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CONTROLLED SYNCHRONIZATION OF TWO 1-DOF COUPLED OSCILLATORS

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Abstract: Two 1-DOF pendulums coupled with a weak spring are considered. This system is a coupled Controlled Hamiltonian Systems that has been studied widely in these days. The control objective is to make pendulums swing synchronously with small input. To this end, Speed Gradient Energy method proposed by Fradkov is adopted to design the controller. Although experimental results showed that the method succeeded in achieving the objective, the mechanism of synchronization was not clear. In this study, the contracted dynamics of the whole system is analyzed and properties of the system are investigated. Through the investigation, a criteria to define the feedback gain of the controller is revealed. Computer simulation and experimental results showed the validity of the method.

Keywords: Synchronization, Energy Control, Nonlinear Systems, Oscillations, Stability

1. INTRODUCTION

Rhythmic phenomena such as oscillation of pendulums, chemical oscillations or circadian rhythm have attracted much interest for a long time. When more than two oscillators are interacted each other, it is known that they may oscillate in same frequency even if their eigen frequencies are different. This mechanism is called ‘synchronization’ or ‘entrainment’, and plays very important role in many applications. For example, since the period of human’s circadian rhythm is not exactly as long as that of earth’s rotation, human’s internal clock may shift from the motion of the sun without any synchronizing mechanism. Phase Locked Loop circuit(Gardner, 1979) is a widely used application of synchronization. Synchronizing chaotic oscillators is applied to secure communication(Wu and Chua, 1993). Author(Kumon and Adachi, 1998) also proposed a path-following control technique for manipulators based on the concept of synchronization.

In this study, two 1-DOF pendulums coupled with a weak spring are considered. Only one of pendulums is actuated to oscillate pendulums synchronously. Fradkov(Boris and Fradkov, 1999) succeeded in oscillating linear pendulums with a controller designed by Speed Gradient Energy method (Fradkov and Pogromsky, 1998; Fradkov, 1999). In the absence of friction, the controller is able to achieve the control objective with arbitrary small input. Fradkov also considered energy consumption caused by frictional forces to design the feedback system. In order to swing the unactuated pendulum in the existence of frictional force, energy is required to be transmitted by the spring attached between pendulums. This fact leads that states of pendulums do not coincide rigorously when frictional forces exist. If one of the pendulums oscillates in slightly different period than the other, two oscillators may not synchronize. Therefore, further investigation of the system is still required. In this paper, the dynamics of the difference between states, which is called ‘phase shift’, is modeled by phase model which

Table 1. Parameters

$m_{1,2}$	$l_{1,2}$	l_0	k
0.3[kg]	0.12[m]	0.02[m]	80[N/m]

was proposed by Kuramoto(Kuramoto, 1984). Relation between the phase shift and frictional forces is investigated numerically. Using this information of the phase shift, more precise computation of the feedback gain can be shown.

The system considered here is a controlled Hamiltonian system that has been studied vigorously. Since Hamiltonian system is a very general form of mechanical systems, the method shown below is applicable to large class.

In Section 2, the dynamical model of the system is shown. In Section 3, the control objectives are defined and the structure of the controller is shown. Section 4 is the main result of this study. Phase model(Kuramoto, 1984) is adopted in order to compute the contracted dynamics of the system. Based on phase model, phase shift, energy of each pendulums and other properties are analyzed. In order to show the validity of the result, computer simulation and experimental results are shown in Section 5. Since the result of the analysis agrees well with results of the simulation and experiment, the efficiency of this study is shown. A brief summary is given in Section 6.

2. COUPLED OSCILLATORS

Two pendulums coupled by a weak spring which is considered in this study are shown in Figure 1. A motor is attached to the left pendulum(pendulum 1) and the joint of the right pendulum(pendulum 2) is passive. The control input generated by the motor is assumed to be small. Values of all physical parameters of the system are shown in Table 1.

