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EFFICIENT ALGORITHMS FOR MISTUNING ANALYSIS [1]

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Abstract: This paper describes two algorithms to calculate bounds on the largest frequency response function that is attained when a large number of parameters is perturbed simultaneously within a box. The algorithms have been created to calculate worst-case responses of mistuned (perturbed) bladed disks, which is an important step in the analysis and optimization of advanced turbomachinery components. Numerical comparisons with other algorithms demonstrate the relative efficiency and accuracy of the proposed algorithms.

Keywords: Algorithms, frequency response, robustness, vibration, convex optimization, rotors.

1. INTRODUCTION

The prediction and optimization of frequency response functions for systems with high modal density and low damping is of great interest to the manufacturers of modern gas turbine aero-engines. This interest is motivated in part by a need to analyze and optimize vibratory responses in new rotor designs which exhibit large number of lightly-damped modes in the operating envelope. Since 1998, we have been working with industry in the development of metrics and effective tools to analyze and optimize real-world rotors. These systems typically posses a degree of spatial symmetry that can be exploited to develop improved methods and tools for analysis and design.

A key problem in this industry is the calculation of the worst-case frequency response that a bladed disk may posses when several (on the order of hundreds) parameters are perturbed. These perturbations, known in the industry as *mistuning*, could cause large increases in vibration amplitudes with respect to the unperturbed or *tuned* condition. The importance of this problem has been discussed in numerous recent publications (Griffin and Richardson, 1995; Thompson, 1999; Slater *et al.*, 1999).

The present paper discusses efficient algorithms for the calculation of worst-case frequency response γ, defined by [2]

$$\gamma = \max_{\delta=(\delta_1,\ldots,\delta_n)} \sigma_{\max}(T(\omega,\delta)) \tag{1a}$$

$$\text{s.t. } |\delta_1| \leq \theta, \ldots, |\delta_n| \leq \theta \tag{1b}$$

where the entries of the frequency response function $T(\omega,\delta)$ depend rationally on the real perturbations $\delta_1,\ldots,\delta_n$ and ω is the frequency.

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[2] The symbol $\sigma_{\max}$ denotes the largest singular value.

It is known that the complexity of computing γ is NP hard (Braatz *et al.*, 1994). This fact suggests that finding γ is computationally intractable except for small or very special problems. For this reason, the usual approach is to compute upper and lower bounds for γ. The upper bound provides a guarantee on the maximal response, while the lower bound is used to asses the conservatism of the upper bound.

To compute an upper bound for γ, we propose the LFTB algorithm (D'Amato and Rotea, 2001; Rotea and D'Amato, 2001). This algorithm computes the bound for rational functions that follows from the results in (Fan *et al.*, 1991). This upper bound is obtained as the solution of a structured semidefinite program (SDP), which is a special kind of convex optimization problem. To the best of our knowledge, LFTB is the most efficient algorithm for solving this structured SDP. Specifically, with n real parameters in (1), LFTB requires $O(n)$ less memory and flops per iteration than generic interior-point solvers for semidefinite programs. Thus in problems with hundreds of parameters, as is the case in mistuning analysis of industrial bladed disks, LFTB offers substantial savings.

To compute a lower bound for γ we propose LFTLB (D'Amato, 2001), which is a minor modification of the cyclic algorithm reported in (Packard *et al.*, 2000). This algorithm actually produces a perturbation δ achieving the lower bound on the worst-case response γ. Our numerical experience indicates that LFTLB gives tighter bounds than those obtained with the methods in (Balas *et al.*, 1995; Coleman *et al.*, 1999) when applied to mistuning analysis problems.

The algorithms are demonstrated in mistuning analysis and optimization of a typical bladed disk model with $n = 56$ real perturbations. In this example, the ratio of the lower bound to the upper bound is better than 0.999, which means that our algorithms estimate the γ with 0.1% relative error. In addition, the calculation of the upper bound is extremely fast, with actual computing times about 100x faster than the times required by the generic solvers in (Gahinet *et al.*, 1995).

