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ON THE GENERALITY OF MULTIPOINT PADÉ APPROXIMATIONS

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Abstract: Multipoint Padé interpolation methods have shown to be very efficient for model reduction of large-scale dynamical systems. The objective of this paper is to analyze the generality of this approach. We mainly focus our attention on the Single Input Single Output case. The generalization of this approach for MIMO model reduction is briefly introduced.

Keywords:
Multipoint Padé, rational interpolation, model reduction, imbedding, large-scale systems

1. INTRODUCTION

Every proper linear time-invariant continuous-time system can be represented by a state-space model :

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}, \tag{1}$$

with input $u(t) \in \mathbb{C}^m$, state $x(t) \in \mathbb{C}^N$ and output $y(t) \in \mathbb{C}^p$. The matrices $A \in \mathbb{C}^{N \times N}$, $B \in \mathbb{C}^{N \times m}$, $C \in \mathbb{C}^{p \times N}$ and $D \in \mathbb{C}^{p \times m}$. Unless specified differently, we assume here that there is only one input and one output, i.e. $m = p = 1$. Without loss of generality, we assume that the system is controllable and observable since otherwise we can always find a smaller dimensional model that is controllable and observable, and that has exactly the same transfer function. In addition to this, we will assume that the system is stable, i.e. the generalized eigenvalues of the matrix A lie in the open left half plane.

When the system order N is too large for solving various control problems within a reasonable computing time, it is natural to consider approximating it by a reduced order system

$$\begin{cases} \dot{\hat{x}} = \hat{A}\hat{x} + \hat{B}u \\ \hat{y} = \hat{C}\hat{x} + \hat{D}u \end{cases} \tag{2}$$

driven with the same input $u(t) \in \mathbb{C}^m$, but having different output $\hat{y}(t) \in \mathbb{C}^p$ and state $\hat{x}(t) \in \mathbb{C}^n$ The matrices $\hat{A} \in \mathbb{C}^{n \times n}$, $\hat{B} \in \mathbb{C}^{n \times m}$, $\hat{C} \in \mathbb{C}^{p \times n}$ and $\hat{D} \in \mathbb{C}^{p \times m}$. For the same reasons as above, we will assume that the realization $(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ of the reduced order model $\hat{T}(s)$ is minimal. The degree n of the reduced order system is also assumed to be much smaller than the degree N of the original system. The objective of the reduced order model is to reduce the dimension of the state-space (of dimension N) of the system to a lower dimension n in such a way that the behavior of the reduced order model is sufficiently close to that of the full order system. For a same input $u(t)$, we thus want $\hat{y}(t)$ to be close to $y(t)$. One shows that in the frequency domain, this is equivalent to imposing conditions on the frequency responses of both systems (Zhou *et al.*, 1996) : we want to find a reduced order model such that the transfer functions of both models, i.e.

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$$T(s) = C(sI_N - A)^{-1}B + D \quad (3)$$
$$\hat{T}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}, \quad (4)$$

are such that the error $\|T(.) - \hat{T}(.)\|$ is minimal for the H_∞ norm. A particular way to construct a reduced order model is the following truncation technique.

Definition 1.1. The transfer function $\hat{T}(s) \doteq \hat{C}(sI_n - \hat{A})^{-1}\hat{B} + \hat{D}$ of Mc Millan degree n, with m inputs and p outputs, is constructed via truncation of the transfer function $T(s) = C(sI_N - A)^{-1}B + D$ (with m inputs and p outputs) of Mc Millan degree N if there exist projecting matrices $Z, V \in \mathbb{C}^{N \times n}$ such that

$$\left\{\hat{A}, \hat{B}, \hat{C}, \hat{D}\right\} = \left\{Z^T AV, Z^T B, CV, D\right\}. \quad (5)$$

Since the matrix D does not depend on the dimension N of the state space and since it cancels out in the difference $T(s) - \hat{T}(s)$, it does not play any role in the model reduction framework. From now on, we therefore assume that $D = \hat{D} = 0$. A rational transfer function $T(s)$ is called strictly proper when $\lim_{s\to\infty} T(s) = 0$, i.e. when $D = 0$. A set of matrices (A, B, C) is called a realization of the strictly proper transfer function $T(s)$ when $T(s) = C(sI - A)^{-1}B$. The structure of this paper is as follows. In the next section, we introduce our notation and recall some results about Krylov subspaces and interpolation. In section 3, we derive the construction of the Multipoint Pade interpolating reduced order transfer function in the SISO case. In section 4, some generalizations are given. We conclude with some remarks in section 5.

