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$\mathcal{H}_2$ CONTROL OF RATIONAL PARAMETER-DEPENDENT CONTINUOUS-TIME SYSTEMS: A PARAMETRIC LYAPUNOV FUNCTION APPROACH [1]

Alexandre Trofino, * **Carlos E. de Souza,** **

* *Department of Automation and Systems*
Universidade Federal de Santa Catarina
88040-900 Florianópolis, SC, Brazil
(*On leave at Lab. Biotechnologie de l'Environnement/INRA, Narbonne, France*)
e-mail: trofino@lcmi.ufsc.br

** *Department of Systems and Control*
Laboratório Nacional de Computação Científica – LNCC/MCT
Av. Getúlio Vargas 333, 25651-070 Petrópolis, RJ, Brazil
e-mail: csouza@lncc.br

Abstract: This paper presents an LMI based design method for static output feedback controllers with a guaranteed $\mathcal{H}_2$ performance. The method can be applied to a class of continuous-time linear systems with either rational or polynomial parameter dependence, provided that the parameters and their rate of variation are bounded by a given polytope. Stability as well as a $\mathcal{H}_2$ performance bound are guaranteed via a rational parameter-dependent Lyapunov function. The method can be used to design: robust controllers, LPV controllers, and non-fragile controllers with prescribed ranges of errors that can be tolerated in the implementation of the designed control gains.

Keywords: Rational systems, uncertain parameters, $\mathcal{H}_2$ control, robust control, LPV control, non-fragile control, parametric Lyapunov functions.

1. INTRODUCTION

The problem of designing robust controllers has been largely studied in the last decade. LMI based design techniques for many important state feedback and full-order observer based control problems are now available (Boyd *et al.*, 1994). Most of these LMI techniques are based on the notion of quadratic stability and the controller is parameterized in terms of some of the LMI decision variables, where typically the matrix of the Lyapunov function is among these variables. Recently, de Oliveira *et al.* (1999) proposed an interesting LMI solution for control problems in which the matrix of the Lyapunov function is not among the decision variables that characterize the controller. This is an important feature because it allows for handling constraints on the controller structure, as for instance, static output feedback and decentralized control requirements, without having to impose hard constraints on the structure of the Lyapunov function. This new controller parameterization is also useful in solving the problem of finding a fixed controller satisfying multiple performance requirements. The idea is that if one let each of the performance objectives be associated with a different Lyapunov function, a less conservative result should be expected. In the case of continuous-time systems, a result along

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the lines of de Oliveira *et al.* (1999) has been recently presented in Shaked (2001) however, to the authors' knowledge, the continuous-time case problem has not yet been fully resolved.

This paper presents an LMI based controller design technique that can be applied to a class of continuous-time systems described by a state-space model with rational parameter dependence and has the following characteristics:

- The matrix of the Lyapunov function is not among the decision variables that characterize the controller.
- The stability of the closed-loop system as well as a $\mathcal{H}_2$ performance bound are established via a rational parameter-dependent Lyapunov function.
- The parameters and their rate of variation are supposed to be bounded by a given polytope.
- The method can be used to design static output feedback controllers, including decentralized control laws.
- The controllers may be of three types: robust (fixed control laws), LPV (gain scheduled control), and non-fragile (robust against a prescribed range of errors tolerated in the implementation of the designed control gains).

The proposed LMI design method is dependent on a scaling parameter in a non-convex manner. To cope with this difficulty this scalar is fixed through a gridding technique. An important feature is that the usual convex LMI conditions, which applies for the nominal system are recovered as a special case when this scalar is sufficiently large.

The paper is organized as follows. The class of systems and the structure of the control law are presented in the next section. Section 3 introduces the $\mathcal{H}_2$ performance index and its upper-bound which will be used in the control design. The main results are presented in Section 4 for the design of static output feedback controllers. These results are then extended in Section 5 for the design of decentralized controllers. Some concluding remarks end the paper.

Notation. The notation used in this paper is quite standard. $\mathbb{R}^n$ denotes the n-dimensional Euclidean space, $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices, $\mathrm{Tr}\{\cdots\}$ denotes matrix trace, and the notation $S>0$ for a real matrix S, means that S is symmetric and positive definite. The notation $\overline{x}$ and $\underline{x}$ refers to a fixed upper and lower bounds on the variable x, respectively. For a symmetric block matrix, the symbol $\star$ is used to represent the block partitions outside the main diagonal block which can be deduced by symmetry.