The algorithms LFTB and LFTLB may be obtained from `roger.ecn.purdue.edu/~rotea`

2. MODELS

We consider frequency domain models for mistuned rotors of the form

$$(-\omega^2 M(\delta) + \mathrm{j}\omega C(\delta) + K(\delta))x = Gu \quad (2a)$$

$$y = Hx \quad (2b)$$

where x is the vector of coordinates of motion, u is the vector of external forces, y is the response or output vector, and δ is a vector of real perturbations or, as called in the industry, the mistuning parameters. The entries of δ have amplitude bounded by θ . The entries of the mass, damping and stiffness matrices $M(\delta)$, $C(\delta)$ and $K(\delta)$ are rational functions of δ. The matrix G is the input matrix, and H is the output matrix.

Model (2) is a fairly general representation of perturbed rotors for forced response analysis and includes the models in (Griffin and Hoosac, 1984; Castanier *et al.*, 1997; Yang and Griffin, 1999).

The frequency response function (FRF) of model (2) is given by

$$T(\omega,\delta) = H(-\omega^2 M(\delta) + \mathrm{j}\omega C(\delta) + K(\delta))^{-1} G \quad (3)$$

It is known that FRFs of any linear system that depends rationally on n parameters $\delta_1, \dots, \delta_n$ can always be expressed in the form (Doyle *et al.*, 1991)

$$T(\omega,\delta) = D + C\Delta(\delta)(I - A\Delta(\delta))^{-1} B \quad (4)$$

where the matrices A, B, C and D are functions of ω and

$$\Delta(\delta) = \mathrm{diag}(\delta_1 I_{r_1}, \dots, \delta_n I_{r_n}) \quad (5)$$

for some integers $r_1, \dots, r_n$. The dependence on the frequency ω of the matrices A, B, C, D is not made explicit to simplify the notation. The representation (4) is also known as linear fractional transformation or LFT and is required by our algorithms.

3. BOUNDS FOR THE WORST-CASE FREQUENCY RESPONSE

3.1 An Upper Bound

The upper bound we propose follows from (Fan *et al.*, 1991). This bound is defined by the following optimization problem

$$\beta = \min w \quad (6a)$$

$$\text{s.t. } X = X^* = \mathrm{diag}(X_1, \dots, X_n) \geq 0 \quad (6b)$$

$$Y = -Y^* = \mathrm{diag}(Y_1, \dots, Y_n) \quad (6c)$$

$$\begin{bmatrix} A & B \\ \theta^{-1} I & 0 \end{bmatrix}^* \begin{bmatrix} X & Y \\ Y^* & -X \end{bmatrix} \begin{bmatrix} A & B \\ \theta^{-1} I & 0 \end{bmatrix} + \begin{bmatrix} C^*C & C^*D \\ D^*C & D^*D - wI \end{bmatrix} \leq 0 \quad (6d)$$

where w is a real scalar variable and the matrix variables X_i and Y_i are $r_i \times r_i$ complex matrices; the integers n, r_1, $\dots$, r_n are from (5). The data of problem (6) is given by the complex matrices A, B, C, and D in (4), and the size θ of the perturbations.

It has been shown (Fan *et al.*, 1991) that γ, defined in (1), and the number β in (6) satisfy

$$\gamma \leq \sqrt{\beta} \quad (7)$$

when T is of the form (4) and the perturbations $\delta_1, \ldots, \delta_n$ are real numbers bounded by θ.

Problem (6) has become a key optimization problem in the analysis of uncertain systems. This problem is a structured semidefinite program, which is a special kind of convex optimization problem. As such, it can be solved, to any desired accuracy, using interior-point methods. To the best of our knowledge, the most efficient method for solving problem (6) is the interior-point algorithm LFTB (D'Amato and Rotea, 2001; Rotea and D'Amato, 2001).

LFTB exploits the specific structure in (6) to lower the memory requirements and floating point operations by a factor of n relative to generic interior-point algorithms. Thus, LFTB achieves substantial savings in problems with large number of real parameters (n between 50 and 500) as is the case in the mistuning analysis of industrial rotors.

The efficiency of LFTB, relative to available solvers, is shown in Fig. 1 for a typical mistuning analysis problem (Rotea and D'Amato, 2001). The figure shows the total time to solve (6) as a function of the number of real parameters n. The solver MINCX is from the LMI Toolbox (Gahinet *et al.*, 1995) and the solver SDPHLF is from the package SDPT3 (Toh *et al.*, 1999). LFTB solves a problem with $n = 50$ real parameters in 50sec, which is much less than the 750sec required with SDPHLF, or the 6300sec with MINCX. The time required by generic solvers grows like $O(n)$x the time required by LFTB. In mistuning problems n is large and, therefore, the savings with LFTB are substantial. Further details on LFTB may be found in (D'Amato and Rotea, 2001; Rotea and D'Amato, 2001), which describe the pseudocode for this algorithm.

