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15th Triennial World Congress, Barcelona, Spain

DIGITAL PID DESIGN FOR MAXIMALLY DEADBEAT AND TIME-DELAY TOLERANCE

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Abstract: This paper presents a new approach to design a digital PID controller for a given LTI plant. By using the Tchebyshev representation of a discrete time transfer function and some new results on root counting with respect to the unit circle, we show how the digital PID stabilizing gains can be directly obtained by solving sets of linear equations. This solution is attractive because it determines the entire set of stabilizing PID gains constructively if exists, Using this characterization of the stabilizing set, we present solutions to two design problems: a) Maximally deadbeat design where we determine the smallest circle within the unit circle wherein the closed loop characteristic roots may be placed by PID control, b) Maximal delay tolerance-where we determine the maximal loop delay that can be tolerated under PID control.

Keywords: Deadbeat, Discrete systems, PID control, Tchebyshev representation

1. INTRODUCTION

There is renewed interest in PID controllers because of two reasons. First, they are extensively used in applications in all industries (Åström and Hägglund, 1995) and second, modern design methods are inapplicable to PID controller design due to their inability to accommodate fixed order or structure (Dorato, 2000). As a result there is much that remains to be done to modernize PID design methods over those developed in the 1940's namely, Ziegler-Nichols and its variations (Goodwin *et al.*, 2001).

Recently, the problem of stabilizing a continuous time linear time invariant (CTLTI) system by using a PID controller was solved (Datta *et al.*, 2000). The complete set of stabilizing controllers was found by developing a generalization of the Hermite Bieler Theorem and applying it to the problem. The result characterized the stabilizing set as the solution of a set of linear inequalities parametrized by the proportional gain.

In this paper, we consider a discrete time linear time invariant (DTLTI) plant to be controlled by a digital PID controller. First, the complex plane image of a real polynomial or rational function over a circle of radius ρ centered at the origin, is determined and expressed in terms of Tchebyshev polynomials of the first and second kinds. For a mathematical treatment of Tchebyshev polynomials (Pólya and Szegó, 1976; Mansour, 1992), they are used in a control problem related to discrete time systems. In terms of this Tchebyshev representation a new formula has been developed for root counting with respect to the unit circle. This formula which differs from the root counting formulas given in (Yamada and Bose, n.d.; Yamada *et al.*, 1998) and is an extension of an initial result presented in (Keel and Bhattacharyya, n.d.), and constitutes the generalization of Hermite Bieler type results for Schur stability. We apply these results to the PID stabilization problems and show how the entire set of stabilizing gains can be found by linear programming after a suitable reparametrization. The solution shows that the stabilizing set for any DTLTI plant, when it is nonempty, consists of unions of convex polygons in the PID gain space.

Our direct solution should be contrasted with the recent result in (Xu *et al.*, 2001) where the set of digital PID stabilizing gains has been determined by applying the bilinear transformation, using the CTLTI results of (Datta *et al.*, 2000) followed by inverse transformation back to the discrete time domain. The direct solution given here allows us to formulate and solve design and performance problems in a transparent fashion whereas the bilinear transformation does not. Specifically we solve two design problems here which are not tractable by the bilinear transformation. The first problem is related to deadbeat control wherein one places all closed loop characteristic roots at the origin so that the transients are zeroed out in a finite number of steps. In general, deadbeat control is not possible using PID and a reasonable goal is to place the closed loop characteristic roots as close to the origin as possible so that the transient error decays quickly. Such designs have been advocated in the literature on sampled data control systems (Ackermann, 1985). We show how the stabilization solution obtained can be refined to give a constructive determination of such "maximally" deadbeat designs. The second problem involves the determination of the maximum delay in the loop that a given plant under PID control can be made to tolerate. We show how our solution can also be extended to determine this maximum delay for a given DTLTI plant.