The dynamics of the system can be written as follows:

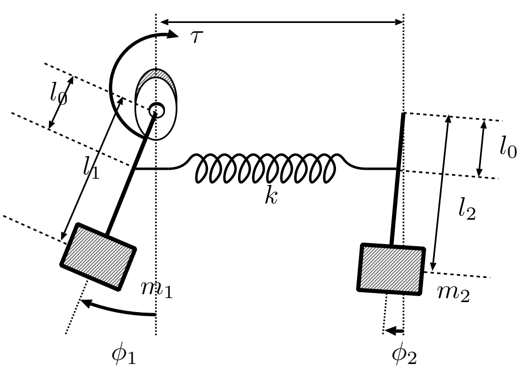

Fig. 1. Coupled Pendulums

$$J_1\ddot{\phi}_1 + \rho_1\dot{\phi}_1 + m_1gl_1\sin\phi_1 = f(\phi_1,\phi_2)\cos\phi_1 + \tau$$
$$J_2\ddot{\phi}_2 + \rho_2\dot{\phi}_2 + m_2gl_2\sin\phi_2 = -f(\phi_1,\phi_2)\cos\phi_2$$
$$f(\phi_1,\phi_2) = -kl_0^2(\sin\phi_1 - \sin\phi_2), \quad (1)$$

where $J_{1,2}$, g and $\rho_{1,2}$ represent inertia, gravity coefficient and frictional coefficients respectively.

Total energy of the system, denoted by H[J], is defined as

$$H = H_1 + H_2 + H_k,$$

where

$$H_i = \frac{1}{2}J_i\dot{\phi_i}^2 + m_igl_i(1-\cos(\phi_i)) \quad (i = 1, 2)$$
$$H_k = \frac{kl_0^2}{2}(\sin\phi_1 - \sin\phi_2)^2.$$

H_i represents energy of oscillator $i(i = 1, 2)$ and H_k represents that of the spring. Since the input τ and the interaction are assumed to be small, motion of two pendulums can be well approximated by independent oscillators. This implies that H_i is a candidate of the constant of motion. In other words, the above system is a weakly coupled controlled Hamiltonian systems with small input. By means of energy control, therefore, motions of oscillators can be controlled.

Although it may be seemed that a very special system is considered in this study, controlled Hamiltonian system is a very common mechanical system and the discussion below is expected to be applicable to large class of systems.

3. CONTROL SCHEME

Following Fradkov's study(Boris and Fradkov, 1999), the control objectives, *excitation* and *synchronization*, are modeled. The first objective is required to make two pendulums swing. Since the energy H demonstrates the motion of the system, the system is excited by achieving the following objective:

(O1) $H(t) \to H_*$ as $t \to \infty$,

where H_* represents the desired energy level.

The second objective is required to synchronize two pendulums. In this study frequencies of eigen oscillation are same because physical parameters of two pendulums are same. Therefore, it is natural to make angular velocities of pendulums coincide in order to synchronize pendulums. To this end, the second objective is denoted mathematically as follows:

(O2) $\dot{\phi}_1(t) - \dot{\phi}_2(t) \to 0$ as $t \to \infty$,

Combining (O1) and (O2), the objective function $Q(t)$ can be given by

$$Q(t) = \alpha(H(t) - H_*)^2 + (1-\alpha)(\dot{\phi}_1(t) - \dot{\phi}_2(t))^2, \quad (2)$$

where $\alpha \in (0,1)$ is a positive constant, and the objective is to make $Q(t) \to 0$ as $t \to \infty$.

Fradkov(Fradkov and Pogromsky, 1998) proposed Speed Gradient Energy method(SGE method in short) and applied the method to linear coupled oscillators with the same control objective(Boris and Fradkov, 1999). As far as (1)'s nonlinearity is negligible, his approach is expected to be applicable. Now, the controller is designed by using SGE method:

$$\tau = -\gamma \Big[\alpha(H(t) - H_*)\dot{\phi}_1 + (1-\alpha)(\dot{\phi}_1(t) - \dot{\phi}_2(t))\Big], \quad (3)$$

where γ is a small positive constant. When $\rho_{1,2} = 0$ and $\alpha = 1$, $H \to H_*$ can be achieved by any small input, that is desirable since the actuator can be small and cheap.

Unfortunately the control goal $Q(\infty) = 0$ cannot be achieved since some of assumptions(Fradkov and Pogromsky, 1998; Shiriaev and Fradkov, 1998) to ensure the convergence of Q to 0 is violated. Although Q can not be converged to 0, the robustness of the controller keeps Q to be small enough after enough time passes. But this fact is not sufficient to ensure synchronization because the difference between ϕ_is may grow large even if angular velocities $\dot{\phi}_i$ differ slightly.

However, using this control law, experimental results showed that controlled synchronization of the system (1) was succeded. In the following section, phase model will be adopted in order to analyze the mechanism of controlled synchronization.

4. SYNCHRONIZATION

4.1 *Phase Model*

Kuramoto(Kuramoto, 1984; Kuramoto, 1991) proposed phase model and studied synchronization of nonlinear oscillators. He contracted the dynamics of an oscillator to only one dimensional first order differential equation in order to simplify the interaction among oscillators.