2. SYSTEM DEFINITION

Consider the system

$$\begin{aligned} \dot{x}(t) &= \Theta_x' A x(t) + \Theta_x' B_w w(t) + \Theta_x' B_u u(t) \\ z(t) &= \Theta_z' C_z x(t) + \Theta_z' D_z u(t) \\ y(t) &= C_y x(t) \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state, $w \in \mathbb{R}^{n_w}$ is the disturbance input, $u \in \mathbb{R}^{n_u}$ is the control input, $z \in \mathbb{R}^{n_z}$ is the $\mathcal{H}_2$ performance output, and $y \in \mathbb{R}^{n_y}$ is the measured output. B_u, C_y and D_z are known constant matrices, whereas the matrices A, B_w and C_z may be affine functions of some possibly time-varying parameters $\theta_1, \ldots, \theta_q$, that will be represented by the vector

$$\theta = \begin{bmatrix} \theta_1 & \ldots & \theta_q \end{bmatrix}' \in \mathbb{R}^q \tag{2}$$

For the sake of notation simplicity, throughout the paper the dependence of θ on the time t will be omitted. The matrices Θ_x and Θ_y are rational matrix functions of the parameters with the following structure

$$\Theta_x = \Theta_a^{-1}\Theta_b \qquad \Theta_z = \Theta_c^{-1}\Theta_d \tag{3}$$

where Θ_a, Θ_c, Θ_b, and Θ_d are affine matrix functions of the parameters. It is supposed that for all admissible values of the parameters, the matrices Θ_a and Θ_c are nonsingular and Θ_b is of full column rank. The former assumption is a regularity condition of the representation (3), whereas the latter is a technical assumption regarding the system representation (1). If θ_b is not of full column rank, it turns out that the state matrix of the closed-loop system resulting from any static output feedback control law will be singular and hence the system (1) will not be stabilizable.

Notice that the class of system (1) is general enough to include the so-called LFT representation (Boyd *et al.*, 1994) and thus any polynomial and rational dependence on the parameters (El Ghaoui and Scarletti, 1996) may be represented in the form as in (1). The state matrix of an LFT representation has the form $\mathcal{A} = A_1 + A_2(I + A_3)^{-1}A_4$ where A_1 and A_4 are fixed matrices and A_2 and A_3 are affine matrix functions of the parameters. With the choice

$$A = \begin{bmatrix} I \\ A_4 \end{bmatrix}, \quad \Theta_a' = \begin{bmatrix} I & 0 \\ 0 & I + A_3 \end{bmatrix},$$

$$\Theta_b' = \begin{bmatrix} A_1 & A_2 \end{bmatrix}$$

it can be easily verified that $\mathcal{A} = \Theta_x' A$.

Further, notice that with suitable partitions of the matrices Θ_a, Θ_b, A, B_u, and B_w it is possible to handle the case where the state, control input, and performance input matrices are subject to different parameters.

In this paper we are concerned with the problem of determining a control law of the type

$$u = \mathcal{K}(\theta)y, \quad \mathcal{K}(\theta) = \Theta_u' K = K_0 + \sum_{i=1}^{q_u \le q} \theta_i K_i \quad (4)$$

where K is a fixed matrix gain to be determined and Θ_u is an affine function of a set of parameters $\{\theta_i\}$. The role of the matrix Θ_u is discussed in the sequel.

With a proper choice of Θ_u we may design a fixed robust controller (if Θ_u is chosen as the identity matrix, or an LPV controller (Apkarian and Gahinet, 1995; Blanchini, 2000; Packard, 1994) if Θ_u contains the scheduling parameters, or even a non-fragile controller if Θ_u contains fictitious uncertain parameters representing the error that can be tolerated in the implementation of the designed control gains. The idea is that the designed control gain is K_i but the implemented one is $K_i\theta_i$ for some θ_i in a given range $\underline{\theta}_i \le \theta_i \le \overline{\theta}_i$ representing the allowed deviations with respect to the designed value, which corresponds to $\theta_i = 1$. For design purpose, these parameters can be viewed as fictitious time-invariant uncertainties. The stability of the closed-loop system must then be assured for all possible values that the applied control gain may take in the specified range. This type of control law is referred to in the literature as non-fragile control (Keel and Bhattacharyya, 1997).