2. PRELIMINARIES

Consider a feedback system consisting of a SISO DTLTI plant $G(z)$ and the unity feedback DTLTI controller $C(z)$. We write

$$G(z) = \frac{N(z)}{D(z)}, \qquad C(z) = \frac{N_C(z)}{D_C(z)}.$$

The *closed loop characteristic polynomial* is

$$\Pi(z) := D_C(z)D(z) + N_C(z)N(z)$$

and a necessary and sufficient condition for stability of the closed loop control system is that the characteristic roots, namely the zeros of $\Pi(z)$ have magnitude less than unity. This condition is commonly referred to as *Schur stability* of $\Pi(z)$.

The stabilization problem can be stated as that of determining $C(z)$ so that for the given $G(z)$, the closed loop characteristic polynomial $\Pi(z)$ is Schur. For a fixed structure controller, such as a PID controller, $C(z)$ is characterized by a set of gains $\mathbf{x}$ and must be chosen to stabilize $\Pi(z)$ if possible. A useful characterization of $\mathcal{S}$ should allow the designer to test the feasibility of imposing various performance constraints and checking their attainability with the controller parameters. Thus if $\mathcal{P}_i$ represents the set of controller parameter values $\mathbf{x}$ attaining a performance specification, the designer should be able to constructively determine $\mathcal{S} \cap \mathcal{P}_i$, if it is nonempty, the subset of $\mathcal{S}$ attaining specifications. We shall show how the stabilizing set $\mathcal{S}$ and the performance specification sets $\mathcal{S} \cap \mathcal{P}_i$ can be constructively determined for the case of digital PID controllers and two different specifications. This solution depends on certain root counting formulas which we need to develop.

3. TCHEBYSHEV REPRESENTATION AND ROOT CLUSTERING

Let us consider a polynomial

$$P(z) = a_n z^n + \cdots + a_0 \tag{1}$$

with real coefficients. Then

$$\begin{aligned} P\left(\rho e^{j\theta}\right) &= R(u,\rho) + j\sqrt{1-u^2}T(u,\rho) \\ &=: P_c(u,\rho) \end{aligned}$$

where

$$\begin{aligned} R(u,\rho) &= a_n c_n(u,\rho) + \cdots + a_1 c_1(u,\rho) + a_0 \\ T(u,\rho) &= a_n s_n(u,\rho) + \cdots + a_1 s_1(u,\rho) \\ s_k(u) &= -\frac{c_k'(u)}{k}, \quad k = 1,2,\cdots \qquad (2) \\ c_{k+1}(u) &= -u c_k(u) - \left(1-u^2\right) s_k(u), \qquad (3) \end{aligned}$$

for $k = 1,2,\cdots$ (Keel and Bhattacharyya, n.d.).

$R(u,\rho)$ and $T(u,\rho)$ are polynomials in u and ρ. The complex plane image of $P(z)$ as z traverses the upper half of the circle $\mathcal{C}_\rho$ can be obtained by evaluating $P_c(u,\rho)$ as u runs from -1 to $+1$.

Let $Q(z)$ be a ratio of two real polynomials $P_1(z)$ and $P_2(z)$. We compute the image of $Q(z)$ on $\mathcal{C}_\rho$ as follows. Let

$$\begin{aligned} P_i(z)|_{z=-\rho u + j\rho\sqrt{1-u^2}} = R_i(u,\rho) \\ +j\sqrt{1-u^2}T_i(u,\rho), \end{aligned}$$

for $i = 1,2$. Then

$$Q(z)|_{z=-\rho u + j\rho\sqrt{1-u^2}} = \left.\frac{P_1(z)}{P_2(z)}\right|_{z=-\rho u + j\rho\sqrt{1-u^2}}$$

$$= \left.\frac{P_1(z)P_2\left(z^{-1}\right)}{P_2(z)P_2\left(z^{-1}\right)}\right|_{z=-\rho u + j\rho\sqrt{1-u^2}}$$

$$= \frac{\overbrace{\left(R_1(u,\rho)R_2(u,\rho) + \left(1-u^2\right)T_1(u,\rho)T_2(u,\rho)\right)}^{R(u,\rho)}}{R_2^2(u,\rho) + \left(1-u^2\right)T_2^2(u,\rho)}$$

$$+j\sqrt{1-u^2}\frac{\overbrace{(T_1(u,\rho)R_2(u,\rho)-R_1(u,\rho)T_2(u,\rho))}^{T(u,\rho)}}{R_2^2(u,\rho)+(1-u^2)T_2^2(u,\rho)}$$
$$=: Q_c(u,\rho). \tag{4}$$