2. PRELIMINARY RESULTS

Here, we recall some results about Krylov subspaces and rational interpolation. Let $T(s) = C(sI - A)^{-1}B$ be a continuous time, linear time invariant transfer function of Mc Millan degree N, with one input and one output. For any pair of matrices (A, B) with compatible dimensions, we define the Krylov subspace of order $k \in \mathbb{N}_0$, written $\mathcal{K}_k(A, B)$, as follows

$$\mathcal{K}_k(A, B) = Im\{B, AB, \ldots, A^{k-1}B\} \quad (6)$$
$$= \{0\} \quad for\ k \leq 0. \quad (7)$$

Two well known matrices of a SISO transfer function $T(s) = C(sI_N - A)^{-1}B$ are the controllability matrix $Contr(A, B) \in \mathbb{C}^{N\times N}$ and the observability matrix $Obs(C, A) \in \mathbb{C}^{N\times N}$ defined as

$$Contr(A, B) = \left[\, B \ldots A^{N-1}B \,\right], \quad (8)$$

$$Obs(C, A) = \begin{bmatrix} C \\ \vdots \\ CA^{N-1} \end{bmatrix}. \quad (9)$$

Definition 2.1. An **interpolation set** I

$$I = \{(s_1, m_1), \ldots (s_r, m_r)\}, \quad (10)$$

is defined as a set of couples (s_i, m_i) where the points $s_i \in \mathbb{C} \cup \infty$ are distinct and the indices $m_i \in \mathbb{N}_0$ are finite. The size of the interpolation set I, denoted by $s(I)$ is defined by

$$s(I) = \sum_{i=1}^{r} m_i. \quad (11)$$

An interpolation set I is called an **admissible interpolation set** for the transfer function $T(s)$ of Mc Millan degree N when no interpolation point s_i is a pole of $T(s)$. Such an interpolation set will be called a $T(s)$-admissible interpolation set. A **minimal $T(s)$-admissible interpolation set** is a $T(s)$-admissible interpolation set of size N, where N is the Mc Millan degree of $T(s)$.

Definition 2.2. A couple of $T(s)$-admissible interpolation sets (I_1, I_2), denoted by

$$I_1 = \{(s_{1,1}, m_{1,1}), \ldots (s_{1,r_1}, m_{1,r_1})\} \quad (12)$$
$$I_2 = \{(s_{2,1}, m_{2,1}), \ldots (s_{2,r_2}, m_{2,r_2})\}, \quad (13)$$

is called a **separation** of I if the set of points of I is the union of those of I_1 and I_2 and if their corresponding indices add up. By that, we mean that for each point $s_k \in I$ belonging to I_1 and I_2 we have

$$s_{1,i} = s_{2,j} = s_k \Rightarrow m_{1,i} + m_{2,j} = m_k, \quad (14)$$

and for each point $s_k \in I$ belonging to only one set I_1 or I_2, we have (e.g. for I_1)

$$s_{1,i} = s_k \Rightarrow m_{1,i} = m_k. \quad (15)$$

As a consequence, we have

$$s(I_1) + s(I_2) = s(I). \quad (16)$$

A separation (I_1, I_2) is called **symmetric** when $s(I_1) = s(I_2)$.

The quantities occurring in $Contr(A, B)$ and $Obs(A, B)$

$$\gamma_{A,B}(\infty, k) \doteq A^{k-1}B \quad \delta_{C,A}(\infty, k) = CA^{k-1} \quad (17)$$

can be seen as "moments" of $(sI - A)^{-1}B$ and $C(sI - A)^{-1}$ about infinity. Similarly, we define the moments about a finite expansion point $\lambda \in \mathbb{C}$

$$\gamma_{A,B}(\lambda, k) \doteq (\lambda I - A)^{-k}B, \quad (18)$$
$$\delta_{C,A}(\lambda, k) \doteq C(\lambda I - A)^{-k}. \quad (19)$$

