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UNIVERSAL FORMULA FOR OUTPUT ASYMPTOTIC STABILIZATION

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Abstract: the work deals with universal control formula which provides output global asymptotic stability for affine in control nonlinear system. Development of this result to input-to-output stability (IOS) is also considered. *Copyright © 2002 IFAC*

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1. INTRODUCTION

In resent decade the problem of output asymptotic stability or IOS for nonlinear systems was in the center of attention (Fradkov, *et al*., 1999; Ingalls and Wang, 2001; Rumyantsev and Oziraner, 1987; Sontag and Wang, 1997b; Sontag and Wang, 1999; Sontag and Wang, 2001; Vorotnikov, 1998). Frequently, in practical tasks the requirement of full state stabilization goes away from natural essence of the system, in such case partial stability (stability with respect to part of state variables) is appeared. Another case is the output stabilization of the system which has output-to-state stability (OSS) property or input-output-to-state stability (IOSS) property (Sontag and Wang, 1997a), then solving task of output asymptotic stabilization, one solves task of full state stabilization or input-to-state stabilization respectively. This supposition is also valid for output-to-input stable systems (strong minimum phase, see (Liberzon, *et al*., 2000)).

In the second section all notations and definitions are introduced. Control construction for output stability is presented in the third section. The proofs of all results are presented in the Appendix.

2. DEFINITIONS AND FORMULATIONS

Let us consider the following nonlinear dynamic system

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},\mathbf{u}),\ \mathbf{y}=\mathbf{h}(\mathbf{x}), \qquad (1)$$

where $\mathbf{x}\in R^n$ is state space vector; $\mathbf{u}\in R^m$ is input vector; $\mathbf{y}\in R^p$ is output vector; $\mathbf{f}$ and $\mathbf{h}$ are locally Lipschitz continuous vector fields, $\mathbf{h}(0)=0$, $\mathbf{f}(0,0)=0$. Euclidean norm will be denoted as $|\mathbf{x}|$, and $\|\mathbf{u}\|_{[t0,t]}$ denotes the L_∞^m norm of the input ($\mathbf{u}(t)$ is measurable and locally essentially bounded function $\mathbf{u}:\mathcal{I}\to R^m$, where $\mathcal{I}$ is a subinterval of R, which contains the origin; if interval $\mathcal{I}$ does not specified, then $\mathcal{I}=R_{\geq 0}$):

$$\|\mathbf{u}\|_{[t_0,T]}=ess\sup\{|\mathbf{u}(t)|, t\in[t_0,T]\},$$

if $T=+\infty$ then we will write simple $\|\mathbf{u}\|$. For initial state $\mathbf{x}_0$ and input $\mathbf{u}$ let $\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})$ be the unique maximal solution of (1) (we will use notation $\mathbf{x}(t)$, if all other arguments of solution are clear from context; $\mathbf{y}(t,\mathbf{x}_0,\mathbf{u})=\mathbf{h}(\mathbf{x}(t,\mathbf{x}_0,\mathbf{u}))$), which is defined on some finite interval $[0,T)$; if $T=+\infty$ for every initial state $\mathbf{x}_0$ and essential bounded input $\mathbf{u}$, then system is called *forward complete*. There exists another one strictly weaker property of system (1), which is closely connected with forward completeness, system (1) has *unboundedness observability* (UO) property, if for each state $\mathbf{x}_0$ and input $\mathbf{u}$ such that $T<+\infty$ necessarily

$$\lim_{t\to T}\sup|\mathbf{y}(t,\mathbf{x}_0,\mathbf{u})|=+\infty .$$

In other words it is possible to observe any unboundedness of the state. The contrapositive statement of this property says that, if $\sup_{t\in[0,T)}|\mathbf{y}(t)|<+\infty$, then $\mathbf{x}(T)$ is defined, so boundedness of UO output means forward completeness. The necessary and sufficient conditions for forward completeness and UO properties were investigated in (Angeli and Sontag, 1999).

As usually, continuous function $\sigma:R_{\geq 0}\to R_{\geq 0}$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\sigma(0)=0$; additionally it belongs to class $\mathcal{K}_\infty$ if it also radially unbounded; and continuous function $\beta:R_{\geq 0}\times R_{\geq 0}\to R_{\geq 0}$ is from class $\mathcal{KL}$, if it is from class $\mathcal{K}$ on the first argument for any fixed second, and strictly decreasing to zero by the second argument for any fixed first one.