To design an LPV controller we let $\{\theta_i\}$ to represent the set of scheduling parameters and Θ_u to be a given affine function of them. The only knowledge the proposed design method requires to solve the problem is a polytope specifying the admissible values of the scheduling parameters and their rate of variations. Observe that the information on the rate of variation is used in the controller design but the controller implementation does not require the on-line knowledge of $\dot{\theta}_i$.

Hereafter, it is assumed that the parameters and their rate of variations belong to a convex bounded polyhedral domain Π.

3. THE $\mathcal{H}_2$ PERFORMANCE INDEX

Let us start by defining the finite-horizon 2-norm of a signal $y(t)$ in the time interval $[t_0, T]$ as

$$\|y\|_{2,[t_0,T]} = \left(\int_{t_0}^{T} y(t)' y(t)\, dt \right)^{\frac{1}{2}} \quad (5)$$

In order to present some preliminary results, consider the parameter-dependent system

$$\begin{array}{l} \dot{x}(t) = \mathcal{A}(\theta)x(y) + \mathcal{B}(\theta)w(t), \quad x(0_-) = 0 \\ y(t) = \mathcal{C}(\theta)x(t) \end{array} \quad (6)$$

where θ is a time-varying parameter vector such that all the admissible $(\theta, \dot{\theta})$ belong to a given polytope Π, and $\mathcal{A}(\theta)$, $\mathcal{B}(\theta)$ and $\mathcal{C}(\theta)$ are bounded matrix functions of θ.

The following lemma, which is an extension of a well known result to the context of parameter-dependent systems, will be the basis for the $\mathcal{H}_2$ control design problems addressed in the next sections.

Lemma 1. Consider the system (6) and let Π be a given polytope of admissible $(\theta, \dot{\theta})$. Suppose that there exists a bounded symmetric positive definite matrix $\mathcal{P}(\theta)$ such that the following inequality is satisfied for all $(\theta, \dot{\theta})$ in Π:

$$-\dot{\mathcal{P}}(\theta) + \mathcal{A}(\theta)\mathcal{P}(\theta) + \mathcal{P}(\theta)\mathcal{A}(\theta)' + \mathcal{B}(\theta)\mathcal{B}(\theta)' < 0 \quad (7)$$

Then the system is (6) exponentially stable for all $(\theta, \dot{\theta}) \in \Pi$, and for any $T > 0$

$$\|y\|^2_{2,[0_+,T]} < \sup_{\theta \in \Pi} \mathrm{Tr}\, [\mathcal{C}(\theta)\mathcal{P}(\theta)\mathcal{C}(\theta)'] \quad (8)$$

The proof of the above lemma follows from (Green and Limebeer, 1995). Observe that $V(x, \theta) = x'\mathcal{P}^{-1}(\theta)x$ is a Lyapunov function for the unforced system of (6).

4. MAIN RESULTS

In the sequel it is presented a condition in which the matrix product $\mathcal{A}(\theta)\mathcal{P}(\theta)$, which appears in the Lemma 1, is replaced by $\mathcal{A}(\theta)\mathcal{G}(\theta)$ where $\mathcal{G}(\theta)$ is a slack variable. The advantage of this over parameterization will be clarified later on in the design results.

Theorem 1. Consider the system (6). There exist bounded matrices $\tilde{\mathcal{N}}(\theta)$ and $\tilde{\mathcal{P}}(\theta) > 0$ such that the following conditions are satisfied

$$-\dot{\tilde{\mathcal{P}}}(\theta) + \mathcal{A}(\theta)\tilde{\mathcal{P}}(\theta) + \tilde{\mathcal{P}}(\theta)\mathcal{A}(\theta)' + \mathcal{B}(\theta)\mathcal{B}(\theta) < 0 \quad (9)$$

$$\tilde{\mathcal{N}}(\theta) - \mathcal{C}(\theta)\tilde{\mathcal{P}}(\theta)\mathcal{C}(\theta)' > 0 \quad (10)$$

if and only if there exist bounded matrices $\mathcal{N}(\theta)$, $\mathcal{P}(\theta) > 0$, $\mathcal{G}(\theta)$ and $\mathcal{S}(\theta)$ and a scalar $\alpha > 0$ such that the following LMIs are satisfied