4. ROOT COUNTING FORMULAS

In this section, we state some formulas for counting the root distribution with respect to the circle $\mathcal{C}_\rho$, for real polynomials and real rational functions. These formulas will be necessary for our solution of the stabilization problem but are also of independent interest. They represent generalizations of earlier results obtained by us in (Keel and Bhattacharyya, n.d.) for the unit circle.

Theorem 1. Let $P(z)$ be a real polynomial with no roots on the circle $\mathcal{C}_\rho$ and suppose that $T(u,\rho)$ has p zeros at $u=-1$. Then the number of roots i of $P(z)$ in the interior of the circle $\mathcal{C}_\rho$ is given by

$$i=\frac{1}{2}\mathrm{Sgn}\left[T^{(p)}(-1,\rho)\right]\Bigg(\mathrm{Sgn}\left[R(-1,\rho)\right]+\left(2\sum_{j=1}^{k}(-1)^j\mathrm{Sgn}\left[R\left(t_j,\rho\right)\right]\right. \tag{5}$$
$$\left.+(-1)^{k+1}\mathrm{Sgn}\left[R(+1,\rho)\right]\right)\Bigg).$$

The result derived above can now be extended to the case of rational functions. Let $Q(z)=\frac{P_1(z)}{P_2(z)}$ where $P_i(z), i=1,2$ are real rational functions. Let $R_i(u,\rho)+j\sqrt{1-u^2}T_i(u,\rho), i=1,2$ denote the Tchebyshev representations of $P_i(z), i=1,2$ and $Q_C(u,\rho)$ denote the Tchebyshev representation of $Q(z)$ on the circle $\mathcal{C}_\rho$. Let $R(u,\rho), T(u,\rho)$ be defined by:

$$R(u,\rho)=R_1(u,\rho)R_2(u,\rho)+(1-u^2)T_1(u,\rho)T_2(u,\rho)$$
$$T(u,\rho)=T_1(u,\rho)R_2(u,\rho)-R_1(u,\rho)T_(u,\rho)$$

Suppose that $T(u,\rho)$ has p zeros at $u=-1$ and let $t_1\cdots t_k$ denote the real distinct zeros of $T(u)$ of odd multiplicity ordered as follows:

$$-1<t_1<t_2<\cdots<t_k<+1.$$

Theorem 2. Let $Q(z)=\frac{P_1(z)}{P_2(z)}$ where $P_i(z), i=1,2$ are real polynomials with i_1 and i_2 zeros respectively inside the circle $\mathcal{C}_\rho$ and no zeros on it. Then

$$i_1-i_2=\frac{1}{2}\mathrm{Sgn}\left[T^p(-1,\rho)\right]\Bigg(\mathrm{Sgn}\left[R(-1,\rho)\right]+2\sum_{j=1}^{k}(-1)^j\mathrm{Sgn}\left[R\left(t_j,\rho\right)\right] \tag{6}$$
$$+(-1)^{k+1}\mathrm{Sgn}\left[R(+1,\rho)\right]\Bigg).$$

5. PARAMETER SEPARATION AND STABILIZING SET COMPUTATION FOR DIGITAL PID CONTROLLERS

In this section, we give a general parametrization of PID controllers in transfer function form. These will be used in the sequel to compute the stabilizing set. Similar parametrizations may also be achieved for PI and PD controllers. The general formula of a discrete PID controller is:

$$C(z)=K_P+K_IT\cdot\frac{z}{z-1}+\frac{K_D}{T}\cdot\frac{z-1}{z}=\frac{\left(K_P+K_I+\frac{K_P}{T}\right)z^2+\left(-K_P-\frac{2K_D}{T}\right)z+\frac{K_D}{T}}{z(z-1)}.$$

We use

$$C(s)=\frac{K_2z^2+K_1z+K_0}{z(z-1)} \tag{7}$$

where

$$K_P=-K_1-2K_0, \qquad K_I=\frac{K_0+K_1+K_2}{T},$$
$$K_D=K_0T.$$

The main idea is to construct a polynomial or rational function such that the controller parameters are separated as much as possible in the real and imaginary parts. By applying the root counting formulas to this function, we can often "linearize" the problem. We emphasize that other root counting formulas such as Jury's test applied to these problems result in difficult nonlinear problem, which are often impossible to solve. In this section we present a complete development along with an example for PID controllers.

