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NONLINEAR OPTIMAL CONTROL OF HVAC SYSTEMS [1]

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Abstract: In this paper we consider the optimum control of heating, ventilation and air-conditioning (HVAC) systems. The objective of the control is to balance energy efficiency, against user comfort. The HVAC system it self is nonlinear, and the cost function to be minimized, non-quadratic. Both adaptive and non-adaptive strategies are given.

Keywords: Nonlinear, Optimal, Adaptive, Applications.

1. INTRODUCTION

A major contributor to energy consumption is environmental conditioning of commercial buildings. In the US it accounts for over a third of the net national consumption (Metha and Thurmann (1991)). In some countries this figure is even higher (Kelley (1992)). It is clear that given this high proportion of energy consumption cost attributable to heating and cooling of commercial buildings, even moderate increase in its efficiency can be expected to result in major energy savings. Consequently, in recent years there has been renewed interest in the design of heating, ventilation and air-conditioning (HVAC) systems that are more energy efficient and do not sacrifice thermal and environmental comfort. This paper focuses on the formulation and evaluation of a nonlinear optimal feedback control scheme that achieves such a balance.

The cost function in this case must include both the energy cost and a cost associated with overall level of comfort. Such a function must penalize at the same time excessive energy consumption and large deviations from user selected conditions that ensure prescribed levels of comfort. One of the key features of the resulting cost function is a non-quadratic term associated with the overall energy cost, specifically that the *cost of fan operation is cubic in the air flow rate (House et. al. (1991))*. Furthermore, the overall HVAC system is it self nonlinear.

The bulk of the previous work in this area has focused on the use of the Linear Quadratic Regulator (LQR) (Barnett and Cameron (1985)). For example, (Zaheeruddin and Patel (1993)) devise a LQR controller, that assumes a linearized model of the HVAC system, and drops the non-quadratic term associated with fan operation. To accommodate potential variations in, and imprecise knowledge of such system parameters as external temperature and thermal load, (Roth et. al. (1994)) formulate an adaptive optimal controller,

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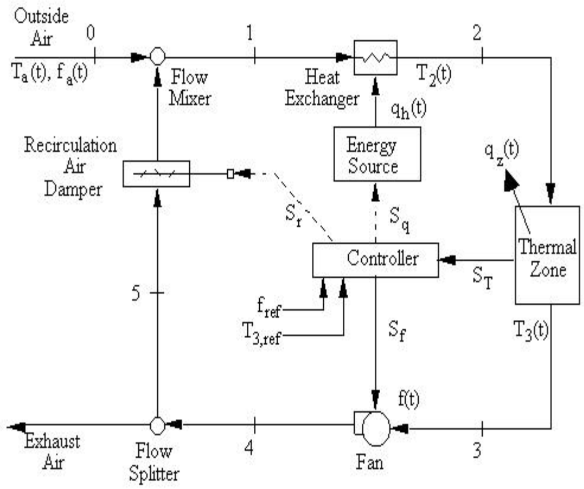

Fig. 1. The HVAC System

again within the LQR framework. An exact optimal controller is provided by (House et. al. (1991)), which is, however non-causal and not amenable to on-line implementation. In this paper we work with the non-quadratic cost function, and solve the non-causality problem by a working with a joint linearization of the state and adjoint equations. Despite the adopted linearization, the overall controller obtained is nonlinear.

2. THE SYSTEM

The HVAC system is depicted in Figure 1, and described by

$$T_1(t) = T_3(t) + [T_a - T_3(t)]\frac{f_a}{f(t)} \quad (1)$$

$$\rho C_p V_h \dot{T}_2(t) = f(t)\rho C_p[T_1(t) - T_2(t)] + q_h(t) \quad (2)$$

$$\rho C_p V_z \dot{T}_3(t) = f(t)\rho C_p[T_2(t) - T_3(t)] + q_z \quad (3)$$

where $T_1(t)$ and $T_2(t)$ are, respectively, the air temperatures prior to and immediately following the heat exchanger, $T_3(t)$ is the temperature of the thermal zone, T_a is the temperature of the outside air, ρ is the air density, C_p is the constant pressure specific heat of air, V_h is the effective volume of the heat exchanger, V_z is the effective thermal space volume, f_a is the flow rate at which the external air enters the system, $q_h(t)$ is the heat input to the heat exchanger, $f(t)$ is the volumetric airflow rate and q_z is the thermal load.

