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DISCRETE TIME DISSIPATIVENESS ON THE DIPOLYNOMIAL RING [1]

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Abstract: In this paper, we consider discrete time dissipativeness on quadratic difference forms and two-variable *dipolynomial* matrices. We show that if a system is dissipativeness for a supply rate on the dipolynomial ring, then there also exist storage functions and dissipation rates on the dipolynomial ring. Moreover, we clarify whether extremal storage functions for a given supply rates on the dipolynomial ring. This generalization is effective to extend dissipation theory in discrete time from the theoretical points of view.

Keywords: Dissipativeness, behavioral approach, dipolynomial matrices, quadratic difference forms, storage functions

1. INTRODUCTION

Dissipativeness is one of the most important properties in dynamical systems (cf. (Willems, 1972),(Trentelman and Willems, 1997),(Willems and Trentelman, 1998), and (Weiland and Willems, 1991)). The reason is that various important system characteristics, e.g., bounded realness, positive realness, and so on, can be formalized as dissipativeness. Recently, Willems and Trentelman developed quadratic differenctial forms and two variable polynomial matrices (Willems and Trentelman, 1998) as mathematical tools to generalize the notion of dissipativeness of linear dynamical systems. And then, many results based on generalized dissipativeness based on quadrtaic differential forms and two variable polynomial matrices has been provided in e.g., (Fagnani and Willems, 1997), (Rapisarda and Willems, 1997) and so on.

The discrete time version of dissipativeness also plays very crucial roles and is used to derive various tools of synthesis and analysis for discrete time systems. Particularly, filtering problems are also one of the specific topics in discrete time. The reason for this is that filters are used to obtain desired signals from noisy discrete time data. If we can develop the discrete time dissipativeness, we obtain useful filtering algorithms which are applicable to the actual discrete time data in a behavioral framework similarly to (Fagnani and Willems, 1997). Moreover, interpolation problems like Nevanlinna-Pick are also deeply related to discrete time dissipativeness. In fact, Rapisarda and Willems consider sub-space Nevanlinna-Pick interpolation problems and the most powerful unfalsified model based on dissipativeness and quadratic differential forms in continuous time (cf. (Rapisarda and Willems, 1997)). The notion of the most powerful unfalsified model is concerned with modeling, which is also a specific topic for discrete

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time data. Thus, discrete time dissipativeness will be useful to solve discrete time interpolation problems so as to obtain effective algorithms that can be applied to actual data. Hence, dissipativeness in discrete time is an important notion as in the continuous time case.

From this motivation, we suggested quadratic difference forms in order to develop the discrete time dissipativeness based on two variable polynomial matrices in (Kaneko and Fujii, 2000) and (Kaneko and Fujii, 1998). Although it is enough to focus on the polynomial ring, the notion of dissipativeness on the *dipolynomial ring* should be also studied for the sake of genertalization of dissipation theory for discrete time systems. Particularly, it is natural to consider that dynamical system can be described by dipolynomials in the discrete time, we must develop dissipation theory so as to fit such repreentations.

In this paper, thus, we generalize the notion of discrete time dissipativeness on dipolynomial rings. Concretely, we provide quadratic difference forms induced by two variable *dipolynomial* matrices. We show that if a system is dissipativeness for a supply rate on the dipolynomial ring, then there also exist storage functions and dissipation rates on the dipolynomial ring. Moreover, we clarify whether extremal storage functions for a given supply rates on the dipolynomial ring. This generalization is effective to extend dissipation theory in discrete time from the theoretical points of view.

2. PRELIMINARIES

Let $\mathbf{Z}$, $\mathbf{R}$, and $\mathbf{C}$ denote the set of integers, real numbers and complex numbers, respectively. The notation $\mathbf{R}^q$ ($\mathbf{C}^q$) denotes the set of real (complex, respectively) vectors of size q. For $\lambda \in \mathbf{C}$, $\bar{\lambda}$ denotes the conjugate of λ. For $a \in \mathbf{R}^q$, $\|a\|^2 := a^T a$. For $a \in \mathbf{C}^q$, a^* denotes the conjugated transpose of a and $\|a\|^2 := a^* a$. Let $\mathbf{R}^{p\times q}$ denotes the set of real matrices of size $p \times q$. Let $\mathbf{R}^{*\times *}$ ($\mathbf{R}^*$) denotes the set of matrices (vectors, respectively) whose size are suitable.

