{D}_{yp} & \mathbf{D}_{yw} & 0 & 0 \end{array}\right]
$$

and $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{Z}$ are arbitrary matrices satisfying $\mathbf{R}_1 = \mathbf{R}_1^T > 0$ and $\|\mathbf{R}_2\| < 1$. Matrices $\overline{\mathbf{B}}_L$, $\overline{\mathbf{B}}_R$, $\overline{\mathbf{C}}_L$ and $\overline{\mathbf{C}}_R$ are any full rank factors such that $\overline{\mathbf{B}} = \overline{\mathbf{B}}_L\overline{\mathbf{B}}_R$ and $\overline{\mathbf{C}} = \overline{\mathbf{C}}_L\overline{\mathbf{C}}_R$. ■

Proof. By using the bounded real lemma, the theorem can be proven using the projection lemma and formulas given in (Iwasaki and Skelton, 1994). Then the robust ROUF (3) is finally obtained by using relations (11), (12) and (38). ❑

Remark 6. Unlike the robust ROUF case which has been solved above as a static output feedback, notice that the full order robust filtering problem can be transformed into a full order dynamic output feedback problem and is then solvable via LMI only (Li and Fu, 1997; Bittanti and Cuzzola, 2000). Then, in full order robust filtering, BMI (3.16c) becomes $\left[\begin{smallmatrix} \mathbf{P} & I_n \\ I_n & \mathbf{Q} \end{smallmatrix}\right] > 0$. That is the main difference with the reduced order filtering where there is an additional non convex constraint (BMI). There is no efficient algorithm to solve this non convex problem which can only have local solutions by means of heuristics such as the cone complementary linearization (El Ghaoui *et al.*, 1997). ■

4. CONCLUSION

This paper has presented a simple solution to the $\mathcal{H}_\infty$ ROUF problem via LMI methods for bilinear systems. After giving conditions for the existence of the ROUF, it is shown that the ROUF design is reduced to a robust state feedback problem in the nominal case and to a static robust output feedback one when the bilinear system is affected by the structured norm-bounded time-varying uncertainties; then there are an additional non convex relation to solve.

5. REFERENCES

Bittanti, S. and F.A. Cuzzola (2000). Unbiased robust $\mathcal{H}_\infty$ filtering by means of LMI optimisation. In: *Proc. IFAC Symposium on Robust Control Design*. Prague, Czech Republic.

Boyd, S.P., L. El Ghaoui, E. Féron and V. Balakrishnan (1994). *Linear Matrix Inequality in Systems and Control Theory*. SIAM. Philadelphia.

El Ghaoui, L., F. Oustry and M. Aït Rami (1997). A cone complementary linearization algorithm for static output-feedback and related problems. *IEEE Trans. Aut. Contr.* **42**, 1171–1176.

Iwasaki, T. and R.E. Skelton (1994). All controllers for the general $\mathcal{H}_\infty$ control problems : LMI existence conditions and state space formulas. *Automatica* **30**, 1307–1317.

Li, H. and M. Fu (1997). A linear matrix inequality approach to robust $\mathcal{H}_\infty$ filtering. *IEEE Trans. Sign. Proc.* **45**, 2338–2350.

van der Schaft, A.J. (1992). $\mathcal{L}_2$-gain analysis of nonlinear systems and nonlinear state-feedback $\mathcal{H}_\infty$ control. *IEEE Trans. Aut. Contr.* **37**, 770–784.