Definition 2.3. Let I be a $T(s)$-admissible interpolation set. For any state-space realization (A,B,C) of $T(s)$, we define the generalized controllability matrix $\mathcal{C}_{A,B}$ to be

$$\mathcal{C}_{A,B}(I) \doteq \left[\, \gamma(s_1,1)\ \gamma(s_1,2)\ \ldots\ \gamma(s_r,m_r)\,\right], \tag{20}$$

and generalized observability matrix to be

$$\mathcal{O}_{T(s)}(I) \doteq \begin{bmatrix} \delta(s_1,1) \\ \delta(s_1,2) \\ \vdots \\ \delta(s_r,m_r) \end{bmatrix}. \tag{21}$$

A proof of the following lemma can be found in (Anderson and Antoulas, 1990).

Lemma 2.1. Let $T(s)$ be a strictly proper SISO LTI transfer function of Mc Millan degree N with a state space realization $T(s) = C(sI-A)^{-1}B$. Let

$$I = \{(s_1,m_1), \ldots (s_r,m_r)\}, \tag{22}$$

be a minimal $T(s)$-admissible interpolation set. Then

(1) $Im\,(\mathcal{C}_{A,B}(I)) = Im\,(Contr(A,B))$.
(2) $Ker\,(\mathcal{O}_{C,A}(I)) = Ker\,(Obs(C,A))$.

Let (A,B,C) be a minimal realization of a SISO strictly proper transfer function of Mc Milan degree N. Let I be a $T(s)$-admissible interpolation set of size $k<N$. A consequence of the above lemma is that

(1) $Rank\,(\mathcal{C}_{A,B}(I)) = k$,
(2) $Rank\,(\mathcal{O}_{C,A}(I)) = k$.

Indeed, I can be seen as a subset of a minimal $T(s)$-admissible interpolation set.

3. RATIONAL INTERPOLATION

As explained earlier, we suppose here that the original transfer function and the reduced order transfer function are both strictly proper, which clearly implies $T(\infty) = \hat{T}(\infty)$. This leads us to the following definition.

Definition 3.1. Let $T(s)$ be a SISO strictly proper transfer function of Mc Millan degree N. Let $\hat{T}(s)$ be a SISO strictly proper transfer function of Mc Millan degree n. We are given one $T(s)$-admissible interpolation set I of size $2n$, denoted by

$$I = \{(s_1,m_1), \ldots (s_r,m_r)\}. \tag{23}$$

We say that $T(s)$ interpolates $\hat{T}(s)$ at I when the following conditions are satisfied :

(1) $\forall\ 1 \le i \le r$ such that $s_i \neq \infty$,

$$T(s_i) = \hat{T}(s_i) + O(s-s_i)^{m_i}, \tag{24}$$

(2) If ∞ is not a point of I, then

$$\lim_{s\to\infty} \left(T(s) - \hat{T}(s)\right) < \infty, \tag{25}$$

(3) If ∞ is a point of I, say $s_k = \infty$, then

$$\lim_{s\to\infty} \left(T(s) - \hat{T}(s)\right) s^{m_k} = 0. \tag{26}$$

Let us consider a minimal realization (A,B,C) of $T(s)$ and a minimal realization $(\hat{A},\hat{B},\hat{C})$ of the reduced order transfer function $\hat{T}(s)$. Writing equation (24) is equivalent to say that the m_i first coefficients of the Taylor expansion of $\hat{T}(s)$ around s_i equal those of $T(s)$, i.e. that, $\forall\ 1 \le k \le m_i$,

$$\hat{C}(s_iI_n - \hat{A})^{-k}\hat{B} = C(s_iI_N - A)^{-k}B. \tag{27}$$

Equation (25) is automatically satisfied when the transfer functions $T(s)$ and $\hat{T}(s)$ are both strictly proper. Equation (26) is equivalent to say that the m_k first Markov parameters of both transfer functions are equal, i.e., $\forall\ 0 \le i \le m_k - 1$,

$$\hat{C}\hat{A}^i\hat{B} = CA^iB. \tag{28}$$

Hence, an interpolation set of size $2n$ corresponds to $2n+1$ interpolation conditions, one of them being trivially satisfied for any couple of strictly proper transfer functions. In general, given a strictly proper transfer function $T(s)$ of Mc Millan degree N and a $T(s)$-admissible interpolation set I of size $2n$ (with $n<N$), the strictly proper solution of minimal Mc Millan degree of the interpolation conditions (24) to (26) is unique and of degree n. For the particular cases, we refer to (Antoulas and Anderson, 1986). For a more complete treatment of the interpolation problem of rational matrix functions, we refer to (Ball *et al.*, 1990) and references therein.