An example of the systems, which admit UO property is OSS systems (Sontag and Wang, 1997a), i.e. for all $\mathbf{x}_0\in R^n$ and all $\mathbf{u}\in\mathcal{M}_\Omega$ (input $\mathbf{u}$ lies in some compact set $\Omega\subset R^m$, and signal $\mathbf{u}:\mathcal{I}\to\Omega$ belongs to class measurable and locally essentially bounded function $\mathcal{M}_\Omega$)

$$|\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})| \le \beta'(|\mathbf{x}_0|,t) + \gamma'(\|\mathbf{y}\|_{[0,T)})$$

holds for all $t \in [0,T)$ for some functions $\beta' \in \mathcal{KL}$ and $\gamma' \in \mathcal{K}$. If the last inequality is satisfied for all $\mathbf{u} \in R^m$, then such property is named as uniformly OSS or UOSS for short. In (Krichman, *et al.*, 2001) was shown, that OSS property is equivalent to global asymptotic stability modulo output (GASMO):

$$|\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})| \ge \nu(|\mathbf{y}(t,\mathbf{x}_0,\mathbf{u})|),\ t \in [0,\tilde{T}) \Rightarrow$$
$$|\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})| \le \eta(|\mathbf{x}_0|,t),\ t \in [0,\tilde{T}),$$

for all $\mathbf{x}_0 \in R^n$, all $\mathbf{u} \in \mathcal{M}_\Omega$, $\tilde{T} < T$ and some functions $\eta \in \mathcal{KL}$ and $\nu \in \mathcal{K}_\infty$. A component of GASMO property is global stability modulo output (GSMO) property:

$$|\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})| \ge \nu(|\mathbf{y}(t,\mathbf{x}_0,\mathbf{u})|),\ t \in [0,\tilde{T}) \Rightarrow$$
$$|\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})| \le \max\{\mu(|\mathbf{x}_0|), \mu(\|\mathbf{u}\|_{[0,\tilde{T})})\},\ t \in [0,\tilde{T}),$$

here functions $\nu \in \mathcal{K}_\infty$, $\mu \in \mathcal{K}$ and $\mathbf{x}_0 \in R^n$, $\mathbf{u} \in R^m$, $\tilde{T} < T$; if $\mathbf{u} \in \mathcal{M}_\Omega$, then term $\mu(\|\mathbf{u}\|_{[0,\tilde{T})})$ can be dropped in the last inequality. It is worth to note, that GSMO property and boundedness of output ensure forward completeness of system (the same as UO).

The generalization of OSS property for system (1) with inputs from not necessary compact set is IOSS property (Krichman, *et al.*, 2001; Sontag and Wang, 1997a), i.e. for all $\mathbf{x}_0 \in R^n$ and all $\mathbf{u} \in R^m$

$$|\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})| \le \beta''(|\mathbf{x}_0|,t) + \gamma_1(\|\mathbf{u}\|_{[0,T)}) + \gamma_2(\|\mathbf{y}\|_{[0,T)})$$

holds for all $t \in [0,T)$ for some functions $\beta'' \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$. The main properties of dynamic systems under consideration in this paper are the following.

Definition 1 (Ingalls and Wang, 2001; Sontag and Wang, 1997b). *A UO system (1) is*:

- *IOS, if there exist* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}$ *so that for all* $\mathbf{x}_0 \in R^n$ *and all* $\mathbf{u}$, *and all* $t \ge 0$:

$$|\mathbf{y}(t,\mathbf{x}_0,\mathbf{u})| \le \beta(|\mathbf{x}_0|,t) + \gamma(\|\mathbf{u}\|); \tag{2}$$

- *OLIOS, or output-Lagrange input-to-output stable if it is IOS and additionally the output-Lagrange (OL) stability property holds*:

$$|\mathbf{y}(t,\mathbf{x}_0,\mathbf{u})| \le \max\{\sigma(|\mathbf{h}(\mathbf{x}_0)|), \sigma(\|\mathbf{u}\|)\},\ \sigma \in \mathcal{K}, \tag{3}$$

for all $\mathbf{x}_0 \in R^n$ *and all* $\mathbf{u}$, *and all* $t \ge 0$. ■

In work (Sontag and Wang, 1999) also several output stability properties were introduced, which were described for forward complete system (1) in (Sontag and Wang, 1999) and for systems with bounded-input-bounded-state (BIBS) property in (Sontag and Wang, 1997b). The BIBS property means that for all $\mathbf{x}_0 \in R^n$, all $\mathbf{u}$, and all $t \ge 0$

$$|\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})| \le \max\{\vartheta(|\mathbf{x}_0|), \vartheta(\|\mathbf{u}\|)\}$$

holds for some function $\vartheta \in \mathcal{K}$. If $\mathbf{h}(\mathbf{x}) = \mathbf{x}$, then OL and BIBS properties are equivalent.