Recall that frictional forces and interaction are assumed to be small. Since the energy of the system converges to some value, oscillator i is attracted to a circular limit cycle in $\omega_i\phi_i, \dot{\phi}_i$ plane, where ω_i represents approximated eigen angular frequency given by $\omega_i = \sqrt{\frac{m_i g l_i}{J_i}}$. Therfore, only one variable is needed to denote the state of the oscillator as far as the state of oscillator is near the limit cycle. Let $\theta_i \in S^1$ be the variable which is defined as follows:

$$\theta_i = -\text{Atan2}(\omega_i\phi_i, \dot{\phi}_i). \quad (4)$$

Conversely, ϕ_i and $\dot{\phi}_i$ can be approximated as a function of θ_i.

$$\phi_i(\theta_i) \approx a_i \cos(\theta_i), \quad (5)$$
$$\dot{\phi}_i = \pm\sqrt{\frac{2m_i g l_i}{J_i}\{\cos(\phi_i(\theta_i)) - \cos(a_i)\}}$$
$$(+\text{: } \theta_i \in (-\pi, 0],\ -\text{: otherwise}),$$

where a_i is the amplitude of the oscillation.

Now the dynamics of θ_i can be computed from the dynamics of ϕ_i.

$$\dot{\theta}_i = \text{grad}_{(\phi_i,\dot{\phi}_i)}\theta_i \cdot \frac{d}{dt}\begin{pmatrix}\phi_i \\ \dot{\phi}_i\end{pmatrix} \quad (6)$$

Substituting (1)(3)(5) into (6), the dynamics of θ_i can be obtained in the closed form with respect to $\theta_{1,2}$. Denote this dynamics as follow:

$$\dot{\theta}_i = \Theta_i(\theta_i, \theta_j) \quad (i,j = 1,2 \quad i \neq j). \quad (7)$$

4.2 *Amplitude of Oscillation*

Now the amplitude of the oscillation a_i is considered. Because the amplitude of oscillator i is a function of the energy H_i H_i is studied in order to compute a_i.

Since the oscillation is well approximated as $\phi_i \approx a_i \cos(\omega_i t)$ and $\dot{\phi}_i \approx -a_i\omega_i \sin(\omega_i t)$, the period of the oscillation, denoted by T_i, can be approximated as $T_i \approx \frac{2\pi}{\omega_i}$. The energy dissipated by frictional forces in one cycle, that is denoted by H_{ρ_i}, can be computed as follows:

$$H_{\rho_i} = \oint \rho_i\dot{\phi}_i d\phi_i = \int_{t_0}^{t_0+T} \rho_i\dot{\phi}_i^{\,2} dt = \rho_i\pi\omega_i a_i^2. \quad (8)$$

Next, assume that the difference between θ_1 and θ_2 is constant, that is, $\theta_2 = \theta_1 + d$, where $d \in [0, \pi)$. The difference d is called 'phase shift' in this paper. Also assume that $a_{1,2}$ are small. Then the following approximation is valid.

$$\sin(\phi_1) = \sin(a_1\cos(\theta_1)) \approx a_1\cos(\theta_1),$$
$$\sin(\phi_2) \approx \sin(a_2\cos(\theta_1 + d)) \approx a_2\cos(\theta_1 + d).$$

Denote the energy transported from the oscillator 1 to 2 in one oscillating cycle of oscillator 1 as H_{12}. H_{12} can be computed as:

$$H_{12} = \oint kl_0 \{\sin(\phi_1) - \sin(\phi_2)\} l_0 d\phi_1$$
$$= kl_0^2 \int_0^{2\pi} \{\sin(a_1\cos(\theta_1)) - \sin(a_2\cos(\theta_1 + d))\} \times \frac{d}{d\theta_1}(a_1\cos(\theta_1))d\theta_1$$
$$= -k_0 l_0 a_1 a_2 \pi \sin(d). \tag{9}$$

Since only the force by the spring injects the energy to the pendulum 2, H_{12} is required to be equal to H_{ρ_2} in order to sustain the oscillation of oscillator 2. By (8) and (9), this requirement leads the following relation:

$$a_2 = \frac{k_0 l_0^2 \sin(d)}{\rho_2\omega_2} a_1. \tag{10}$$

Remark 1. Since the energy transmits only from pendulum 1 to 2, d is limited to $[0, \pi)$.