$$\begin{bmatrix} H(\theta) + \alpha I & \mathcal{A}(\theta)\mathcal{G}(\theta) \\ \star & \alpha[\mathcal{S}(\theta) - \mathcal{G}(\theta) - \mathcal{G}(\theta)'] \end{bmatrix} < 0 \quad (11)$$

$$\begin{bmatrix} \mathcal{S}(\theta) & \mathcal{G}(\theta) - \mathcal{P}(\theta) \\ \star & I \end{bmatrix} > 0 \quad (12)$$

$$\begin{bmatrix} \mathcal{N}(\theta) & \mathcal{C}(\theta)\mathcal{G}(\theta) \\ \star & \mathcal{G}(\theta) + \mathcal{G}(\theta)' - \mathcal{P}(\theta) \end{bmatrix} > 0 \quad (13)$$

where

$$H(\theta) = -\dot{\mathcal{P}}(\theta) + \mathcal{A}(\theta)\mathcal{G}(\theta) + \mathcal{G}(\theta)'\mathcal{A}(\theta)' + \mathcal{B}(\theta)\mathcal{B}(\theta)'$$

Moreover, under the above condition, $V(x,\theta) = x'\mathcal{P}^{-1}(\theta)\,x$ is a Lyapunov function for the unforced system of (6) and

$$\|y\|^2_{2,[0,T]} < \mathrm{Tr}\,[\mathcal{N}(\theta)] \qquad (14)$$

Proof. Firstly, note that as for any matrices $\mathcal{G}(\theta)$ and $\mathcal{S}(\theta) \geq 0$,

$$[\mathcal{S}(\theta) - \mathcal{G}(\theta)]'\mathcal{S}^{-1}(\theta)[\mathcal{S}(\theta) - \mathcal{G}(\theta)] \geq 0 \qquad (15)$$

it results that

$$\mathcal{G}(\theta)'\mathcal{S}^{-1}(\theta)\mathcal{G}(\theta) \geq \mathcal{G}(\theta) + \mathcal{G}(\theta)' - \mathcal{S}(\theta) \qquad (16)$$

Further, by denoting

$$\Psi = \sqrt{\alpha}\,I + \frac{1}{\sqrt{\alpha}}[\mathcal{G}(\theta) - \mathcal{P}(\theta)]'\mathcal{A}(\theta)' \qquad (17)$$

the inequality $\Psi'\Psi \geq 0$ implies that for any $\alpha > 0$ and for any matrices $\mathcal{G}(\theta)$ and $\mathcal{P}(\theta)$

$$[\mathcal{G}(\theta)-\mathcal{P}(\theta)]'\mathcal{A}(\theta)' + \mathcal{A}(\theta)[\mathcal{G}(\theta)-\mathcal{P}(\theta)] + \alpha I$$
$$+\,\alpha^{-1}\mathcal{A}(\theta)[\mathcal{G}(\theta)-\mathcal{P}(\theta)]\,[\mathcal{G}(\theta)-\mathcal{P}(\theta)]'\mathcal{A}(\theta)' \geq 0 \qquad (18)$$

To show the sufficiency, suppose that the LMIs of (11)-(13) are satisfied. Then, we must have $\mathcal{S}(\theta) > 0$ and $\mathcal{G}(\theta)$ invertible. Applying Schur's complement to the inequality (12) and using (16) one gets

$$[\mathcal{G}(\theta) - \mathcal{P}(\theta)]\,[\mathcal{G}(\theta) - \mathcal{P}(\theta)]' < \mathcal{S}(\theta)$$
$$< \mathcal{G}(\theta)[\mathcal{G}(\theta) + \mathcal{G}(\theta)' - \mathcal{S}(\theta)]^{-1}\mathcal{G}(\theta)' \qquad (19)$$

Considering the above inequality and applying Schur's complement to (11) it follows that

$$H + \alpha I + \frac{\mathcal{A}(\theta)[\mathcal{G}(\theta)-\mathcal{P}(\theta)]\,[\mathcal{G}(\theta)-\mathcal{P}(\theta)]'\mathcal{A}(\theta)'}{\alpha} < 0 \qquad (20)$$

Subtracting the term $\mathcal{A}(\theta)\mathcal{P}(\theta) + \mathcal{P}(\theta)\mathcal{A}(\theta)'$ from both sides of (20) leads to

$$-\Psi'\Psi > -\dot{\mathcal{P}}(\theta) + \mathcal{A}(\theta)\mathcal{P}(\theta) + \mathcal{P}(\theta)\mathcal{A}(\theta)'$$
$$+\,\mathcal{B}(\theta)\mathcal{B}(\theta)' \qquad (21)$$

and therefore (9) is satisfied with $\tilde{\mathcal{P}}(\theta) = \mathcal{P}(\theta)$. Further, since $\mathcal{P}(\theta) > 0$ it follows that the system (6) is exponentially stable as $V(x,\theta) = x'\mathcal{P}^{-1}(\theta)x$ is a Lyapunov function for that system.