In the sequel, the external air flow rate will be kept at its minimum allowable value

$$f_a = f_{am} > 0.$$

Table 1. Physical Parameters

Parameter	Value
ρ	1.19 kg/m^3
C_p	1005 J/kg- C
V_h	25.5 m^3
V_z	255 m^3
α_2	4.86×10^{-3} \$/min- C^2
α_3	5.39×10^{-10} \$/min- W^2
α_4	5.20×10^{-5} \$-min/m^6
α_5	1.22×10^{-6} \$-min^2/m^9
f_{am}	0.05 m^3/sec

The overall goal is to devise a feedback law that uses the temperatures $T_2(t)$ and $T_3(t)$ to modulate the two quantities $q_h(t)$ and $f(t)$ so as to minimize a performance index that balnaces comfort level with the cost needed to attain it. Specifically over a period of operation that extends over the time interval $[0, t_f]$ the total cost is

$$J_{t_f} = \int_0^{t_f} \Big[\alpha_2(T_3(t) - T_r)^2 + \alpha_3 q_h^2(t) + \alpha_4(f(t) - f_r)^2 + \alpha_5 f^3(t)\Big]\,\mathrm{d}t \quad (4)$$

where the α_i's are cost weighting factors, T_r and f_r reflect the respective values of $T_3(t)$ and $f(t)$ corresponding to the maximum level of comfort. Further, $T_2(t)$ and $T_3(t)$ constitute the system states and $q_h(t)$ and $f(t)$ the control inputs. Accordingly, one defines the state vector

$$x(t) = [x_1(t)\ \ x_2(t)]' = [T_2(t)\ \ T_3(t)]' \quad (5)$$

and the input vector

$$u(t) = [u_1(t)\ \ u_2(t)]' = [q_h(t)\ \ f(t)].' \quad (6)$$

The physical parameters in the foregoing are given in Table 1. The minimization must be performed subject to the additional constraint that

$$f(t) \geq f_{am}.$$

The first and the third terms in the cost function of (4) are comfort costs due to temperature mismatch and the level of draft, respectively, while the second and the fourth are energy costs deriving from heat exchange and fan operation, respectively. The optimization of this cost function thus affects a trade off between comfort level and the resulting operating cost of the HVAC system.

It is noteworthy that the cost function to be minimized here is *non-quadratic*. By way of comparison one can cite the Optimal Control law embedded in the Adaptive Optimal Control algorithm of (Barnett and Cameron (1985)). Appealing as it does to Linear Quadratic Regulator (LQR) theory, (Barnett and Cameron (1985)) drops the non-quadratic term $\alpha_5 f^3(t)$ in (4) and derives a linear controller obtained on the basis of a linearized version of the model equations (1), (2) and (3).

3. THE CONTROLLER

3.1 *A reformulation*

Our objective is to consider an infinite horizon controller, i.e. one which minimizes J_{t_f} as $t_f \to \infty$. For such an optimization problem to be well posed, there must exist a control input and state values for which the system equations (1), (2) and (3) are at steady state and, the integrand of the cost function to be minimized is zero. Otherwise the cost function would be infinite over the infinite horizon of its operation and consequently the optimization problem would not have a solution. It is readily verified that the integrand in (4) cannot in general be zero. To circumvent this difficulty one may pose the alternative problem of minimizing:

$$J_{ex} = \int_0^\infty \left[\alpha_2 (T_3(t) - T_r)^2 + \alpha_3 q_h^2(t) + \alpha_4 (f(t) - f_r)^2 - \alpha_5 f^3(t) - J_e\right] \mathrm{d}t \quad (7)$$

where J_e is a constant representing the minimum steady state value that the integrand in (4) can assume; i.e.

$$J_e = \min\left[\alpha_2 (T_3(t) - T_r)^2 + \alpha_3 q_h^2(t) + \alpha_4 (f(t) - f_r)^2 - \alpha_5 f^3(t)\right] \quad (8)$$

subject to

$$\dot{T}_2(t) = 0, \qquad \dot{T}_3(t) = 0, \text{ and } f(t) \geq f_{am}. \quad (9)$$

Indeed, while technically one can have a lower value of the integrand in (7) by setting $T_3(t) = T_r$, $q_h(t) = 0$ and

$$f(t) = \max\left\{ f_{am}, \frac{-\alpha_4 + \sqrt{\alpha_4^2 + 6\alpha_4\alpha_5 f_r}}{3\alpha_5} \right\},$$

at these values the derivative of $T_3(t)$ will not be zero. Consequently, these values themselves cannot be sustained.