Remark 2. When the pendulum 2 has no friction i.e. $\rho_2 = 0$, (10) can not be computed. However both a_1 and a_2 should be bounded since the total energy of the system is well controlled. This implies that $\dfrac{\sin(d)}{\rho_2}$ must be bounded and that $d = 0$ when $\rho_2 = 0$.

Because of frictional forces, $H(t)$ may not converge to H_* but to another value. Let H'_* be the value to which $H(t)$ converges and let H_u be the energy that is injected by the controller in one oscillating cycle.

$$H_u = \oint -\alpha\gamma(H'_* - H_*)\dot{\phi}_1 d\phi$$
$$= -\alpha\gamma(H'_* - H_*) \int_{t_0}^{T+t_0} a_1^2\omega_1^2 \sin(\omega_1 t)dt$$
$$= \alpha\gamma(H'_* - H_*)a_1^2\omega_1^2\pi \tag{11}$$

The injected energy H_u must be equal to the consumed energy $H_{\rho_1} + H_{\rho_2}$ in order to sustain the oscillations . Therefore

$$\alpha\gamma\Delta H a_1^2\omega_1^2\pi = \sum_{i=1,2} \rho_1\pi\omega_i a_i^2,$$

where $\Delta H = H'_* - H_*$. This leads

$$\Delta H = \frac{\rho_1\omega_1 + \rho_2\omega_2 A^2}{\alpha\gamma\omega_1}, \tag{12}$$

where $A = \frac{k_0 l_0^2 \sin(d)}{\rho_2\omega_2}$. Because the interaction is small, the energy stored by the spring can be neglected. Then,

$$H'_* = H - \Delta H \tag{13}$$
$$= m_1 g l_1(1 - \cos(a_1)) + m_2 g l_2(1 - \cos(a_2))$$
$$= c_1 - g(m_1 l_1\cos(a_1) + m_2 l_2\cos(Aa_1)),$$
$$\approx c_1 - g\left\{m_1 l_1(1 - \frac{a_1^2}{2}) + m_2 l_2(1 - \frac{(Aa_1)^2}{2})\right\}$$
$$= \frac{1}{2}g(m_1 l_1 + m_2 l_2 A^2)a_1^2$$

where $c_1 = g(m_1 l_1 + m_2 l_2)$. By (13)

$$a_1 = \sqrt{\frac{2(H_* - \Delta H)}{g(m_1 l_1 + m_2 l_2 A^2)}}, \tag{14}$$

and a_2 is given by (10) and (14).

Remark 3. Since $0 < A < \frac{k_0 l_0^2}{\rho_2\omega_2}$, (8) is useful to tune controller's parameter γ. If the energy loss ΔH should be smaller than a given positive constant δ, then the sufficient condition for γ is given as follows:

$$\gamma \geq \frac{\rho_1\omega_1 + \dfrac{k_0^2 l_0^4}{\rho_2\omega_2}}{\alpha\delta\omega_1}.$$

4.3 *Phase Shift*

Although $a_{1,2}$, A and ΔH are computed as functions of phase shift d, the above discussion can be validated even when d changes slowly. In order to study the dynamics of d, the dynamics (7) is rewritten as

$$\dot{\theta}_1 - \dot{\theta}_2 = \Theta_1(\theta_1, \theta_2) - \Theta_2(\theta_2, \theta_1).$$

Recalling the definition of d, the above differential equation can be written as follows:

$$\dot{d} = -\Theta(\theta_1, d), \tag{15}$$

where $\Theta(\theta_1, d) \equiv \Theta_1(\theta_1, \theta_1 + d) - \Theta_2(\theta_1 + d, \theta_1)$. Since d is assumed to vary slowly, the actual effect of (15) is investigated by averaging (15) through one oscillating cycle.

$$\Delta d = \frac{-1}{2\pi}\oint \Theta(\theta_1, d)d\theta_1, \tag{16}$$

and the amount of phase shift can be obtained by solving $\Delta d = 0$ with respect to d. However, the solutions may not be unique. Indeed, Figure 2 shows Δd vs d when $\gamma = 0.5$, $\alpha = 0.99$, $\rho_{1,2} = 0.001$[Nms] and $H_* = 0.1$[J]. There exist three solutions i.e. $d_1 \approx 0.2205$, $d_2 \approx 2.381$ and $d_3 \approx 2.818$. In order to select only meaningful solutions, stability of the solution is considered, such that the solution can be stable when $d\Delta d < 0$ around a solution. Because (16) is a difference equation, this criteria is not valid in general. In this case, however, this criteria is still useful since

d varies slowly. Figure 3 shows the solutions when coefficients of friction varies. Other parameters are same as in Figure 2. $\circ$ represents a stable solution and $\times$ represents a solution that is not stable. The vertical and horizontal axes represent coefficients $\rho_1 = \rho_2$ and d respectively.