Next, it will be shown that the inequality (13) implies (10) with $\tilde{\mathcal{P}}(\theta) = \mathcal{P}(\theta)$ and $\tilde{\mathcal{N}}(\theta) = \mathcal{N}(\theta)$. The result follows by applying Schur's complement to (13) and taking into consideration that

$$\mathcal{P}(\theta) < \mathcal{G}(\theta)[\mathcal{G}(\theta) + \mathcal{G}(\theta)' - \mathcal{P}(\theta)]^{-1}\mathcal{G}(\theta)' \qquad (22)$$

Conversely, if (9) and (10) are satisfied, let us choose the scalar α sufficiently large such that

$$\alpha^2[-\dot{\tilde{\mathcal{P}}}(\theta) + \mathcal{A}(\theta)\tilde{\mathcal{P}}(\theta) + \tilde{\mathcal{P}}(\theta)\mathcal{A}(\theta)']$$
$$+\,\mathcal{B}(\theta)\mathcal{B}(\theta)' + \alpha[I + \mathcal{A}(\theta)\tilde{\mathcal{P}}(\theta)\mathcal{A}(\theta)'] < 0 \qquad (23)$$

Hence, it can be readily verified that the conditions of (11)-(13) are satisfied with $\mathcal{S}(\theta) = \mathcal{G}(\theta) = \mathcal{P}(\theta) = \alpha^2\tilde{\mathcal{P}}(\theta)$ and $\mathcal{N}(\theta) = \alpha^2\tilde{\mathcal{N}}(\theta)$, which completes the proof. $\nabla\nabla\nabla$

In order to present the control design result, let the following notation. Given the system (1) and matrices P, G, F, N and S to be determined later on, define the matrices

$$\Psi_1 = \begin{bmatrix} \alpha I & 0 & \phi_1' & 0 \\ 0 & \alpha(S-G-G') & \phi_1' & 0 \\ \phi_1 & \phi_1 & B_wB_w' - \dot{P} & -P \\ 0 & 0 & -P & 0 \end{bmatrix} \qquad (24)$$

$$\Psi_2 = \begin{bmatrix} \Theta_b & 0 & -\Theta_a & 0 \\ -\dot{\Theta}_b & 0 & \dot{\Theta}_a & \Theta_a \end{bmatrix} \qquad (25)$$

$$\Psi_3 = \begin{bmatrix} N & 0 & 0 & 0 \\ 0 & G+G' & G'\phi_3' & 0 \\ 0 & \phi_3 G & 0 & 0 \\ 0 & 0 & 0 & -P \end{bmatrix} \qquad (26)$$

$$\Psi_4 = \begin{bmatrix} \Theta_d & 0 & -\Theta_c & 0 \\ \Theta_b & 0 & 0 & -\Theta_a \end{bmatrix} \qquad (27)$$

$$\Psi_5 = \begin{bmatrix} S & G & 0 & 0 \\ G' & I & 0 & 0 \\ 0 & 0 & 0 & -P \\ 0 & 0 & -P & 0 \end{bmatrix} \qquad (28)$$

$$\Psi_6 = \begin{bmatrix} \Theta_d & 0 & -\Theta_c & 0 \\ 0 & \Theta_b & 0 & -\Theta_a \end{bmatrix} \qquad (29)$$

$$\phi_1 = AG + B_u\Theta_u' F C_y \qquad (30)$$

$$\phi_3 = C_z G + D_z\Theta_u' F C_y \qquad (31)$$

where Θ_u is a given affine matrix function of θ associated with the control law (4).