Construction of the solution

In this paper, we are given a SISO strictly proper transfer function of Mc Millan degree N and a $T(s)$-admissible interpolation set I of size $2n$ and we want to find the strictly proper transfer function $\hat{T}(s)$ of Mc Millan degree n that interpolates $T(s)$ at I. The objective consists of finding the projecting matrices Z and V such that $\hat{T}(s)$ can be constructed from truncation of $T(s)$. From now on, we suppose therefore that there is only one solution of Mc Millan degree n of the interpolation conditions given in Definition 3.1, with $s(I) = 2n$. We call this solution $\hat{T}(s) = \hat{C}(sI-\hat{A})^{-1}\hat{B}$.

Lemma 3.1. Let $T(s) = C(sI_N - A)^{-1}B$ be a strictly proper SISO transfer function. Let I be a $T(s)$-admissible interpolation set. Let $\hat{T}(s) = \hat{C}(sI_n - \hat{A})^{-1}\hat{B}$ be a strictly proper SISO transfer function of Mc Millan degree n. $\hat{T}(s)$ interpolates $T(s)$ at I if and only if either of the two following equivalent conditions hold :

(1)
$$C\mathcal{C}_{A,B}(I) = \hat{C}\mathcal{C}_{\hat{A},\hat{B}}(I), \quad (29)$$
(2)
$$\mathcal{O}_{C,A}(I)B = \mathcal{O}_{\hat{C},\hat{A}}(I)\hat{B}. \quad (30)$$

Proof : It is simply another way to write down the interpolation conditions of definition 3.1. □

Lemma 3.2. Let $T(s) = C(sI - A)^{-1}B$ be a SISO strictly proper transfer function of Mc Millan degree N. Let I be a $T(s)$-admissible interpolation set of size $2n$. Let (I_1, I_2) be a symmetric separation of I. Suppose that the SISO strictly proper transfer function of Mc Millan degree n, $\hat{T}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}$ interpolates $T(s)$ at I. Then

$$\mathcal{O}_{C,A}\mathcal{C}_{A,B} = \mathcal{O}_{\hat{C},\hat{A}}\mathcal{C}_{\hat{A},\hat{B}} \quad (31)$$
$$\mathcal{O}_{C,A}A\mathcal{C}_{A,B} = \mathcal{O}_{\hat{C},\hat{A}}\hat{A}\mathcal{C}_{\hat{A},\hat{B}}. \quad (32)$$

Sketch of the proof :
Let us verify the second equation. If $s_i \neq \infty$, then define the matrices $A_i \in \mathbb{C}^{N\times N}$ and $B_i \in \mathbb{C}^{N\times 1}$ by
$$A_i = (s_i I - A)^{-1}, \quad B_i = (s_i I - A)^{-1}B. \quad (33)$$
If $s_i = \infty$, then define
$$A_i = A, \quad B_i = B. \quad (34)$$
Let us consider again one element of the matrix equality (32). We have to prove that
$$CA_i^{k_1}AA_j^{k_2}B = \hat{C}\hat{A}_i^{k_1}\hat{A}\hat{A}_j^{k_2}\hat{B}. \quad (35)$$
The idea is that it is always possible to rewrite equation (35) as a linear combination of the scalar elements of the matrix equations (29), (30) and (31) by partial fraction expansion.
The point at infinity requires more care. Let us show it for instance when $A_1 = A$. From Definition 2.3, this implies that one of the points of I_1, say $s_{1,1}$ is equal to ∞. Then, $\forall\ u,\ 1 \leq u \leq m_{1,1}$,
$$CA^{u-1}B = \hat{C}\hat{A}^{u-1}\hat{B}. \quad (36)$$
If $A_j = A$, then the point ∞ is also a point of I_2, say $s_{2,1} = \infty$. Then, $\forall\ v,\ 1 \leq v \leq m_{2,1}$
$$CA^{v-1}B = \hat{C}\hat{A}^{v-1}\hat{B}. \quad (37)$$
Clearly, the point ∞ must be a point of I, say $s_1 = \infty$. Because (I_1, I_2) is a separation of I, $m_{1,1} + m_{2,1} = m_1$, and $\forall\ w,\ 1 \leq w \leq m_1$,
$$CA^{w-1}B = \hat{C}\hat{A}^{w-1}\hat{B}. \quad (38)$$
Now, $k_1 + 1 + k_2 \leq m_{1,1} + m_{2,1} - 1 = m_1 - 1$, and equality (35) follows from equation (38). This concludes the proof for the case $A_i = A_j = A$. Suppose now that $A_j = (s_k I - A)^{-1}$ and $A_i = A$. Then, $\forall\ v,\ 1 \leq v \leq m_{2,k}$,
$$CA_j^{v}B = CA_j^{v}B. \quad (39)$$
From partial fraction expansion, it follows that
$$CA^{k_1}AA_j^{k_2}B = -CA^{k_1}A_j^{k_2-1}B + s_k CA^{k_1}A_j^{k_2}B. \quad (40)$$
Equation (35) now follows from Lemma 3.1. This concludes the proof for $A_i = A$. The case $A_i \neq A$ is easier and the details are omitted here. □