Definition 2 (Ingalls and Wang, 2001). *For system (1), a smooth function* V *and a function* $\lambda : R^n \to R_{\ge 0}$ *are called respectively an IOS-Lyapunov function and auxiliary modulus if there exist* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ *so that expression*

$$\alpha_1(|\mathbf{h}(\mathbf{x})|) \le V(\mathbf{x}) \le \alpha_2(|\mathbf{x}|) \tag{4}$$

holds and there exist $\chi \in \mathcal{K}$ *and* $\alpha_3 \in \mathcal{KL}$ *such that*

$$V(\mathbf{x}) > \chi(|\mathbf{u}|) \Rightarrow DV(\mathbf{x})\mathbf{f}(\mathbf{x},\mathbf{u}) \le -\alpha_3(V(\mathbf{x}), \lambda(\mathbf{x})) \tag{5}$$

for all $\mathbf{x}_0 \in R^n$ *and all* $\mathbf{u} \in R^m$, *and there exist some* $\delta \in \mathcal{K}$ *such that for any* $T \ge 0$

$$V(\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})) > \chi(|\mathbf{u}(t)|),\ t \in [0,T) \Rightarrow$$
$$\lambda(\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})) \le \max\{\delta(|\mathbf{x}_0|), \delta(\|\mathbf{u}\|)\}.$$

The function V *is called an OLIOS-Lyapunov function if it is an IOS-Lyapunov function, and in addition, inequality (4) can be strengthened to*

$$\alpha_1(|\mathbf{h}(\mathbf{x})|) \le V(\mathbf{x}) \le \alpha_2(|\mathbf{h}(\mathbf{x})|), \tag{6}$$

for all $\mathbf{x} \in R^n$. ■

In (Sontag and Wang, 2001) IOS- and OLIOS-Lyapunov functions were introduced for BIBS system (1). In this way one can use $|\mathbf{x}|$ as auxiliary modulus λ (see Remark 3 in (Ingalls and Wang, 2001)) and inequality (5) can be rewritten as follows:

$$V(\mathbf{x}) > \chi(|\mathbf{u}|) \Rightarrow DV(\mathbf{x})\mathbf{f}(\mathbf{x},\mathbf{u}) \le -\alpha_3(V(\mathbf{x}), |\mathbf{x}|).$$

Theorem 1 (Ingalls and Wang, 2001). *Suppose that system (1) is UO.*

(1) *The following are equivalent for the system:*
  – *it is IOS;*
  – *it admits an IOS-Lyapunov function;*

(2) *The following are equivalent for the system:*
  – *it is OLIOS;*
  – *it admits an OLIOS-Lyapunov function.* ■

Assume that inputs $\mathbf{u}$ take values in compact set $\Omega \subset R^m$, in this case there are other characterizations of output stability.

Definition 3 (Ingalls and Wang, 2001; Sontag and Wang, 2001). *A forward complete system (1) with inputs from* $\mathcal{M}_\Omega$ *is*:

- *UOS, or uniformly output stable if there exist* $\beta \in \mathcal{KL}$ *and* $\gamma \in \mathcal{K}$ *so that for all* $\mathbf{x}_0 \in R^n$ *and all* $\mathbf{u} \in \mathcal{M}_\Omega$, *and all* $t \ge 0$:

$$|\mathbf{y}(t,\mathbf{x}_0,\mathbf{u})| \le \beta(|\mathbf{x}_0|,t); \tag{7}$$

- *OLUOS, or output-Lagrange uniformly output stable if it is UOS and additionally the uniformly output-Lagrange (UOL) stability property holds*:

$$|\mathbf{y}(t,\mathbf{x}_0,\mathbf{u})| \le \max\{\sigma(|\mathbf{h}(\mathbf{x}_0)|)\},\ \sigma \in \mathcal{K}, \tag{8}$$

for all $\mathbf{x}_0 \in R^n$ *and all* $\mathbf{u} \in \mathcal{M}_\Omega$, *and all* $t \ge 0$. ■

In work (Ingalls and Wang, 2001) OLUOS property was named as output-Lagrange output uniformly global asymptotic stable and its Lyapunov characterization was given.