Theorem 2. Consider the system (1) and let Π be a given polytope of admissible $(\theta,\dot{\theta})$ and Θ_u a given affine matrix function of θ associated with the control law (4). Suppose that there exist fixed matrices F, M, G, L_2, L_4 and L_6, matrix functions P, N and S which are affine in θ, and a scalar $\alpha > 0$ such that the following LMI problem is feasible at all the vertices of Π:

$$\Psi_1 + L_2\Psi_2 + \Psi_2'L_2' < 0 \qquad (32)$$

$$\Psi_3 + L_4\Psi_4 + \Psi_4'L_4' > 0 \qquad (33)$$

$$\Psi_5 + L_6\Psi_6 + \Psi_6'L_6' > 0 \qquad (34)$$

$$C_1 G = M C_1 \qquad (35)$$

Then the control law

$$u(t) = \mathcal{K}(\theta)y(t) \qquad (36)$$

where

$$\mathcal{K}(\theta) = \Theta_u' K, \quad K = FM^{-1} \qquad (37)$$

ensures that the resulting closed-loop system is exponentially stable for all $(\theta,\dot\theta)\in\Pi$ and for all $T>0$

$$\|z\|^2_{2,[0,T]} < \mathrm{Tr}\,(N), \quad \forall\,(\theta,\dot\theta)\in\Pi \tag{38}$$

Moreover,

$$V(x,\theta)=x'\mathcal{P}^{-1}(\theta)x, \quad \mathcal{P}(\theta)=\Theta_x' P\Theta_x \tag{39}$$

is a Lyapunov function for the unforced closed-loop system.

Proof. The closed-loop system of (1) with the control law of (36) and (37) is given by

$$\begin{aligned} \dot x(t) &= \mathcal{A}(\theta)x(t)+\mathcal{B}(\theta)w(t)\\ z(t) &= \mathcal{C}(\theta)x(t) \end{aligned} \tag{40}$$

where

$$\begin{aligned} \mathcal{A}(\theta) &= \Theta_x' A+\Theta_x' B_u\Theta_u' K C_y\\ \mathcal{C}(\theta) &= \Theta_z' Cz+\Theta_z' D_z\Theta_u' K C_y\\ \mathcal{B}(\theta) &= \Theta_x' B_w \end{aligned} \tag{41}$$

Note that (35), (37) and (24)-(31) imply that

$$\mathcal{A}(\theta)G=\Theta_x'\phi_1, \quad \mathcal{C}(\theta)G=\Theta_y'\phi_3. \tag{42}$$

The idea of the proof is to show that the conditions (11)-(13) of Theorem 1 are satisfied with $\mathcal{A}(\theta),\mathcal{B}(\theta)$ and $\mathcal{C}(\theta)$ as above and

$$\begin{aligned} \mathcal{P}(\theta)=\Theta_x' P\Theta_x, \quad \mathcal{G}(\theta)=G\\ \mathcal{N}(\theta)=N, \quad \mathcal{S}(\theta)=S \end{aligned} \tag{43}$$

To show this, define the matrices

$$\Psi_7=\begin{bmatrix} I & 0\\ 0 & I\\ \Theta_x & 0\\ \dot\Theta_x & 0\end{bmatrix}, \quad \Psi_8=\begin{bmatrix} I & 0\\ 0 & I\\ \Theta_z & 0\\ 0 & \Theta_x\end{bmatrix}, \quad \Psi_9=\begin{bmatrix} I & 0\\ 0 & I\\ \Theta_x & 0\\ 0 & \Theta_x\end{bmatrix} \tag{44}$$

It should be noted that

$$\Psi_2\Psi_7=0, \quad \Psi_4\Psi_8=0, \quad \Psi_6\Psi_9=0\,. \tag{45}$$

Further, observe that

$$\dot{\mathcal{P}}(\theta)=\dot\Theta_x' P\Theta_x+\Theta_x' P\dot\Theta_x+\Theta_x'\dot P\Theta_x \tag{46}$$

$$\dot\Theta_x=\Theta_a^{-1}\dot\Theta_b-\Theta_a^{-1}\dot\Theta_a\Theta_a^{-1}\Theta_b \tag{47}$$

In the light of (45), it results from (32)-(34) that

$$\Psi_7'\Psi_1\Psi_7<0, \quad \Psi_8'\Psi_3\Psi_8>0, \quad \Psi_9'\Psi_5\Psi_9>0\,. \tag{48}$$

Now, considering (41)-(44), (46), (47) and performing straightforward matrix manipulations it can be established that the inequalities of (48) imply that the LMIs (11)-(13) are satisfied. The proof is then completed by taking into account the Theorem 1. $\nabla\nabla\nabla$

The equality constraint (35) plays an important role in static output feedback control problems (Crusius and Trofino, 1999). Note that this constraint is trivially satisfied when state feedback is used. Indeed, it turns out that for $C_y=I$, (35) reduces to a simple change of variable $M=G$. In the next section it will be shown how this equality constraint can be used to handle decentralized control problems.