Consider the discrete plant $P(z)=\frac{N(z)}{D(z)}$ with $N(z), D(z)$ being real polynomials of $\deg[D(z)]\geq\deg[N(z)]$ with the PID controller:

$$C(z)=\frac{K_2z^2+K_1z+K_0}{z(z-1)} \tag{8}$$

The characteristic polynomial becomes

$$\delta(z)=z(z-1)D(z)+\left(K_2z^2+K_1z+K_0\right)N(z). \tag{9}$$

Multiplying the characteristic polynomial by $z^{-1}N\left(z^{-1}\right)$, we have

$$z^{-1}\delta(z)N\left(z^{-1}\right)=(z-1)D(z)N\left(z^{-1}\right)+\left(K_2z+K_1+K_0z^{-1}\right)N(z)N\left(z^{-1}\right). \tag{10}$$

Recall the Tchebyshev representation, and the facts that

$$z=e^{j\theta}=-u+j\sqrt{1-u^2} \tag{11}$$
$$z^{-1}=e^{-j\theta}=-u-j\sqrt{1-u^2} \tag{12}$$

we have

$$z^{-1}\delta(z)N\left(z^{-1}\right)$$
$$=-(u+1)P_1(u)-\left(1-u^2\right)P_2(u)-\left[\left(K_0+K_2\right)u-K_1\right]P_3(u)$$
$$+j\sqrt{1-u^2}\Big[-(u+1)P_2(u)+P_1(u)\left(K_2-K_0\right)P_3(u)\Big]$$
$$=R\left(u,K_0,K_1,K_2\right)+j\sqrt{1-u^2}T\left(u,K_0,K_2\right)$$

where

$$
\begin{aligned}
P_1(u) &= R_D(u)R_N(u) \\
&\qquad + \left(1-u^2\right)T_D(u)T_N(u) \\
P_2(u) &= R_N(u)T_D(u) - T_N(u)R_D(u) \qquad (13)\\
P_3(u) &= R_N^2(u) + \left(1-u^2\right)T_N^2(u).
\end{aligned}
$$

Now let $K_3 := K_2 - K_0$.

$$
K_P = -K_1 - 2K_0, \qquad K_I = \frac{K_0+K_1+K_2}{T},
$$
$$
K_D = K_0 T.
$$

Hence we rewrite $R\left(u, K_0, K_1, K_2\right)$ and $T\left(u, K_0, K_2\right)$ as follows.

$$
\begin{aligned}
R\left(u, K_0, K_1, K_2\right) &= -(u+1)P_1(u) \\
&\quad - \left(1-u^2\right)P_2(u) - \left[\left(2K_2 - K_3\right)u - K_1\right]P_3(u) \\
T\left(u, K_3\right) &= P_1(u) - (u+1)P_2(u) + K_3P_3(u)
\end{aligned}
$$

We observe the parameter separation achieved above: K_3 appears only in the imaginary part and K_1, K_2, K_3 appear linearly in the real part. Thus by applying root counting formulas to the rational function on the left, and imposing the stability requirement yields linear inequalities in the parameters for fixed K_3. The solution is completed by sweeping over the range of K_3 for which an adequate number of real roots t_k exist. We illustrate with an example.