3.2 A Lower Bound

The algorithm proposed for calculating a lower bound for the worst-case response γ defined in (1) is based on the approach presented in (Packard *et al.*, 2000). This algorithm produces a sequence of vectors $\{\delta^k\}_{k=1}^{\infty}$ with non-decreasing values of the cost function in (1), i.e.

$$\sigma_{\max}(T(\omega, \delta^k)) \leq \sigma_{\max}(T(\omega, \delta^{k+1})) \quad k = 1, 2, \ldots$$

At the k-th iteration, the estimate δ^k of the worst-case perturbation is computed as follows. Certain randomly chosen entries of δ^k are fixed at their values in the previous iteration. The remaining entries are allowed to vary along a randomly chosen pattern.

To be more precise, let J^k denote the set of indices for the entries of δ^k that are fixed at their previous value. Let π^k denote the real vector defining the random pattern for the remaining entries of δ^k. The perturbation δ^k is computed from

$$\delta^k = \phi^k + \alpha_{\text{opt}} \pi^k \tag{8a}$$

where the i-th entry of vectors ϕ^k and π^k is given by

$$\phi_i^k = \begin{cases} \delta_i^{k-1} & \text{if } i \in J^k \\ 0 & \text{if } i \notin J^k \end{cases} \tag{8b}$$

$$\pi_i^k = \begin{cases} 0 & \text{if } i \in J^k \\ \text{random number} & \text{if } i \notin J^k \end{cases} \tag{8c}$$

$$\max_{1 \leq i \leq n} |\pi_i^k| = \theta \tag{8d}$$

and α_{opt} is an optimal step size whose computation is explained below.

If $\sigma_{\max}(T(\omega, \delta^k)) \geq \sigma_{\max}(T(\omega, \delta^{k-1}))$ we accept the new perturbation in (8). Otherwise, we set $\delta^k = \delta^{k-1}$. In either case, we repeat the process until $\sigma_{\max}(T(\omega, \delta^k))$ does not increase for a pre-specified number of iterations.

The scalar α_{opt} is the quantity that maximizes the cost function $\sigma_{\max}(T(\omega, \phi^k + \alpha\pi^k))$. That is

$$\alpha_{\text{opt}} = \arg\max_\alpha \; \sigma_{\max}(G(\alpha)) \quad \text{s.t. } \alpha \in [-1, 1] \tag{9}$$

where

$$G(\alpha) = T(\omega, \phi^k + \alpha\pi^k) \tag{10}$$

The rational function $G(\alpha)$ in (10) can be expressed in the form

$$G(\alpha) = D_g + C_g \alpha (I - A_g \alpha)^{-1} B_g \tag{11}$$

using elementary operations on linear fractional transformations. From this representation and the results in (Packard *et al.*, 2000) it can be shown that problem (9) can be solved by solving a sequence of eigenvalue problems.

The precise result from is as follows (Packard *et al.*, 2000). Define the complex matrix

$$\mathcal{H}(\rho) = \begin{bmatrix} A_g & B_g B_g^* \\ 0 & A_g^* \end{bmatrix} + \begin{bmatrix} B_g D_g^* \\ C_g^* \end{bmatrix} (\rho^2 I - D_g D_g^*)^{-1} \begin{bmatrix} C_g & D_g B_g^* \end{bmatrix}$$

where all matrices are from (11). Define the set

$$\mathcal{R} = \{\rho \mid \mathcal{H}(\rho) \text{ has a real eigenvalue } \lambda \text{ satisfying } |\lambda| \geq 1\} \tag{12}$$

Let ρ_s denote the supremum of the set $\mathcal{R}$. Let λ_s denote a real eigenvalue of $\mathcal{H}(\rho_s)$ with magnitude one or bigger. Then $\alpha_{\text{opt}} = \lambda_s^{-1}$.

The proposed lower bound algorithm is coined LFTLB. Its pseudocode is in Table 1. LFTLB stops when the cost function makes no further improvement for a pre-specified number of consecutive iterations. Further details on LFTLB may be found in (D'Amato, 2001).