It should be remarked that stability as well as the $\mathcal{H}_2$ performance bound of (38) are based on the parameter-dependent Lyapunov function (39) which is rational in the system parameters. With a proper choice of the structure of the matrix P in the Lyapunov function and an appropriate choice of the matrices Θ_a and Θ_b of the system description, the result of Theorem 2 can handle systems with polynomial parameter-dependence as well as polynomial-type Lyapunov functions, or even a parameter-independent Lyapunov function, as in the usual quadratic stability approach.

Finally, it is important to notice that (32) is not convex with respect to the scalar α. From the proof of the Theorem 1, it follows that for large values of α the slack matrix $\mathcal{G}(\theta)$ approaches the matrix $\mathcal{P}(\theta)$ of the Lyapunov function and thus much of the degrees of freedom in $\mathcal{G}(\theta)$ are likely to be lost for too large values of α. On the other hand, for small values of α the condition (32) will fail to be satisfied. A gridding technique seems to be an appropriate way for selecting the best value of the scalar α.

5. DECENTRALIZED CONTROL

The Theorem 2 can be extended to handle structural constraints on the control gains that can be expressed in the form

$$u(t)=\mathcal{K}(\theta)x(t)=\sum_{i=1}^{n_c} R_i\mathcal{K}_i(\theta)S_i x(t) \tag{49}$$

where R_i and S_i are given matrices representing the structure associated with the control gain $\mathcal{K}_i(\theta)$. This type of control law appears, for instance, in decentralized control problems (Siljak, 1978). In this case the matrices R_i and S_i are used to specify that the i-th actuator is allowed to receive information only from the i-th sensor. The conditions of the Theorem 2 can be easily adapted to the design of a control law of the type (49). To this end, we just need to redefine ϕ_1 and ϕ_3 of (30) and (31) as

$$\phi_1=AG+B_u\sum_{i=1}^{n_c} R_i\Theta_{u_i}' F_i S_i \tag{50}$$

$$\phi_3 = C_2 G + D_u \sum_{i=1}^{n_c} R_i \Theta'_{u_i} F_i S_i \qquad (51)$$

and to replace the equality constraint (35) by the following set of equality constraints

$$S_i G = M_i S_i, \;\; i = 1, \ldots, n_c \qquad (52)$$

The $\mathcal{H}_2$ control law is now given by (49) with

$$\mathcal{K}_i(\theta) = \Theta'_{u_i} F_i M_i^{-1}, \;\; i = 1, \ldots, n_c \qquad (53)$$

The matrices Θ_{u_i} in the above control law have the same role of the matrix Θ_u in the previous Section 4.

6. CONCLUDING REMARKS

The contribution of this paper is twofold. It proposes, for a class of continuous-time systems with either rational or polynomial parameter dependence, a $\mathcal{H}_2$ control design method in which the matrix of the Lyapunov function is not among the LMI decision variables that characterize the controller. The advantages of this feature was emphasized in de Oliveira *et al.* (1999) in the context of linear discrete-time system with affine parameter dependence and for constant parameters. Another important feature of the proposed method is that it can be used to design static output feedback controllers of the following three types, including decentralized control laws: (i) robust control; (ii) LPV control; and (iii) non-fragile control (robust against a prescribed range of errors that can be tolerated in the implementation of the designed control gains). Moreover, stability and $\mathcal{H}_2$ performance of the closed-loop system are based on a rational parameter-dependent Lyapunov function.

The proposed design method is given in terms of LMIs and can handle time-varying parameters with admissible values and rates of variation bounded by a given polytope. In contrast with its discrete-time counterpart (Trofino *et al.*, 2001), the results of this paper are not globally convex as the design conditions depend on a scaling factor in a non-convex manner. To handle this difficulty, a possible solution is to fix this scalar through a gridding technique.

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