Example 3.

$$
G(z) = \frac{1}{z^2 - 0.25}
$$

Then

$$
R_D(u) = 2u^2 - 1.25, \qquad T_D(u) = -2u
$$
$$
R_N(u) = 1, \qquad T_N(u) = 0
$$
$$
P_1(u) = 2u^2 - 1.25, P_2(u) = -2u, P_3(u) = 1.
$$

Recall eq. (10). Since $G(z)$ is of order 2 and C(z) is of order 2, the number of roots of $\delta(z)$ inside the unit circle is required to be 4 for stability. From Theorem 1,

$$
i_i - i_2 = \underbrace{\left(i_\delta + i_{N_r}\right)}_{i_1} - \underbrace{(l+1)}_{i_2}
$$

where i_δ and i_{N_r} are the numbers roots of $\delta(z)$ and the reverse polynomial of $N(z)$, respectively. l is the order of $N(z)$ and 1 came from the term z^{-1}. Since the required i_δ is 4, $i_{N_r} = 0$, and $l = 0$, $i_1 - I_2$ is required to be 3. To illustrate the example in detail, we first fix $K_3 = 1.3$. Then the real roots of $T\left(u, K_3\right)$ in $(-1, 1)$ are -0.4736 and -0.0264. Thus, $\mathcal{U} = \{-1 \;\; -0.4736 \;\; -0.0264 \;\; 1\}$. Furthermore, we have $\mathrm{Sgn}[T(-1)] = 1$. From Theorem 1 and $i_1 - i_2 = 3$, we have the only one valid sequence for

$$
\frac{1}{2}\mathrm{Sgn}[T(-1)]\Big(\mathrm{Sgn}[R(-1)] - 2\mathrm{Sgn}[R(-0.4736)] + 2\mathrm{Sgn}[R(-0.0264)] - \mathrm{Sgn}[R(1)]\Big) = 3.
$$

Sgn[R(-1)]	Sgn[R(-0.4736)]	Sgn[R(-0.0264)]
1	-1	1
Sgn[R(1)]	$2\left(i_1 - i_2\right)$	
-1	6	

From this valid sequence, we have the following set of linear inequalities.

$$
\begin{aligned}
-1.3 + K_1 + 2K_2 &> 0 \\
-0.9286 + K_1 + 0.9472 &< 0 \\
1.1286 + K_1 + 0.0528K_2 &> 0 \\
-0.2 + K_1 - 2K_2 &< 0.
\end{aligned}
$$

This set of inequalities characterize the stability region in (K_1, K_2) space for the fixed $K_3 = 1.3$. By repeating this procedure for the range of K_3, we obtain the the stability region shown in the left of Figure 1. Consider the following relation.

$$
\begin{bmatrix} K_P \\ K_I \\ K_D \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 \\ \frac{1}{T} & \frac{1}{T} & \frac{1}{T} \\ T & 0 & 0 \end{bmatrix} \begin{bmatrix} K_0 \\ K_1 \\ K_2 \end{bmatrix}
$$
$$
= \begin{bmatrix} -1 & -2 & -2 \\ \frac{1}{T} & \frac{2}{T} & -\frac{1}{T} \\ 0 & T & -T \end{bmatrix} \begin{bmatrix} K_1 \\ K_2 \\ K_3 \end{bmatrix}.
$$

Using this relation, we plot the region in (K_P, K_I, K_D) space in the left of Figure 1.

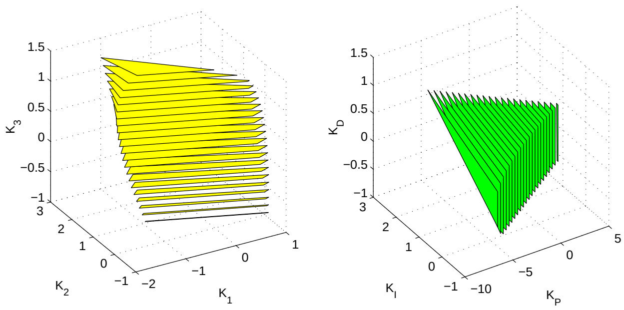

Fig. 1. Stability regions in (K_1, K_2, K_3) space (left) and (K_P, K_I, K_D) space (right)

6. MAXIMALLY DEADBEAT CONTROL VIA PID CONTROLLERS

An important design technique in digital control is deadbeat control wherein one places all closed loop poles at the origin. If this is used in conjunction with integral control the tracking error is zeroed out in a finite number of sampling steps. Deadbeat control requires in general that we be able to control all the poles of the system. However, such a pole placement design is in general not possible when a lower order controller is used. Thus, we are motivated to design a PID controller that places the closed loop as close to the origin as possible. The transient response of such a system will decay out faster than any other design and therefore the fastest possible convergence of the error under PID control will be achieved.