Let $(\mathbf{R}^q)^Z$ denote the set of real time series vectors of size q. For $w \in (\mathbf{R}^q)^Z$, the shift operator σ is defined by $(\sigma w)(t) := w(t+1)$. By using σ, the backward shift of w is also defined by $(\sigma^{-1}w)(t) := w(t-1)$. The notation l_2^q is the set of square summable time series vectors of size q, i.e., $w \in l_2^q$ means that $\sum_{t=-\infty}^{t=\infty} \|w(t)\|^2 < \infty$.

Let $\mathbf{R}[\xi]$ denote the set of polynomials in the indeterminate ξ with coefficients in $\mathbf{R}$. Similarly, $\mathbf{R}[\zeta, \eta]$ denote the set of two-variable polynomials in the indeterminates ζ and η with coefficients in $\mathbf{R}$. In addition, let $\mathbf{R}[\xi^{-1}, \xi]$ denote the set of (one variable) dipolynomials (or two-sided polynomials) with coefficients in $\mathbf{R}$, i.e., an element of $\mathbf{R}[\xi^{-1}, \xi]$ consists of not only nonnegative but also negative powers of indeterminate ξ. Similarly, let $\mathbf{R}[\zeta^{-1}, \zeta, \eta^{-1}, \eta]$ denote the set of two-variable dipolynomials in the indeterminates ζ and η with coefficients in $\mathbf{R}$. The set of matrix version of them are written respectively by $\mathbf{R}^{p\times q}[\xi]$, $\mathbf{R}^{p\times q}[\zeta, \eta]$, $\mathbf{R}^{p\times q}[\xi^{-1}, \xi]$ and $\mathbf{R}^{p\times q}[\zeta^{-1}, \zeta, \eta^{-1}, \eta]$ for real coefficient matrices of size $p \times q$. Note that the determinant of unimodular matrix of $\mathbf{R}^{q\times q}[\xi^{-1}, \xi]$ is descried by $\alpha\xi^d$ with $\alpha \neq 0 \in \mathbf{R}$ and $d \in \mathbf{Z}$. We use the symbol 'λ' in order to denote element of $\mathbf{C}$ and 'ξ' in order to denote the indereminate of one-variable polynomials and dipolynomials. For a nonsingular polynomial matrix $D(\xi) \in \mathbf{R}^{*\times *}[\xi]$, we call it Hurwitz (anti-Hurwitz) if $\det(D(\lambda)) \neq 0$ for all $\lambda \in \mathbf{C}$ such that $|\lambda| \geq 1$ ($|\lambda| \leq 1$, respectively).

3. QUADRATIC DIFFERENCE FORMS AND TWO-VARAIBLE DIPOLYNOMIAL MATRICES

Quadratic difference forms are appropriate mathematical tools related to discrete time dissipativeness. They are used throughout this paper, so we introduce some necessary definitions and properties briefly. The following materials are similar to those introduced in (Willems and Trentelman, 1998) and (Kaneko and Fujii, 2000) the continuous time case and the discrete time case, respectively. In the discrete time case, the set of polynomials were treated. However, in order to obtain more general result from the theoretical point of view, we should consider dipolynomials as well as polynomials . Thus, we expand quadratic difference forms induced by two-variable polynomials to those induced by two-variable dipolynomials as follows.

An element of $\mathbf{R}^{p\times q}[\zeta^{-1}, \zeta, \eta^{-1}, \eta]$ is described by

$$\Phi(\zeta^{-1}, \zeta, \eta^{-1}, \eta) = \sum_{k,l} \Phi_{kl} \zeta^k \eta^l. \tag{1}$$