D e f i n i t i o n 4 (Ingalls and Wang, 2001; Sontag and Wang, 2001). *For system (1), a smooth function V and a function $\lambda: R^n \to R_{\geq 0}$ are called respectively an UOS-Lyapunov function and auxiliary modulus if there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ so that (4) holds and there exist $\chi \in \mathcal{K}$ and $\alpha_3 \in \mathcal{KL}$ such that*

$$DV(\mathbf{x})\mathbf{f}(\mathbf{x},\mathbf{u}) \leq -\alpha_3(V(\mathbf{x}), \lambda(\mathbf{x})) \quad (9)$$

is satisfied for all $\mathbf{x}_0 \in R^n$ and all $\mathbf{u} \in \mathcal{M}_\Omega$, and λ is locally Lipschitz on the set $\{\mathbf{x}: V(\mathbf{x}) > 0\}$ and $\lambda(\mathbf{x}(t, \mathbf{x}_0, \mathbf{u})) \leq \lambda(\mathbf{x}_0)$. The function V is called an OLUOS-Lyapunov function if it is an UOS-Lyapunov function, and in addition, inequality (4) can be strengthened to (6). ■

T h e o r e m 2 (Ingalls and Wang, 2001; Sontag and Wang, 2001). *Suppose that system (1) is forward complete and $\mathbf{u} \in \mathcal{M}_\Omega$.*

(1) *The following are equivalent for the system:*
- *it is UOS;*
- *it admits an UOS-Lyapunov function;*

(2) *The following are equivalent for the system:*
- *it is OLUOS;*
- *it admits an OLUOS-Lyapunov function.* ■

In work (Ingalls and Wang, 2001) only OLUOS case was considered. Obvious generalization of results in (Ingalls and Wang, 2001) and (Sontag and Wang, 2001) gives theorem 2.

Problem of output stability of nonautonomous system

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \quad (10)$$

where $\mathbf{f}$ is continuous locally Lipschitz vector field, was investigated in (Fradkov, *et al.*, 1999; Rumyantsev and Oziraner, 1987; Vorotnikov, 1998). The sufficient Lyapunov characterization was given as follows: differential positive definite and radially unbounded function V provides output global asymptotic stability for forward complete system (10), if (6) holds and for all $\mathbf{x} \in R^n$

$$\dot{V} \leq -\alpha(|\mathbf{y}|), \quad (11)$$

where continuous function α is positive definite. It is clear that from (11) follows (9) (one can choose $\alpha_3(s,r) = \alpha(s)/(1+r)$) and moreover inequality (11) gives only sufficient condition for output stability: there exist output global asymptotic stable systems, which admit Lyapunov characterization in form (9) and fail in (11) (see Remark 2.2 in (Sontag and Wang, 2001)).

To approach more closely to this work, let us consider the task of asymptotic stabilization of system (1) without output function. For solving this problem one can use integral controller (the reasons are mentioned in (Jiang and Mareels, 2000), see also references therein), common system can be described by differential equations:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{v}); \\ \dot{\mathbf{u}} = \tilde{\mathbf{u}}, \quad (12)$$

where $\tilde{\mathbf{u}}$ is new control that should be synthesized; $\mathbf{v} \in R^m$ reflects unknown disturbances in the right hand side of integral controller, signal $\mathbf{v}(t)$ is measurable and locally essentially bounded function of time. If $\mathbf{v}$ is absent, then task global asymptotic stabilization of (1) is equivalent to UOS stabilization of (12) for output $\mathbf{y} = \mathbf{x}$. In the presence of disturbance $\mathbf{v}$ this task can be formalized as IOS stabilization of (12). Note that (12) is affine in new control $\tilde{\mathbf{u}}$, so this problem reduces to task of UOS or IOS control synthesis for affine system:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{v}) + \mathbf{G}(\mathbf{x})\mathbf{u}; \; \mathbf{y} = \mathbf{h}(\mathbf{x}), \quad (13)$$

which will be considered in the next section.

3. MAIN RESULTS

First of all note that GSMO property in some sense lies between UO and BIBS properties:

P r o p o s i t i o n 1. *For system (1) the following implications hold:*
1. BIBS $\Rightarrow$ GSMO $\Rightarrow$ UO;
2. BIBS $\Leftrightarrow$ GSMO & Output boundedness (A1). ■

According to this result it is possible to change the requirement of UO property in statements of theorems 1 and 2 to GSMO property, that allows to specify auxiliary modulus function:

P r o p o s i t i o n 2. *Suppose that system (1) is GSMO and $\mathbf{u} \in R^m$.*

(1) *The following are equivalent for the system:*
- *it is IOS;*
- *it admits an IOS-Lyapunov function $\lambda(\mathbf{x}) = |\mathbf{x}|$;*

(2) *The following are equivalent for the system:*
- *it* is OLIOS;
- *it admits an OLIOS-Lyapunov function $\lambda(\mathbf{x}) = |\mathbf{x}|$.*

Suppose that system (1) is GSMO and $\mathbf{u} \in \mathcal{M}_\Omega$.