In the mistuning problems considered, LFTLB has shown superior accuracy (higher lower bound) than available algorithms for solving this problem such as the power iteration method (Balas *et al.*, 1995) or the specialized functions in (Coleman *et al.*, 1999). A numerical comparison is in section 4.

Table 1. LFTLB pseudocode.

```
choose a random perturbation vector δ^0 with |δ_i| ≤ θ
set k = 1
while no stopping criterion
  choose the set of indices J^k randomly
  calculate φ^k and π^k using (8)
  find a representation (11) for G(α) defined in (10)
  calculate the supremum ρ_s of the set R in (12)
  calculate a real eigenvalue λ_s of H(ρ_s) such that
      |λ_s| ≥ 1 and set α_opt = λ_s^{-1}
  if σ_max(T(ω, δ^{k-1})) ≤ ρ_s
      δ^k = φ^k + α_opt π^k
  end
  k ← k + 1
end
```

4. APPLICATION EXAMPLE

The rotor example is a mass-spring model with $n = 56$ sectors as shown in Fig. 2. There are two degrees of freedom per sector, one DOF represents blade motion and the other DOF disk motion. All model parameters are known except for the blade-alone stiffness which, for the i-th blade, is modeled as

$$k_{1i} = 1 + \delta_i \tag{13}$$

where $k_1 = 1$ is the nominal (tuned) blade-alone stiffness and δ_i is the unknown but bounded stiffness perturbation or mistuning. The equation for this model is of the form shown in (2). Damping is added to the model assuming that the tuned modes (obtained with all perturbations δ_i set to zero) have uniform damping ratio ζ.

Let $F_i(\omega, \delta)$ denote the transfer function from forces at the blade degrees of freedom to the displacement x_{bi} of blade i. The objective is to study the FRF

$$T_i(\omega, \delta) = F_i(\omega, \delta) f \tag{14}$$

where f is a prescribed (complex) forcing vector. Thus, the FRF T_i represents the complex displacement of blade i in response to the excitation $f \exp(\mathrm{j}\omega t)$.

The combination of nominal symmetry and low damping produces a practical device sensitive to symmetry-breaking perturbations. Tiny blade-to-blade variations lead to large variations of the blade response $T_i(\omega, \delta)$.

The high sensitivity of these devices to real-world perturbations (e.g., due to manufacturing tolerances and wear) has created a strong need to predict the worst-case response that the rotor blades may posses. That is to calculate $\gamma(\theta)$ defined by

$$\gamma(\theta) = \max_{1 \le i \le 56} \max_{\omega \in \Omega} \max_{\delta} |T_i(\omega, \delta)| \tag{15}$$
$$\text{s.t. } |\delta_1| \le \theta, \ldots, |\delta_{56}| \le \theta$$

where $T_i(\omega, \delta)$ denotes the FRF of the i-th blade, Ω a given frequency grid, and θ is the size of the perturbations.

The most inner maximization in (15) can be estimated using the algorithms LFTB and LFTLB discussed in section 3. The maximization on the frequency grid Ω may be done by repeated evaluation at each frequency. Repeated evaluation can also be used for the outer maximization. However, this is not necessary as the symmetry of the tuned rotor can be used to prove that all blades have the same worst-case response and same bounds on it. Thus, the outer maximization is eliminated completely and the worst-case response bounds of a single blade are calculated.

The results are in Fig. 3, which shows the variation of the worst-case response $\gamma(\theta)$ with the mistuning size θ for three different damping ratios. The plot shows the response normalized by the response of the tuned rotor, which is obtained as $\gamma(\theta)/\gamma(0)$.

Each point in the plot is obtained by solving a problem of the form (15) over a grid Ω with 30 frequencies. There are 1260 upper bounds and 42 lower bounds, taking a total time of 3.4 hours on a Pentium III PC at 600 MHz with 128 MB of memory. On average, each upper bound computation took 6 seconds and each lower bound 1.7 minutes. This is a very reasonable computing time for a problem with 56 simultaneous real perturbations.

The quality of the bounds is excellent in the sense that the ratio of the lower to upper bound is very close to one for all cases. In fact, in all cases this ratio was bigger than 0.999, which in turn implies that we have computed $\gamma(\theta)$ with a relative accuracy better than 0.1% in all cases.