The design scheme to be developed will attempt to place the closed loop poles in a circle of minimum radius ρ. Let $\mathcal{S}_\rho$ denote the set of PID controllers achieving such a closed loop root cluster. We show below how $\mathcal{S}_\rho$ can be computed for fixed ρ. The minimum value of ρ can be found by determining the value ρ^* for which $\mathcal{S}_{\rho^*} = \phi$ but $\mathcal{S}_\rho \neq \phi, \rho > \rho^*$. Now let us again consider the PID controller

$$C(z) = \frac{K_2 z^2 + K_1 z + K_0}{z(z-1)} \tag{14}$$

and the characteristic polynomial

$$\delta(z) = z(z-1)D(z) + \left(K_2 z^2 + K_1 z + K_0\right) N(z).$$

Note that

$$D(z)|_{z=-\rho u + j\rho\sqrt{1-u^2}} = R_D(u,\rho) + j\sqrt{1-u^2}T_D(u,\rho)$$
$$N(z)|_{z=-\rho u + j\rho\sqrt{1-u^2}} = R_N(u,\rho) + j\sqrt{1-u^2}T_N(u,\rho)$$

and

$$N\left(\rho^2 z^{-1}\right)\big|_{z=-\rho u + j\rho\sqrt{1-u^2}} = N(z)|_{z=-\rho u - j\rho\sqrt{1-u^2}}$$
$$= R_N(u,\rho) - j\sqrt{1-u^2}T_N(u,\rho).$$

We now evaluate

$$\rho^2 z^{-1}\delta(z) N\left(\rho^2 z^{-1}\right) = \rho^2 z^{-1} \underbrace{\left[z(z-1)D(z) + \left(K_2 z^2 + K_1 z + K_0\right) N(z)\right]}_{\delta(z)} \cdot N\left(\rho^2 z^{-1}\right)$$

over the circle $\mathcal{C}_\rho$

$$\rho^2 z^{-1}\delta(z) N\left(\rho^2 z^{-1}\right)\big|_{z=-\rho u + j\rho\sqrt{1-u^2}}$$
$$= -\rho^2(\rho u + 1)P_1(u,\rho) - \rho^3\left(1-u^2\right)P_2(u,\rho)$$
$$- \left[\left(K_0 + K_2\rho^2\right)\rho u - K_1\rho^2\right] P_3(u,\rho)$$
$$+ j\sqrt{1-u^2}\left[\rho^3 P_1(u,\rho) - \rho^2(\rho u + 1)P_2(u,\rho) + \left(K_2\rho^2 - K_0\right)\rho P_3(u,\rho)\right]$$

where $P_1(u,\rho)$, $P_2(u,\rho)$, and $P_3(u,\rho)$ are defined as in eq. (13). By letting

$$K_3 := K_2\rho^2 - K_0,$$

we have

$$\rho^2 z^{-1}\delta(z) N\left(\rho^2 z^{-1}\right)\big|_{z=-\rho u + j\rho\sqrt{1-u^2}}$$
$$= -\rho^2(\rho u + 1)P_1(u,\rho) - \rho^3\left(1-u^2\right)P_2(u,\rho)$$
$$- \left[\left(2K_2\rho^2 - K_3\right)\rho u - K_1\rho^2\right] P_3(u,\rho)$$
$$+ j\sqrt{1-u^2}\left[\rho^3 P_1(u,\rho) - \rho^2(\rho u + 1)P_2(u,\rho) + K_3\rho P_3(u,\rho)\right]$$

To determine the set of controllers achieving root clustering inside a circle of radius ρ we proceed as before: Fix K_3, use the root counting formula of 2, develop linear inequalities in K_2, K_3 and sweep over the requisite range of K_3. This procedure is then performed as ρ decreases until the set of stabilizing PID parameters just disappears. The following example illustrates this scheme.