(3) *The following are equivalent for the system:*
- *it is UOS*;
- *it admits an UOS-Lyapunov function $\lambda(\mathbf{x}) = |\mathbf{x}|$;*

(4) *The following are equivalent for the system:*
- *it* is OLUOS;
- *it admits an OLUOS-Lyapunov function $\lambda(\mathbf{x}) = |\mathbf{x}|$.* ■

The result of proposition 2 is a special case of theorem 1 and 2. According to proposition 1 the GSMO systems are UO, but converse is in general false, so proposition 2 deals with more restrictive class of system (1). However, in this case the upper bound of derivative of IOS–UOS Lyapunov function V has more constructive form like in (Sontag and Wang, 2001), where BIBS systems were considered. This advance will be demonstrated now during control synthesis phase.

It is well known "universal" formula for full state global asymptotic stabilization of affine system (Sontag, 1989) and input-to-state stabilization or integral input-to-state stabilization of (13) (Liberzon, *et al*., 2001). First of all let us consider the case then $\mathbf{v}(t)\equiv 0$ for all $t\geq 0$

$$\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x})+\mathbf{G}(\mathbf{x})\mathbf{u}\ ;\ \mathbf{y}=\mathbf{h}(\mathbf{x}). \qquad (14)$$

In (Lin and Sontag, 1995) was presented "universal" formula for compact set stabilization and for bounded/positive control stabilization. The task of output asymptotic stabilization can be considered as task of asymptotic stabilization of *non compact* set $\mathcal{Z}=\{\mathbf{x}:\mathbf{h}(\mathbf{x})=0\}$. The problem of "universal" control construction is closely connected with task of control Lyapunov function (CLF) choosing. The definition of CLF with respect to closed invariant not necessary compact set was introduced in (Lin and Sontag, 1995). As discussed in example 4.2 (Lin and Sontag, 1995) this definition of CLF does not suit well for case of non compact set. Hence, here we present another one definition of UOS CLF.

D e f i n i t i o n 5 . *An UOS CLF for system (14) and control* $\mathbf{u}\in\mathcal{U}\subseteq R^m$ *is a differentiable function* $V:R^n\rightarrow R_{\geq 0}$ *satisfying*:

1. *there exist some* $\alpha_1,\alpha_2\in\mathcal{K}_\infty$ *such that for all* $\mathbf{x}\in R^n$ *(4) holds.*
2. *for all* $\mathbf{x}\in R^n$, $|\mathbf{h}(\mathbf{x})|\neq 0$

$$\inf_{\mathbf{u}\in\mathcal{U}}\{a(\mathbf{x})+\mathbf{b}(\mathbf{x})\mathbf{u}\}<0, \qquad (15)$$

$a(\mathbf{x})=\nabla V(\mathbf{x})\mathbf{f}(\mathbf{x})$, $\mathbf{b}(\mathbf{x})=\nabla V(\mathbf{x})\mathbf{G}(\mathbf{x})$.

If instead (4) such function V *admits condition (6), then it is an OLUOS CLF for system (14) with respect to control in* $\mathcal{U}$.

Function V *is said to satisfy small control property with respect to output if for any* $\varepsilon>0$ *there is an* $\delta>0$ *such that, for any* $\mathbf{h}(\mathbf{x})\neq 0$, $|\mathbf{h}(\mathbf{x})|<\delta$ *there exists a control* $\mathbf{u}\in\mathcal{U}$ *with* $|\mathbf{u}|<\varepsilon$ *and* $a(\mathbf{x})+\mathbf{b}(\mathbf{x})\mathbf{u}<0$. ■

Note that hard verified condition (15) can be changed to another one:

$$\mathbf{b}(\mathbf{x})\equiv 0\ \Rightarrow\ a(\mathbf{x})<0 \text{ for all } \mathbf{x}\in R^n,\ |\mathbf{h}(\mathbf{x})|\neq 0. \qquad (16)$$

The condition (16) forms the main restriction on UOS CLF for system (14). This work considers the same formula for "universal" control as usually (Sontag, 1989):

$$\mathbf{u}=\kappa\left(a(\mathbf{x}),|\mathbf{b}(\mathbf{x})|^2\right)\mathbf{b}(\mathbf{x})^T, \qquad (17)$$

where $\kappa(s,r)=-\dfrac{s+\sqrt{s^2+r^2}}{r}$. According to fact, that UOS property generalizes global asymptotic stability property, then following theorem develops sufficient part of another one from (Sontag, 1989).

T h e o r e m 3 . *If function V is UOS (OLUOS) control Lyapunov function (it admits (4) (or (6) in OLUOS case) and (16) conditions) satisfying small control property with respect to output and controls in* R^m, *then control (17) is continuous on* R^n *and it provides for GSMO system (14),(17) UOS (respectively OLUOS) property.* ■

Indeed, if one considers special case $\mathbf{h}(\mathbf{x})=\mathbf{x}$, then GSMO property can be dropped (boundedness of all state immediately follows from the fact, that $\dot{V}\leq 0$ for all $\mathbf{x}\in R^n$) and all conditions of theorem 4 coincide with corresponded one from (Sontag, 1989). The smoothness property can be obtained with assumption that functions $\mathbf{f}$, $\mathbf{G}$ and V are smooth. As remarked above, it seems that GSMO property is rather restrictive requirement for the system. The important class of GSMO system (14) is uniformly OSS system (14), i.e. such kind of the system, that for any input $\mathbf{u}\in R^m$ OSS property holds.