The figure shows that amount of the phase shift also becomes large as frictional forces become large. This result is rational because of the following fact. When the consumption of H_2 by the frictional force becomes large, the spring is required to transmit larger amount of energy from pendulum 1 to 2. Therefore, the spring must be deformed largely and this implies the increase of d.

When ρ is small, there exist two stable solutions. The one is almost in-phase, but the other is anti-phase. This diagram implies that the first order transition of d may be observed when ρ is small.

5. SIMULATION AND EXPERIMENT

In order to show the validity of the above result, a computer simulation was executed. Parameters of the controller are same in Figure 2. Coefficients of friction are $\rho_{1,2} = 0.001$. Pendulums are stayed $\phi_1 = 1.0$[rad], $\phi_2 = 0$[rad] at rest when $t = 0$.

Figure 4 shows the time response of ϕ_1 and ϕ_2. The control synchronization is seemed to be achieved but the figure shows that pendulums are excited and oscillated synchronously with a small phase shift. By figure 2 or 3, the phase shift is estimated to be 0.2205. Figure 5 shows

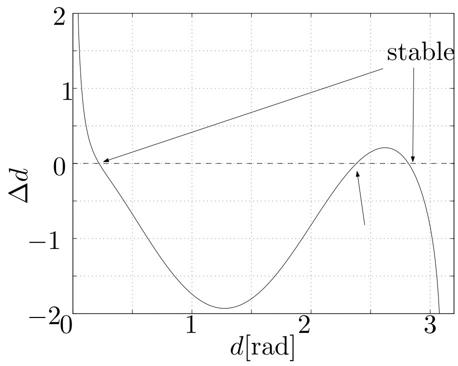

Fig. 2. Δd vs d

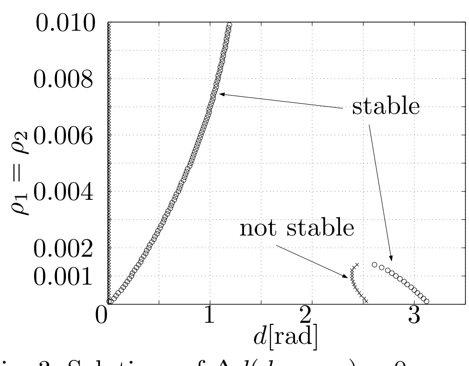

Fig. 3. Solutions of $\Delta d(d, \rho_1, \rho_2) = 0$

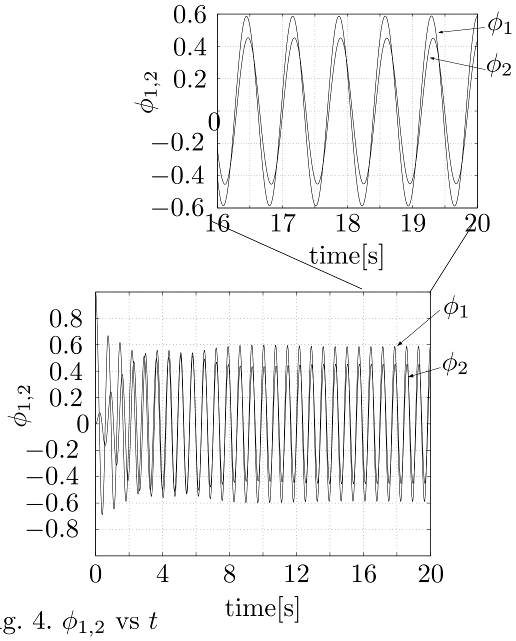

Fig. 4. $\phi_{1,2}$ vs t

the time response of d and that the estimation above is valid. Substituting $d = 0.2205$ and other parameters,

$$A = \frac{80 \times 0.02 \sin(0.2205)}{0.001 \times 9.0416} \approx 0.774,$$

and

$$\Delta H = \frac{0.001 \times 9.0416 + 0.001 \times 9.0416 \times A^2}{0.99 \times 0.5 \times 9.0416} \approx 0.0032.$$

By (14) and (10),

$$a_1 = \sqrt{\frac{2(0.1 - 0.0032)}{9.81(0.3 \times 0.12 + 0.3 \times 0.12 A^2)}} \approx 0.5854$$

$$a_2 = A a_1 \approx 0.4531$$

Amplitudes of oscillations are well estimated(Figure 4). Since $H'_* = H_* - \Delta H$,

$$H'_* \approx 0.1 - 0.0032 = 0.0968.$$

Figure 6 shows the time response of the energy H[J] and validates the result of the analysis.