The problem of minimizing (8) subject to (9) is readily solvable and the constant determined. Further, as J_e is a constant, the control law that minimizes (7) also minimizes (4). Henceforth J_{ex} will be referred as the *excess cost.*

3.2 *A noncausal controller*

Proceeding in a standard fashion, (Barnett and Cameron (1985)) and using the notations (5) and (6), we work with the cost function

$$J_e(t_f) = \int_0^\infty F(x,u,t)\mathrm{d}t = \int_0^\infty \left[\alpha_2 (x_2(t) - T_r)^2 + \alpha_3 u_1^2(t) + \alpha_4 (u_2(t) - f_r)^2 \ \alpha_5 u_2^3(t) - J_e\right] \mathrm{d}t \quad (10)$$

formulate the two dimensional costate vector $p(t)$=[$p_1(t), p_2(t)$]$'$ and the Hamiltonian,

$$H = F(x,u,t) + p'(t)g(x,u,t) = \alpha_2 (x_2(t) - T_{3,ref})^2 + \alpha_3 u_1^2(t) + \alpha_4 (u_2(t) - f_r)^2 + \alpha_5 u_2^3(t) - J_e + p_1(t)\dot{x}_1(t) + p_2(t)\dot{x}_2(t) \quad (11)$$

and obtain the control law we use

$$[\dot{p}_1(t) \ \ \dot{p}_2(t)] = -\frac{\partial H}{\partial x}, \quad p(\infty) = 0 \quad (12)$$

and $\frac{\partial H}{\partial u_i} = 0$, $i = 1, 2$. First observe that with $k = \rho C_p$, (1), (2) and (3) can be rewritten as

$$\dot{x}_1(t) = \frac{1}{V_h}\left[u_2(t)\left(x_2(t) - x_1(t)\right) + f_a\left(T_a - x_2(t)\right) + \frac{u_1(t)}{k}\right] \quad (13)$$

and

$$\dot{x}_2(t)(t) = \frac{u_2(t)}{V_z}\left(x_1(t) - x_2(t)\right) + \frac{q_z}{kV_z}. \quad (14)$$

Moreover (12) becomes

$$\dot{p}_1(t) = u_2(t)\left(\frac{p_1(t)}{V_h} - \frac{p_2(t)}{V_z}\right) \quad (15)$$

and

$$\dot{p}_2(t) = -2\alpha_2\left(x_2(t) - T_r\right) + u_2(t)\left(\frac{p_2(t)}{V_z} - \frac{p_1(t)}{V_h}\right) + f_a\frac{p_1(t)}{V_h}. \quad (16)$$

Then

$$\frac{\partial H}{\partial u_1} = 2\alpha_3 u_1(t) + \frac{p_1(t)}{kV_h} = 0$$

and

$$\frac{\partial H}{\partial u_2} = 2\alpha_4[u_2(t) - f_r] + \frac{p_1(t)}{V_h}[x_2(t) - x_1(t)] + 3\alpha_5 u_2^2(t) + \frac{p_2(t)}{V_z}[x_1(t) - x_2(t)] = 0$$

Thus,

$$u_1(t) = \frac{-p_1(t)}{2\alpha_3 kV_h}. \tag{17}$$

Further, define

$$h(x,p) = (x_2(t) - x_1(t))\left(\frac{p_1(t)}{V_h} - \frac{p_2(t)}{V_z}\right), \tag{18}$$

and

$$\hat{u}(t) = \frac{-\alpha_4 + \sqrt{\alpha_4^2 + 3\alpha_5\left[2\alpha_4 f_r - h(x,p)\right]}}{3\alpha_5}. \tag{19}$$

Then

$$u_2(t) = \begin{cases} f_{am} & \hat{u}(t) \text{ complex or } \hat{u}(t) < f_{am} \\ \hat{u}(t) & \text{else} \end{cases}. \tag{20}$$

Observe, the determination of $u_1(t)$ and $u_2(t)$ requires the knowledge of both $p(t)$ and $x(t)$. As the former is the solution of a differential equation for which a boundary, as opposed to an initial condition, is provided, the above control law is unimplementable online and in fact may not even have a closed form solution.

3.3 *An approximation*

To remove the noncausality, we introduce an approximation. One can show, that there exist, p^*, x^* such that with $p = p^*$ and $x = x^*$ the integrand in (7), the $\dot{p}_i(t)$ and $\dot{x}_i(t)$ are all zero under (17) and (20). Now linearize (13) - (16), under (17) and (20). Defining, $\Delta p(t) = p(t) - p^*$ and $\Delta x(t) = x(t) - x^*$, as we will show presently, the linearized equations are

$$\begin{bmatrix} \dot{\Delta x}(t) \\ \dot{\Delta p}(t) \end{bmatrix} = \begin{bmatrix} A & -B \\ -C & -A' \end{bmatrix} \begin{bmatrix} \Delta x(t) \\ \Delta p(t) \end{bmatrix} \tag{21}$$