C o r o l l a r y 1 . *If function V is UOS CLF satisfying small control property with respect to output and controls in* R^m, *then control (17) is continuous on* R^n *and it provides for UOSS system (14) global asymptotic stability property with respect to origin.* ■

It is well known results of global asymptotic stabilization of affine system with input-to-state stable internal dynamic (Isidori, 1989; Isidori, 2000). As pointed out in (Liberzon, *et al*., 2000), such systems are UOSS, if they have globally defined normal form (for stabilization also relative degree property is necessary (Isidori, 1989; Isidori, 2000)). In the corollary neither of this conditions are not needed.

Now let us consider the case $\mathbf{v}(t)\neq 0$ for all $t\geq 0$. To specify conditions and "universal" control formula as (17), which provide IOS property for controlled system, we should look for suitable CLF formulation. One of them is as follows.

D e f i n i t i o n 6 . *An IOS CLF for system (13) and control* $\mathbf{u}\in\mathcal{U}\subseteq R^m$ *is a differentiable function* $V:R^n\rightarrow R_{\geq 0}$ *satisfying*:

1. *there exist some* $\alpha_1,\alpha_2\in\mathcal{K}_\infty$, *such, that for all* $\mathbf{x}\in R^n$ *(4) holds.*
2. *there exists some function* $\chi\in\mathcal{K}_\infty$, *such, that*

$$V(\mathbf{x})>\chi(|\mathbf{v}|)\ \Rightarrow\ a(\mathbf{x},\mathbf{v})\leq\psi(\mathbf{x}),\ \psi\in C_0$$

and for all $\mathbf{x}\in R^n$, $|\mathbf{h}(\mathbf{x})|\neq 0$

$$\inf_{\mathbf{u}\in\mathcal{U}}\{\psi(\mathbf{x})+\mathbf{b}(\mathbf{x})\mathbf{u}\}<0,$$

where $a(\mathbf{x},\mathbf{v})=\nabla V(\mathbf{x})\mathbf{f}(\mathbf{x},\mathbf{v})$, $\mathbf{b}(\mathbf{x})=\nabla V(\mathbf{x})\mathbf{G}(\mathbf{x})$.

If instead (4) such function V *admits condition (6), then it is an OLUOS CLF for system (13) with respect to control in* $\mathcal{U}$.

Function V *is said to satisfy small control property with respect to output if for any* $\varepsilon>0$ *there is an* $\delta>0$ *such that, for any* $\mathbf{h}(\mathbf{x})\neq 0$, $|\mathbf{h}(\mathbf{x})|<\delta$ *there exists a control* $\mathbf{u}\in\mathcal{U}$ *with* $|\mathbf{u}|<\varepsilon$ *and* $\psi(\mathbf{x})+\mathbf{b}(\mathbf{x})\mathbf{u}<0$. ■

The condition 2 of the last definition can be rewritten as follows:

$$V(\mathbf{x})>\chi(|\mathbf{v}|) \Rightarrow a(\mathbf{x},\mathbf{v})\le\psi(\mathbf{x}),\ \psi\in C_0; \quad (18)$$

$$|\mathbf{b}(\mathbf{x})|=0 \Rightarrow \psi(\mathbf{x})<0 \text{ for all } \mathbf{x}\in R^n,\ |\mathbf{h}(\mathbf{x})|\ne 0. \quad (19)$$

So, expressions (18), (19) are the main requirements for IOS or OLIOS CLF. The "universal" control formula is the same as (17):

$$\mathbf{u}=\kappa\left(\psi(\mathbf{x}),|\mathbf{b}(\mathbf{x})|^2\right)\mathbf{b}(\mathbf{x})^T, \quad (20)$$

where κ coincides with another one from (17).

T h e o r e m 4 . *If function V is IOS (OLIOS) control Lyapunov function (it admits (4) (or (6) in OLUOS case) and (18), (19) conditions) satisfying small control property with respect to output and controls in R^m, then control (20) is continuous on R^n and it provides for GSMO system (13),(20) IOS (respectively OLIOS) property.* ■

It is worth to note, that as in theorem 3, if $\mathbf{h}(\mathbf{x})=\mathbf{x}$, then GSMO property is unnecessary and this theorem is very closely connected with another one from (Liberzon, *et al*., 2001). The class of systems, which admits GSMO property in this task, includes IOSS systems and the following corollary can be proposed.