Theorem 3.1. Let $T(s) = C(sI - A)^{-1}B$ be a SISO strictly proper transfer function of Mc Millan degree N. Let I be a $T(s)$-admissible interpolation set of size $2n$. If there exists one transfer function $\hat{T}(s) \doteq \hat{C}(sI - \hat{A})^{-1}\hat{B}$ of Mc Millan degree n that interpolates $T(s)$ at I, then it is the unique interpolating transfer function of Mc Millan degree n and there is no interpolating transfer function of smaller Mc Millan degree at I. Finally Let (I_1, I_2) be a symmetric separation of I. Then $\hat{T}(s)$ can be obtained by truncation of $T(s)$ with the following projecting matrices :

$$Z^T = \mathcal{O}_{\hat{C},\hat{A}}(I_1)^{-1}\mathcal{O}_{C,A}(I_1) \quad (41)$$
$$V = \mathcal{C}_{A,B}(I_2)\mathcal{C}_{\hat{A},\hat{B}}(I_2)^{-1}. \quad (42)$$

Proof :
Clearly, I_1 and I_2 are two minimal $\hat{T}(s)$-admissible interpolation sets. From Lemma 2.1, the matrices $\mathcal{O}_{\hat{C},\hat{A}}(I_1)$ and $\mathcal{C}_{\hat{A},\hat{B}}(I_2)$ are invertible. From Lemmata 3.1 to 3.2, it is easy to check that the conditions of Definition 1.1 are satisfied. Uniqueness follows. □

4. GENERALIZATION OF THE RESULTS

Generalized state-space model

Every linear time-invariant system can be represented by the following generalized state-space model :
$$\begin{cases} E\dot{x} = Ax + Bu \\ \quad y = Cx + Du \end{cases}, \quad (43)$$
with the matrix $E \in \mathbb{C}^{N\times N}$. Such a representation arises naturally in many applications. As usual, we assume that the transfer function
$$T(s) \doteq C(sE - A)^{-1}B + D \quad (44)$$
(of Mc Millan degree N) is stable, i.e. the generalized eigenvalues of the pencil $sE - A$ lie in the open left half plane (this also implies that E is non-singular). For large-scale systems, one wants to keep sparsity. So, inverting E to come back to the classical state-space model (1) should be avoided. The important point is that all the developments given above remain true for generalized state-space models with some modifications. Let us consider an expansion of $T(s)$ about a point σ that is *not* a pole of $T(s)$. It then follows that

$\sigma E - A$ is invertible and one obtains the following formal series expansion :

$$T(s) = C\left(\sigma E - A - (\sigma - s)E\right)^{-1} B$$
$$= C\left(I - (\sigma E - A)^{-1}E(\sigma - s)\right)^{-1}(\sigma E - A)^{-1}B$$
$$= \sum_{j=0}^{+\infty} C\left((\sigma E - A)^{-1}E\right)^{j}(\sigma E - A)^{-1}B \cdot (\sigma - s)^{j}$$
$$\doteq \sum_{j=0}^{+\infty} T_{\sigma}^{(j)} \cdot (\sigma - s)^{j} \quad (45)$$