where $B = GG'$ is positive definite symmetric and $C = c'c$ is positive semidefinite symmetric. Then (Barnett and Cameron (1985)), provided $[A, G]$ is completely controllable, and $[A, c]$ is completely observable, the solution to (21) can be written as

$$\Delta p(t) = P\Delta x(t)$$

where P is the unique positive definite symmetric solution to the Riccati equation

$$A'P + PA + C - PBP = 0. \tag{22}$$

Once A, B and C are obtained, (22) is easily solved. The control law we propose then is (17), (20) and

$$p(t) = P\left(x(t) - x^*\right) + p^*. \tag{23}$$

Further, under these conditions the linearized equations are also exponentially stable. Observe, despite the linearization in the design process, one has a nonlinear controller.

To find x^* and p^* one must set the right hand sides of (13)- (16) to zero subject to

$$\frac{\partial H}{\partial x_i} = \frac{\partial H}{\partial u_i} = 0 \ , \ i = 1, 2.$$

This reduces to

$$S = \sqrt{\alpha_4^2 + 6\alpha_4\alpha_5 f_r}, \tag{24}$$

$$X = \begin{bmatrix} 0 & 2\alpha_2 & -f_a/V_h & 0 \\ -f_s & f_s - f_a & 0 & 1/k \\ f_s & -f_s & 0 & 0 \\ 0 & 0 & 1/kV_h & 2\alpha_3 \end{bmatrix}^{-1} \begin{bmatrix} 2\alpha_2 T_r \\ -f_a T_a \\ q_z/k \\ 0 \end{bmatrix},$$

$$x^* = \begin{bmatrix} X_1 & X_2 \end{bmatrix}^T \text{ and } p^* = X_3 \begin{bmatrix} 1 & V_z/V_h \end{bmatrix}^T \tag{25}$$

where X_i is the i-th element of X and

$$f_s = u_2^*(t) = \max\left\{f_{am}, \frac{-\alpha_4 + S}{3\alpha_5}\right\}$$

is the steady-state value of the volumetric airflow rate. Again, for all meaningful values of the system parameters

$$\frac{-\alpha_4 + S}{3\alpha_5} \geq f_{am}.$$

Whence one can write

$$f_s = \frac{-\alpha_4 + S}{3\alpha_5}.$$

By linearizing (13) - (16) around $[x^{*T}, p^{*T}]^T$ given by (24)-(25), one finds that

$$A = \begin{bmatrix} -f_s/V_h & f_s/V_h - f_a/V_h \\ f_s/V_z & -f_s/V_z \end{bmatrix} \qquad (26)$$

$$B = \begin{bmatrix} \frac{1}{2\alpha_3 k^2 V_h^2} + \frac{(x_1^* - x_2^*)^2}{2V_h^2 S} & -\frac{(x_1^* - x_2^*)^2}{2V_h V_z S} \\ -\frac{(x_1^* - x_2^*)^2}{2V_h V_z S} & \frac{(x_1^* - x_2^*)^2}{2V_z^2 k^2 S} \end{bmatrix}$$

and $C = 2\alpha_2[0, \quad 1]^T[0, \quad 1]$.

Indeed, since $c = [0, \sqrt{2\alpha_2}]$ forms an observable pair with A, and B is positive definite, the conditions we had earlier stated are met. The overall optimal controller is therefore as in (17) and (20), with $p(t)$ computed by (23)-(25), and P, the unique positive definite symmetric solution of (22), with A, B and C as above.

need to be computed online.

4. ADAPTIVE CONTROL

In this section we provide a Recursive Least Squares (RLS) based adaptive controller to cope with the fact that some of the key system parameters, such as external temperature and thermal load vary over time, and are difficult to accurately measure. To this end consider (13) and (14). Define

$$\overline{x}(t) = \dot{w}(t) = -w(t) + x(t), \qquad (27)$$

$$V_1(t) = \dot{\eta}_1(t) = -\eta_1(t) + [u_1(x_2 - x_1), 1, -x_2 + u_1]',$$

and

$$V_2(t) = \dot{\eta}_1(t) = -\eta_1(t) + [u_1(x_1 - x_2), 1]'.$$

Also define the system parameter vectors

$$\theta_1 = \left[\frac{1}{V_h}, f_a T_a, f_a, \frac{1}{k}\right]'$$

and

$$\theta_2 = \left[\frac{1}{V_z}, +\frac{q_z}{kV_z}\right]'.$$

Observe, the HVAC system model can be written in the form

$$\begin{cases} \overline{x}_1(t) = V_1^T(t)\theta_1 \\ \overline{x}_2(t) = V_2^T(t)\theta_2 \end{cases}. \qquad (28)$$