C o r o l l a r y 2 . *If function V is IOS control Lyapunov function satisfying small control property with respect to output and controls in R^m, then control (20) is continuous on R^n and it provides for IOSS system (13) the input-to-state stability property with respect to origin.* ■

4. CONCLUSION

In the paper the definitions of UOS and IOS control Lyapunov function are proposed. The "universal" control formulas, that ensures for the system discussed properties, are presented. The applying of those obtained formulas for global asymptotic stabilization of UOSS systems and input-to-state stabilization of IOSS systems is carried out. The computer simulation confirms the all claims of theoretical part of the work.

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APPENDIX

P r o o f o f p r o p o s i t i o n 1 . It is clear, that BIBS property provides GSMO for any function ν. The GSMO system state space variables can not increase infinitely while output function stays bounded, hence we can "observe" any unboundedness of the state. Conversely if output of GSMO system is bounded, i.e. for all $\mathbf{x}_0 \in R^n$, all $\mathbf{u}$ there is a $\varsigma(\mathbf{x}_0)>0$, so that

$$|\mathbf{y}(t,\mathbf{x}_0,\mathbf{u})| \le \max\{\varsigma(\mathbf{x}_0), \varsigma(\|\mathbf{u}\|)\},\ t\ge 0,\ \varsigma\in\mathcal{K}, \quad \text{(A1)}$$

then state is also bounded (from UO property follows forward completeness). Indeed, there are two cases:

1. $|\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})| \ge \nu(|\mathbf{y}(t,\mathbf{x}_0,\mathbf{u})|),\ t\in[0,\tilde{T}) \Rightarrow$
$$|\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})| \le \max\{\mu(|\mathbf{x}_0|), \mu(\|\mathbf{u}\|_{[0,\tilde{T})})\},\ t\in[0,\tilde{T}),$$
for all $\mathbf{x}_0\in R^n$, all $\mathbf{u}$ and $\tilde{T}<+\infty$;

2. For all $\mathbf{x}_0\in R^n$, all $\mathbf{u}$, $t\in[0,\tilde{T})$ and $\tilde{T}<+\infty$:
$$|\mathbf{x}(t,\mathbf{x}_0,\mathbf{u})| < \nu(|\mathbf{y}(t,\mathbf{x}_0,\mathbf{u})|) \le \max\{\nu\circ\varsigma(\mathbf{x}_0), \nu\circ\varsigma(\|\mathbf{u}\|)\}.$$

Hence system possesses BIBS property for $\vartheta(s)=\max\{\mu(s), \nu\circ\tau(s)\}$. ■

P r o o f o f p r o p o s i t i o n 2. Let us consider statements (1) and (2). Inequalities (4), (5) ensure that IOS (OLIOS) Lyapunov function V and output are bounded:

$$V(\mathbf{x}(t)) > \chi(|\mathbf{u}(t)|) \Rightarrow$$
$$\dot{V}(t)\le 0 \text{ and } |\mathbf{y}(t)| \le \alpha_1^{-1}\circ\alpha_2(|\mathbf{x}_0|),\ t\in[t_0,\tilde{T})$$

or

$$V(\mathbf{x}(t)) \le \chi(|\mathbf{u}(t)|) \Rightarrow |\mathbf{y}(t)| \le \alpha_1^{-1}\circ\chi(\|\mathbf{u}\|),\ t\in[t_0,\tilde{T}),$$

where $t_0, \tilde{T}\le+\infty$. It means output boundedness property as (A1) with function

$$\varsigma(s)=\max\{\alpha_1^{-1}\circ\chi(s), \alpha_1^{-1}\circ\alpha_2(s)\}.$$

The GSMO and (A1) properties provide BIBS property for system (1) (proposition 1) and norm of state space vector $\mathbf{x}$ can be chosen as auxiliary modulus. If instead (4) the condition (6) is satisfied, then the last one inequality can be rewritten as follows:

$$V(\mathbf{x}(t)) > \chi(|\mathbf{u}(t)|) \Rightarrow$$
$$\dot{V}(t)\le 0 \text{ and } |\mathbf{y}(t)| \le \alpha_1^{-1}\circ\alpha_2(|\mathbf{y}(0)|),\ t\in[t_0,\tilde{T}),$$

so, OL stability property additionally holds. The parts 3 and 4 of the proposition can be proved the same. ■