which defines the so-called *moments*

$$T_{\sigma}^{(j)} \doteq C\left((\sigma E - A)^{-1}E\right)^{j}(\sigma E - A)^{-1}B \quad (46)$$

about an expansion point σ. These moments exist for every σ for which $(\sigma E - A)$ is non-singular. This leads us to replace $\gamma_{A,B}(\lambda, k)$ and $\delta_{A,B}(\lambda, k)$ when λ is finite by

$$\gamma_{A,B,E}(\lambda, k) \doteq \left((\lambda E - A)^{-1}E\right)^{k-1}(\lambda E - A)^{-1}B$$
$$\delta_{C,A,E}(\lambda, k) \doteq C(\lambda E - A)^{-1}\left(E(\lambda E - A)^{-1}\right)^{k-1},$$

and when λ is not finite by

$$\gamma_{A,B,E}(\infty, k) \doteq (E^{-1}A)^{k-1}B$$
$$\delta_{C,A,E}(\infty, k) \doteq C(AE^{-1})^{k-1}.$$

With such a modification, Proposition 3.1 remains true for generalized state-space models. In 1997, Grimme already found by a different approach the following result :

Theorem 4.1. If

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{b_k}}((\sigma_k E - A)^{-1}E, (\sigma_k E - A)^{-1}B) \subseteq Im(V) \quad (47)$$

and

$$\bigcup_{k=1}^{K} \mathcal{K}_{J_{c_k}}((\sigma_k E - A)^{-T}E^T, (\sigma_k E - A)^{-T}C^T) \subseteq Im(Z) \quad (48)$$

where the interpolation points σ_k are chosen such that the matrices $\sigma_k E - A$ are invertible $\forall k \in \{1, \ldots, K\}$ then the moments of the systems (1) and (2) at the points σ_k satisfy

$$T_{\sigma_k}^{(j_k)} = \hat{T}_{\sigma_k}^{(j_k)} \quad (49)$$

for $j_k = 1, 2, \ldots, J_{b_k} + J_{c_k}$ and $k = 1, 2, \ldots, K$, provided these moments exist, i.e. provided the matrices $\sigma_k \hat{E} - \hat{A}$ are invertible.

Proof :
This is a consequence of Theorem 3.1. Another proof can be found in (Gallivan *et al.*, 1998) and (Gallivan *et al.*, 2002*b*), and implicitly also in (de Villemagne and Skelton, 1987). □

A note about truncation

A straightforward consequence of Proposition 3.1 is the following fact. Let $T(s)$ be a SISO strictly proper transfer function of Mc Millan degree N and $\hat{T}(s)$ a SISO strictly proper transfer function of Mc Millan degree $n < N$. If the error transfer function $E(s) = T(s) - \hat{T}(s)$ has a Mc Millan degree greater or equal to $2n + 1$, then we can find a $T(s)$-admissible interpolation set I of size $2n$ such that $\hat{T}(s)$ interpolates $T(s)$ at I. From Proposition 3.1, this implies that $\hat{T}(s)$ can be obtained from truncation of $T(s)$. Actually, it is possible to prove the following Theorem.

Theorem 4.2. Choose $T(s)$, an arbitrary SISO strictly proper transfer function of Mc Millan degree N with the minimal state-space realization $T(s) = C(sI - A)^{-1}B$. Choose $\hat{T}(s)$, an arbitrary SISO strictly proper transfer function of Mc Millan degree $n < N$ with the minimal state-space realization $\hat{T}(s) = \hat{C}(sI - \hat{A})^{-1}B$. Then $\hat{T}(s)$ can be constructed via truncation of $T(s)$.

This rather striking result is not true anymore in the MIMO case.

4.1 *The MIMO case*

The Multipoint Padé technique can be generalized for MIMO systems in two different ways. Maybe the simplest way is to use a block version of the *Multipoint Padé* technique to construct the interpolating transfer function. This implies e.g. that we need to impose the error transfer function $T(s) - \hat{T}(s)$ to be zero at certain points. Such an approach is discussed in (Gallivan *et al.*, 2002*b*), but it may be too constraining. A more natural way is to generalize the concept of *moment matching* to *tangential interpolation*. The idea is the following : Given a transfer function $T(s)$ of Mc Millan degree N with m inputs and p outputs, one wants to construct a transfer function $\hat{T}(s)$ with p inputs and m outputs of Mc Millan degree $n < N$ by imposing three types of interpolation condition :