Figure 3 demonstrates a known characteristic of these rotors–the blade response is very sensitive to perturbations. For example, with a damping ratio of 0.5%, $\theta = 2\%$ mistuning is enough to increase the response by a factor of 3 in the worst-case.

Figure 4 provides a comparison between LFTLB and two other methods for estimating a lower bound: the power iteration algorithm (PIA) from (Balas *et al.*, 1995) and a subspace trust region method (STIR) implemented in (Coleman *et al.*, 1999). The figure shows the worst-case frequency response function used to get the point $\theta = 3\%$ mistuning size, $\zeta = 0.5\%$ damping ratio, in Fig. 3. For all 30 points, the result from LFTLB is on top of the upper bound from LFTB. This is not the case for the PIA and STIR methods.

Intentional mistuning is known to reduce the sensitivity of the response to unintentional perturbations. Intentional perturbations, that break the nominal symmetry of a tuned rotor, are introduced to reduce worst-case response in the presence of the unintentional perturbations. Due to their efficiency, our algorithms can quickly determine an optimal intentional mistuning solution in which the worst-case response is minimized.

As an example, consider the problem of calculating nominal blade-alone stiffness k_{1i} that minimize the worst-case blade response, where the worst-case is calculated over the entire set of unintentional perturbations $\delta_1, \ldots, \delta_{56}$. It is assumed that there are two distinct nominal blades only, type A and type B, arranged in the alternating pattern ABABAB . . . AB.

The model is as before, see Fig. 2, except that the blade-alone stiffness k_{1i} is now modeled as

$$k_{1i} = 1 - \rho + \delta_i \text{ and } k_{1i} = 1 + \rho + \delta_i \qquad (16)$$

for type A and type B blades, respectively. In (16), ρ is the intentional mistuning parameter. If $\rho = 0$ there is no intentional mistuning and the model is tuned. If $\rho > 0$, there are 28 type A blades and 28 type B blades. As a result, the intentionally mistuned rotor has a unit average stiffness; thus, preserving the average stiffness of the tuned rotor.

Figure 5 shows the variation of nominal and worst-case response with intentional mistuning ρ for the case with 0.5% damping ratio. Both responses are normalized by the tuned nominal response; i.e., the response for $\rho = \theta = 0$. For each $\rho > 0$, and $\theta = 3\%$, the worst-case response is quickly (and accurately) estimated using LFTB.

Notice that the response of the nominal rotor with no perturbations δ_i and intentional mistuning $\rho > 0$ is worse than the tuned rotor response obtained with $\rho = 0$. However, nonzero intentional mistuning $\rho > 0$ does reduce the worst-case response; the optimum is $\rho = 0.05$. Thus intentional mistuning makes the rotor more robust to perturbations.

5. CONCLUSION

Two efficient algorithms (LFTB and LFTLB) for the calculation of worst-case frequency responses have been presented. These algorithms have been created to address vibration analysis problems for uncertain models with a large number of real perturbations. LFTB is believed to be the most efficient algorithm to date for calculating a well-known upper bound to the worst-case response. LFTLB follows from an existing algorithm and it works well in our rotor dynamics applications in the sense that it provides a tight lower bound.

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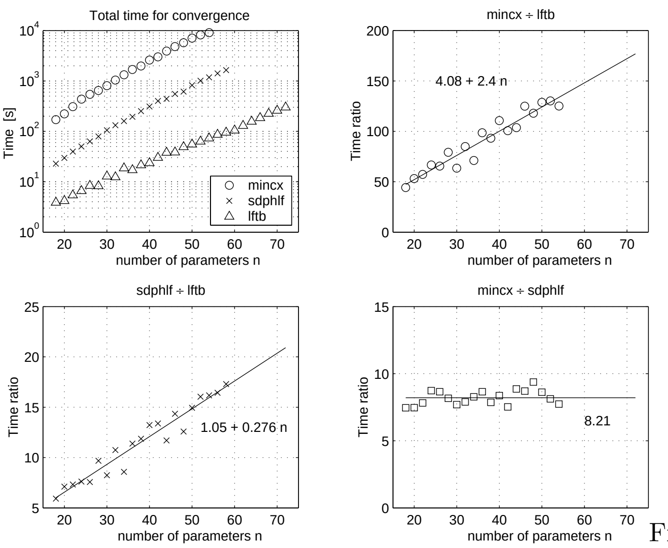

Fig. 1. Time for convergence as function of the number of parameters n for three algorithms. LFTB is $2.4n$x ($0.28n$x) faster than MINCX (SDPHLF).