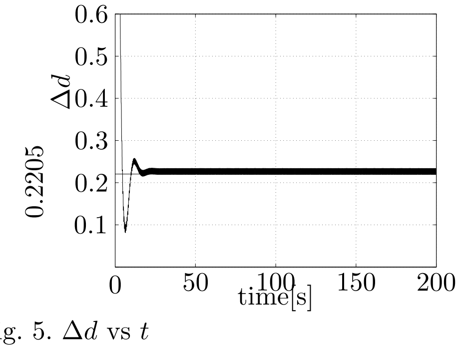

Fig. 5. Δd vs t

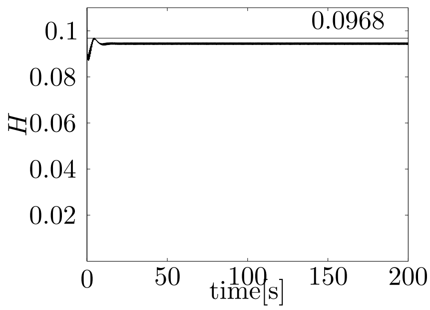

Fig. 6. H vs t

Table 2. Parameters

$m_{1,2}$	$l_{1,2}$	l_0
0.302[kg]	0.150[m]	0.0200[m]
k	ρ_1	J_1
119[N/m]	0.0130[Ns/m]	0.00706[kgm^2]
J_2	γ	α
0.00680[kgm^2]	0.9	0.999

A simple experimental device is shown in Figure 7. Parameters of the system and those of controller are shown in Table 2. Figure 8 shows the result of stability analysis and experimental result. ∇ shows experimental data. The figure shows that experimental results agree well with theoretical results when ρ_2 is smaller than 0.01. Since it is difficult to measure frictional coefficients precisely when ρ_2 is large, experimental values slightly differ from theoretical results when ρ_2 is larger than 0.01.

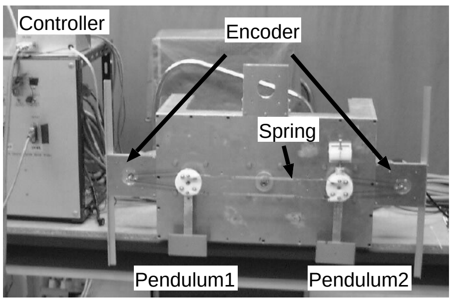

Fig. 7. Experimental Setup

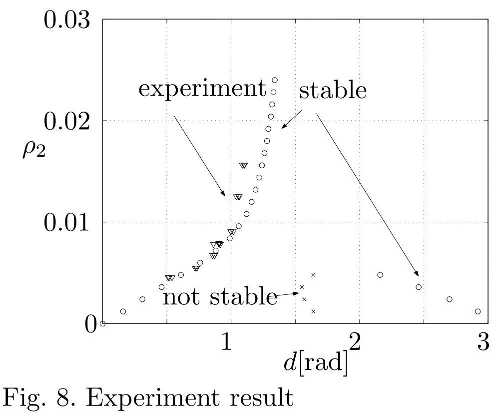

Fig. 8. Experiment result

6. CONCLUSION

Two 1-DOF pendulums coupled with a weak spring are controlled by SGE controller in order to make these pendulums oscillate synchronously. In this study, the mechanism of synchronization is studied by using Kuramoto's phase model and following results are revealed.

- Although the system is nonlinear and there exist frictional forces, the control objective can be achieved.
- Although the objective function is designed in order to make the angular velocities of pendulums same, it can not be achieved. Synchronization is achieved with phase shift.
- Taking the consumption by the friction into account, the total energy, phase shift and amplitudes of oscillations are well estimated.
- A sufficient condition of the parameter of the controller, γ, is shown.

Results of computer simulation and experiments show that all quantities estimated by the analysis are well estimated and that the analysis is valid.

Although, Figure 3 implies that there exists more than one stable phase shift, it was not easy to make the system synchronize at almost anti-phase oscillation. Since only the stability of the averaged phase shift is considered, the basin of the entrainment is needed to be studied in order to find a realizable anti-phase oscillation.

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