Example 4. We consider the same plant used in Example 3. Figure 2 (left) shows the stabilizing set in the PID gain space at $\rho = 0.275$. For a smaller value of ρ, the stabilizing region in PID parameter space disappears. This means that there is no PID controller available to push all closed loop poles inside the circle of radius smaller than 0.275. From this we select a point inside the region that is

$$K_0 = 0.0048, K_1 = -0.3195, K_2 = 0.6390 \quad \text{and}$$
$$K_P = 0.3099, K_I = 0.3243, K_D = 0.0048.$$

Figure 2 (right) shows the closed loop poles that lie inside the circle of radius $\rho = 0.275$. The roots are $0.2500 \pm j0.1118$ and $0.2500 \pm j0.0387$.

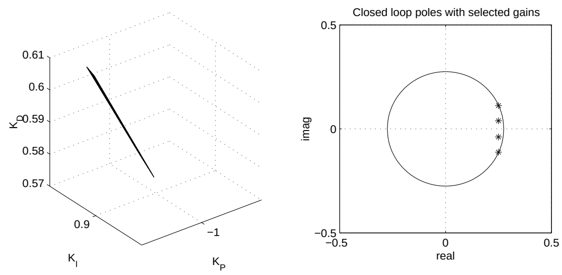

Fig. 2. Stability regions with $\rho = 0.275$ (left). Closed loop poles of the selected PID gains (right)

To illustrate further, we select several sets of stabilizing PID parameters from the set obtained in Example 3 (i.e., $\rho = 1$) and compare the step responses between them. Figure 3 shows that the maximally deadbeat design produces nearly deadbeat response.

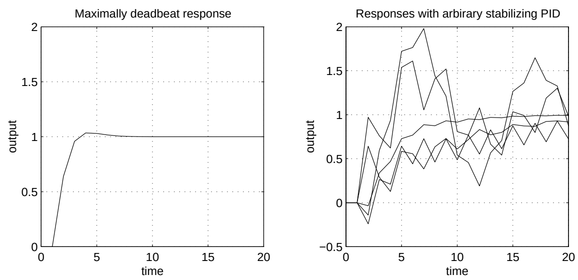

Fig. 3. Maximally deadbeat design (left). Arbitrary stabilization (right)

7. MAXIMUM DELAY TOLERANCE DESIGN

In some control systems an important design parameter is the delay tolerance of the loop, that is the maximum delay that can be inserted into the loop without destabilizing it. In digital control a delay of k sampling instants is represented by z^{-k}. We use this to determine the maximum delay that a control loop under PID control can be designed to tolerate. This gives the limit of delay tolerance achievable for the given plant under PID control. Let the plant be

$$G(z) = \frac{N(z)}{D(z)}. \tag{15}$$

We consider the problem of finding the maximum delay L^* such that the plant can be stabilized by a PID controller. In other words, finding the maximum values of L^* such that the stabilizing PID gain set for the plant

$$z^{-L}G(z) = \frac{N(z}{z^L D(z)}, \qquad \text{for } L = 0, 1, \cdots, L^*$$

is not empty. Let $\mathcal{S}_i$ be the set of PID gains that stabilizes the plant $z^{-i}G(z)$. Then it is clear that

$$\cap_{i=0}^{L} \mathcal{S}_i \text{ stabilizes } z^i G(z), \quad \forall\ i = 0, 1, \cdots, L.$$

We illustrate this computation by an example.

Example 5. Consider the same example we used before. Figure 4 (left) shows the stabilizing PID gains when there is no delay (i.e, $L = 0$). The right figure shows the stabilizing PID gains when $L = 0, 1$. As seen in the figure, the size of the set is reduced as the delay increases.