Define, the vector $\theta' = [\theta_1' \ \theta_2']$ and the matrix

$$V'(t) = \begin{bmatrix} V_1(t)' & 0 \\ 0 & V_2(t)' \end{bmatrix}.$$

As, $\overline{x}(t) = V'(t)\theta$ the observation signals can be generated without explicit differentiation. Then the parameter vector θ can be estimated by

$$\hat{\theta}(t) = \left[\int_0^t e^{-\alpha(t-\tau)} V(\tau)V'(\tau)d\tau\right]^{-1} \left[\int_0^t e^{-\alpha(t-\tau)} V(\tau)\overline{x}(\tau)d\tau\right].$$

Here α represents a forgetting factor. In practice of course one implements this identifier with a well known recursive algorithm, (Astrom and Wittenmark (1989)).

4.1 *Simulations*

We now present a simulation result to validate both the adaptive and non-adaptive algorithms. Figures 2 and 3 depict these results. Specifically, fig. 2 gives $J_{ex}(t)$ in (29) as a function of t, while fig. 3 gives $J_T(t)$ in (30) as a function of t. The solid lines are the result of the nonadaptive algorithm, when all parameters are perfectly known. The dashed lines represent the case where the parameters are uncertain, and the controller of the previous section, uses the parameter estimates provided by the identification algorithm.

$$J_{ex}(t_f) = \int_0^{t_f} \left[\alpha_2(T_3(t) - T_r)^2 + \alpha_3 q_h^2(t) - \alpha_5 f^3(t) + \alpha_4(f(t) - f_r)^2 - J_e\right] \mathrm{d}t \qquad (29)$$

$$J_T(t_f) = \int_0^{t_f} \left[\alpha_2(T_3(t) - T_r)^2 + \alpha_3 q_h^2(t) + \alpha_4(f(t) - f_r)^2 - \alpha_5 f^3(t)\right] \mathrm{d}t \qquad (30)$$

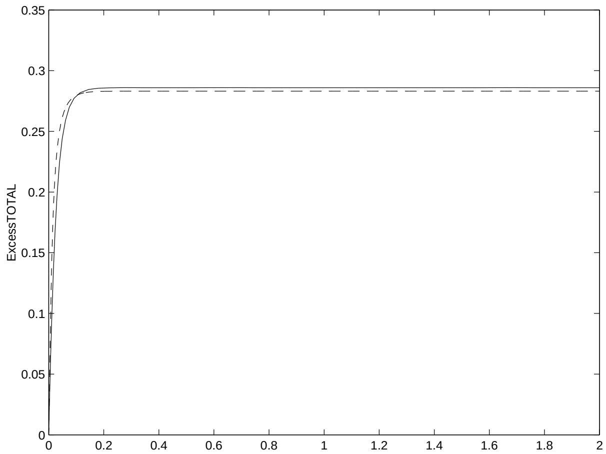

Fig. 2. The Excess Cost vs. Time in hours.

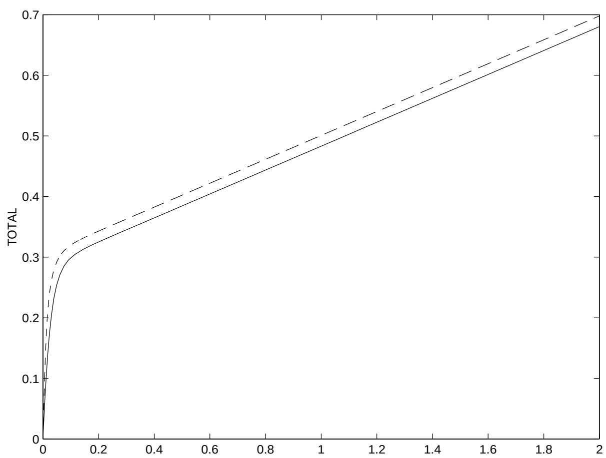

Fig. 3. The Total Cost vs. Time in hours.

As expected $J_{ex}(t)$ quickly converges to a constant value, reflecting the fact that the control law forces the system to converge to the trajectory correponding to a zero value of the integrand in (29). The fact that the total cost rises linearly after this period reflects the fact that the integrand in (30) converges to the non-zero value of J_e. The additional costs in the adaptive case can be traced to the learning time involved in the identification process.

5. CONCLUSION

We have presented an optimal control law for HVAC systems that minimizes a non-quadratic cost function. Though obtained by a linearizing approximation of the state and adjoint equations, the law itself is non-linear. Both adaptive and nonadptive versions of the algorithm are examined by simulations.

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