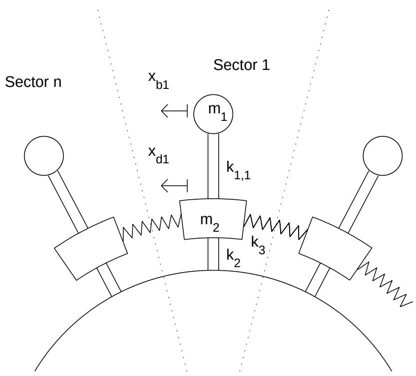

Fig. 2. Schematics of example rotor.

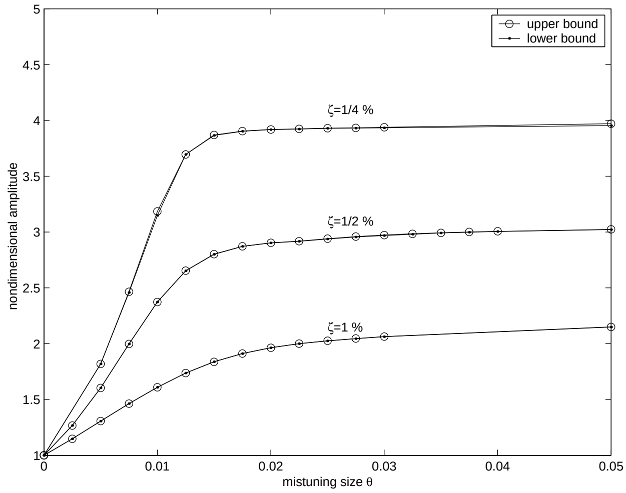

Fig. 3. Variation of worst-case frequency response $\gamma(\theta)/\gamma(0)$ with mistuning size θ, upper bound ($\circ$), lower bound ($\cdot$).

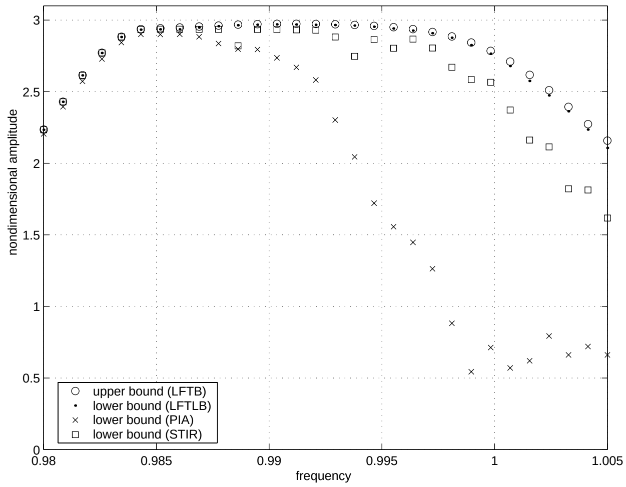

Fig. 4. Bounds on worst-case frequency response. Upper bound: LFTB ($\circ$). Lower bounds: LFTLB ($\cdot$), STIR ($\square$), PIA (x).

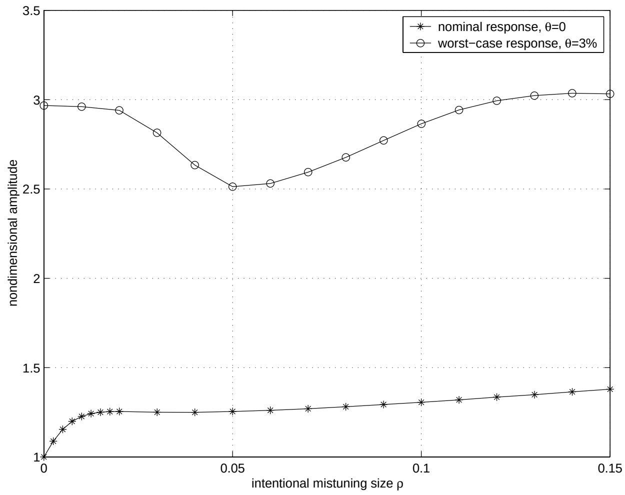

Fig. 5. Effect of the intentional mistuning parameter ρ on the nominal (*) and worst-case ($\circ$) responses.