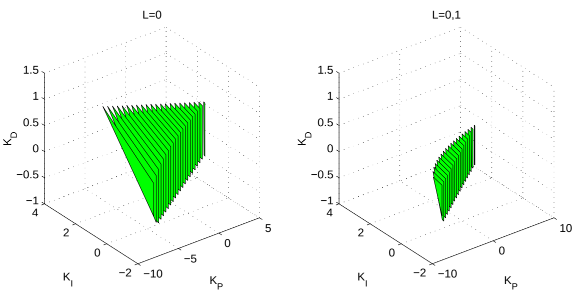

Fig. 4. Stability region for delayed systems

In many systems, the set disappears for a large value of L^*. This is the maximum delay that can be stabilized by any PID controllers. To illustrate better, we fix $K_3 = 1$. Figure 5 shows that the stability region reduces when the required time delay increases and for the system with the delay $L \in (0, 3)$ the region vanishes.

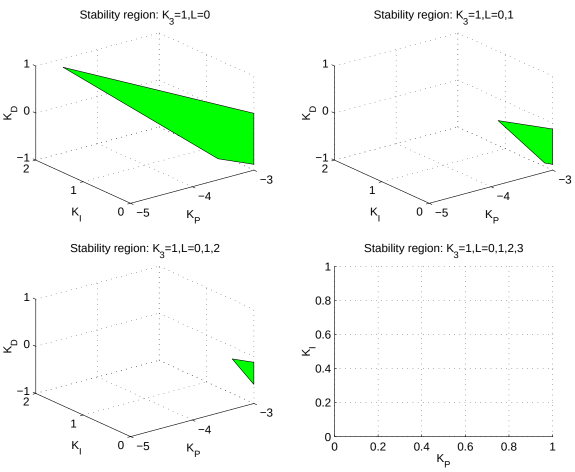

Fig. 5. Stability region for delayed systems

8. CONCLUDING REMARKS

In this paper, we have given a solution to the problem of stabilization of a digital control system using PID controllers. The solution is complete in the sense that a constructive yes or no answer to whether stabilization is possible, is given and in case it is possible the entire set is determined by solving sets of linear equations. These solution sets open up the possibility of improved and optimal design using PID controllers. The questions of loop shaping time domain response shaping and robust designs are important candidates for research.

9. REFERENCES

Ackermann, J. (1985). *Sampled-data Control Systems: Analysis and Synthesis, Robust System Design.* Springer-Verlag. New York, NY.

Åström, K. and T. Hägglund (1995). *PID Controllers: Theory, Design, and Tuning.* 2nd ed.. Instrument Society of America. Research Triangle Park, NC.

Datta, A., M.T. Ho and S.P. Bhattacharyya (2000). *Structure and Synthesis of PID Controllers.* Springer-Verlag. London.

Dorato, P. (2000). *Analytic Feedback System Design: An Interpolation Approach.* Books Cole Publishing. Pacific Grove, CA.

Goodwin, G.C., S.F. Graebe and M.E. Salgado (2001). *Control System Design.* Prentice Hall PTR. Upper Saddle River, NJ.

Keel, L.H. and S.P. Bhattacharyya (n.d.). Root counting, phase unwrapping, stability and stabilization of discrete time systems. *Linear Algebra and its Application.* (To appear).

Mansour, M. (1992). Robust stability in systems described by rational functions. In: *Control and Dynamic Systems* (C.T. Leandes, Ed.). Vol. 51. pp. 79 – 128. Academic Press. New York.

Pólya, G. and G. Szegó (1976). *Problems and Theorems in Analysis II.* reprint of the 1976 ed.. Springe-Verlag. New York, NY.

Xu, H., A. Datta and S.P. Bhattacharyya (2001). Computation of all stabilizing PID gains for digital control systems. *IEEE Transactions on Automatic Control* **46**(4), 647 – 652.

Yamada, I. and N.K. Bose (n.d.). Algebraic phase unwrapping and zero distribution of polynomial for continuous time systems. *IEEE Transactions on Automatic Control.* (submitted).

Yamada, I., K. Kurosawa, H. Hasegawa and K. Sakaniwa (1998). Algebraic multidimensional phase unwrapping and zero distribution of complex polynomials - characterization of multivariate stable polynomials. *IEEE Transactions on Signal Processing* **46**(6), 1639 – 1664.