P r o o f o f t h e o r e m 3. In control (17) function $\kappa(s,r)$ is a root of the polynomial

$$F(p)=r\,p^2+2s\,p-r.$$

The root $\kappa(s,r)$ rushes to zero, then r tends to zero for non positive s, and provides negativity property for derivative of the polynomial:

$$F'(p)=dF/dp=2(r\,p+s),$$
$$F'(\kappa(s,r))=-2\sqrt{s^2+r^2}.$$

Note that with substitution $s=a(\mathbf{x})$ and $r=|\mathbf{b}(\mathbf{x})|^2$ derivative $0.5\,F'(\kappa(\cdot,\cdot))$ coincides with time derivative of function V for system (14), (17):

$$\dot{V}=a(\mathbf{x})-\mathbf{b}(\mathbf{x})\kappa\left(a(\mathbf{x}),|\mathbf{b}(\mathbf{x})|^2\right)\mathbf{b}(\mathbf{x})^T = -\sqrt{a(\mathbf{x})^2+|\mathbf{b}(\mathbf{x})|^4}. \quad \text{(A2)}$$

The condition (16) with small control property with respect to output ensure that if $r=|\mathbf{b}(\mathbf{x})|^2$ goes to zero then $s=a(\mathbf{x})$ is not positive and control (17) is continuous. Inequality (A2) means that condition $\dot{V}\le 0$ holds for all $\mathbf{x}\in R^n$ and all $t\ge 0$, according to (4) and results from (Fradkov, *et al.*, 1999; Rumyantsev and Oziraner, 1987) this claims output boundedness property as (A1):

$$|\mathbf{y}(t)| \le \alpha_1^{-1}\circ\alpha_2(|\mathbf{x}(0)|) \text{ for all } t\in[0,T);$$

for OLUOS case from (6) additionally UOL property (8) can be obtained:

$$|\mathbf{y}(t)| \le \alpha_1^{-1}\circ\alpha_2(|\mathbf{y}(0)|) \text{ for all } t\in[0,T),$$

here T defines time interval of solution (14), (17) definition. The boundedness of output ensures for GSMO system (14) forward completeness and consequently $T=+\infty$. Form (A2) and (16) also follows that

$$\dot{V}<0 \text{ for all } \mathbf{x}\in R^n,\ |\mathbf{h}(\mathbf{x})|\ne 0.$$

In Lemma A.5 (Sontag and Wang, 2001) was proven, that in this case more stronger inequality holds for the system:

$$\dot{V}\le-\varpi(V(\mathbf{x}),|\mathbf{x}|) \text{ for all } \mathbf{x}\in R^n,\ \varpi\in\mathcal{KL}.$$

Hence, desired conclusion immediately follows from proposition 2 and system (14), (17) is UOS and (7) holds. The OLUOS case can be proved in the same way (UOL property was obtained above). ■

P r o o f o f c o r o l l a r y 1. It is clear, that UOSS system (14) grants OSS property for system (14), (17) and from OSS property follows GSMO. Therefore theorem 3 can be applied and system (14), (17) is UOS. In (Sontag and Wang, 1997a) was mentioned, that OSS and UOS imply global asymptotic stability. ■

P r o o f o f t h e o r e m 4. The continuity property of control (20) can be proved in the same line as in he theorem 3. While $V(\mathbf{x})>\chi(|\mathbf{u}|)$ the time derivative of function V for system (13), (20) has form

$$\dot{V}=a(\mathbf{x},\mathbf{v})-\mathbf{b}(\mathbf{x})\kappa\left(\psi(\mathbf{x}),|\mathbf{b}(\mathbf{x})|^2\right)\mathbf{b}(\mathbf{x})^T = -\sqrt{\psi(\mathbf{x})^2+|\mathbf{b}(\mathbf{x})|^4}. \quad \text{(A3)}$$

Form (A3) follows that

$$V(\mathbf{x})>\chi(|\mathbf{u}|) \Rightarrow \dot{V}<0 \text{ for all } \mathbf{x}\in R^n,\ |\mathbf{h}(\mathbf{x})|\ne 0.$$

In Lemma A.5 (Sontag and Wang, 2001) was proven, that in this case more stronger inequality holds for the system:

$$V(\mathbf{x})>\chi(|\mathbf{u}|) \Rightarrow \dot{V}\le-\varpi(V(\mathbf{x}),|\mathbf{x}|)$$

for all $\mathbf{x}\in R^n$, $\varpi\in\mathcal{KL}$. Hence, desired conclusion immediately follows from proposition 2 and system (13), (17) is IOS and inequality (2) holds. The OLIOS case can be proved in the same way (condition (3) follows from boundedness property of function V (A3)). ■

P r o o f o f c o r o l l a r y 2. If system (13), (20) is IOSS, then it has also GSMO property and theorem 4 can be applied. In proposition 3.1 of (Jiang, *et al.*, 1994) was shown, that IOS and IOSS system is input-to-state stable. ■