Left interpolation conditions : Let $x(s)$ be a transfer function with 1 output and p inputs, and the point $\alpha \in \mathbb{C}$, we impose that

$$\left.\frac{d^{i-1}}{ds^{i-1}}\{x(s)T(s)\}\right|_{s=\alpha} = \left.\frac{d^{i-1}}{ds^{i-1}}\{x(s)\hat{T}(s)\}\right|_{s=\alpha}, \quad (50)$$

Right interpolation conditions : Let $u(s)$ be a transfer function with 1 input and m outputs, and the point $w \in \mathbb{C}$, we impose that

$$\left.\frac{d^{i-1}}{ds^{i-1}}\{T(s)u(s)\}\right|_{s=w} = \left.\frac{d^{i-1}}{ds^{i-1}}\{\hat{T}(s)u(s)\}\right|_{s=w}, \quad (51)$$

Two-sided interpolation conditions :

$$\frac{d^{f+g-1}}{ds^{f+g-1}} \left\{x^{(f)}(s)T(s)u^{(g)}(s)\right\}\Big|_{s=\xi} = \frac{d^{f+g-1}}{ds^{f+g-1}} \left\{x^{(f)}(s)\hat{T}(s)u^{(g)}(s)\right\}\Big|_{s=\xi}. \quad (52)$$

It is possible to generalize the technique developed in section 3 to such a framework. As in the SISO case (see for instance (Gallivan *et al.*, 2002*b*)), it can be shown that

$$Im(V) = Im(\tilde{V}) \quad and \quad Im(Z^T) = Im(\tilde{Z}^T), \quad (53)$$

where $\tilde{Z}$ and $\tilde{V} \in \mathbb{C}^{N\times n}$ are solutions of the following Sylvester equation :

$$A\tilde{V} + \tilde{V}F_B + BG_B = 0 \quad (54)$$

$$A^T\tilde{Z} + \tilde{Z}F_C + C^TG_C = 0. \quad (55)$$

To see this, describe the solutions $\tilde{Z}$ and $\tilde{V}$ of (54) and (55) when the matrices F_B and F_C are in Jordan canonical form. Then, the interpolation points appear to be the opposites of the poles of the matrices F_B and $F_C \in \mathbb{C}^{n\times n}$. For more details, see for instance (Gallivan *et al.*, 2002*a*). General results about tangential interpolation may be found in (Antoulas *et al.*, 1990) and (Ball *et al.*, 1990).

5. CONCLUDING REMARKS

In this paper, we have shown the generality of *Multipoint Padé* technique to construct interpolating reduced-order transfer functions. Indeed, generically, given a SISO strictly proper transfer function $T(s)$ of Mc Millan degree N and a $T(s)$-admissible interpolation set I of size $2n$ (with $n < N$), there is only one transfer function $\hat{T}(s)$ of Mc Millan degree n that interpolates $T(s)$ at I, and this transfer function can be constructed via *Multipoint Padé*.

A big advantage of *Multipoint Padé* compared to others model reduction technique is its low computational cost. Hence, it can be applied to large-scale linear systems. A weakness of *Multipoint Padé* is that there exists no global error bound between the original and the reduced-order model.

At first sight, we can think that a reduced order transfer function $\hat{T}(s)$ constructed from $T(s)$ via *Multipoint Padé* must be close to $T(s)$ because it interpolates it at an interpolation set I. Actually, this is false. Indeed, take a SISO strictly proper transfer function $\hat{T}(s)$ of Mc Millan degree $n < N$ such that the error transfer function $E(s) = T(s) - \hat{T}(s)$ has more than $2n + 1$ zeroes, then $\hat{T}(s)$ can be constructed from $T(s)$ via Multipoint Padé. Clearly, the error may be arbitrarily large. Nevertheless, for a practical point of vue, *Multipoint Padé* gives good results for random points of interpolation.

The generality of *Multipoint Padé* in the MIMO case is still under investigation and will appear in a later paper. Finding interpolating conditions such that there exists a global bound between the original and the reduced-order transfer function is an open question. For instance, we could look at well-known model reduction techniques such as balanced truncation or optimal Hankel norm approximation and try to characterize the interpolation points of reduced-order transfer function constructed via these techniques.

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