The sum in Eq.(1) ranges over not only the nonnegative integers but also the negative integers and is assumed to be finite, and $\Phi_{kl} \in \mathbf{R}^{q\times q}$. For $\Phi(\zeta^{-1}, \zeta, \eta^{-1}, \eta) \in \mathbf{R}^{p\times q}[\zeta^{-1}, \zeta, \eta^{-1}, \eta]$, let $\mathbf{R}_s^{q\times q}[\zeta, \eta]$ denote the set of two-variable polynomial matrices satisfying $\Phi(\zeta^{-1}, \zeta, \eta^{-1}, \eta) = \Phi(\eta^{-1}, \eta, \zeta^{-1}, \zeta)^T$. For all $w \in (\mathbf{R}^q)^Z$, $\Phi(\zeta^{-1}, \zeta, \eta^{-1}, \eta) \in \mathbf{R}_s^{q\times q}[\zeta^{-1}, \eta^{-1}, \zeta, \eta]$ induces a quadratic difference form $Q_\Phi : (\mathbf{R}^q)^Z \mapsto \mathbf{R}^Z$ as defined by

$$Q_\Phi(w)(t) := \sum_{k,l} w(t+k)^T \Phi_{kl} w(t+l). \qquad (2)$$

Given $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta) \in \mathbf{R}_s^{q\times q}[\zeta^{-1},\eta^{-1},\zeta,\eta]$, by replacing the indeterminates ζ and η with ξ^{-1} and ξ, respectively, we obtain a one-variable dipolynomial matrix $\Phi(\xi,\xi^{-1},\xi^{-1},\xi) \in \mathbf{R}^{q\times q}[\xi^{-1},\xi]$. In order to avoid the confusion, we use the notation defined by

$$\partial\Phi(\xi^{-1},\xi) := \Phi(\xi,\xi^{-1},\xi^{-1},\xi) \in \mathbf{R}^{q\times q}[\xi^{-1},\xi]. \quad (3)$$

For a given $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta) \in \mathbf{R}_s^{q\times q}[\zeta^{-1},\eta^{-1},\zeta,\eta]$, we define the following indecies

$$N(\Phi)^{(+)} := \min\{n' \in \mathbf{Z} \text{ s.t. } \Phi_{kl} = 0, ^\forall k,l > n'\}$$
$$N(\Phi)^{(-)} := \max\{m' \in \mathbf{Z} \text{ s.t. } \Phi_{kl} = 0, ^\forall k,l < m'\}.$$

The nonnegativity of quadratic difference forms induced by $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta) \in \mathbf{R}_s^{q\times q}[\zeta^{-1},\eta^{-1},\zeta,\eta]$ are defined by

$$\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta) \geq 0 :\Leftrightarrow Q_\Phi(w)(t) \geq 0 ^\forall w \in (\mathbf{R}^q)^Z \text{ and } ^\forall t \in Z. \qquad (4)$$

4. DISCRETE TIME DISSIPATIVENESS ON THE DIPOLYNOMIAL RING

4.1 *Supply rates, storage functions, and dissipation rates*

At first, $Q_\Phi(w)$ induced by $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta) \in \mathbf{R}_s^{q\times q}[\zeta^{-1},\eta^{-1},\zeta,\eta]$ can be regarded as the power entering into the physical system $\Sigma = (\mathbf{Z},\mathbf{R}^q,\mathcal{B})$. The reason for this is that the power will be described by characterizing a quadratic expression involving system variables and its shifted variables, in terms of the dynamics of system variables similarly to the continuous time case. By using quadratic difference forms, we can formalize dissipativeness of discrete time dynamical system as follows.

Definition 4.1. Let $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta) \in \mathbf{R}_s^{q\times q}[\zeta^{-1}, \eta^{-1}, \zeta, \eta]$ induce a supply rate $Q_\Phi(w)$.

1. $\Sigma = (\mathbf{Z},\mathbf{R}^q,\mathcal{B})$ is said to be dissipative for a supply rate $Q_\Phi(w)$ if $\sum_{t=-\infty}^{t=\infty} Q_\Phi(w)(t) \geq 0$, for all $w \in \mathcal{B}\bigcap l_2^q$.
2. $Q_\Psi(w)$ induced by $\Psi(\zeta^{-1},\zeta,\eta^{-1},\eta) \in \mathbf{R}_s^{q\times q}$ $[\zeta^{-1}, \eta^{-1}, \zeta, \eta]$ is said to be a forward storage function of $\Sigma = (\mathbf{Z},\mathbf{R}^q,\mathcal{B})$ for the supply rate $Q_\Phi(w)$, if
$$Q_\Psi(w)(t+1) - Q_\Psi(w)(t) \leq Q_\Phi(w)(t) \qquad (5)$$
for all $t \in \mathbf{Z}$ and for all $w \in \mathcal{B}$.
3. $Q_\Delta(w)$ induced by $\Delta(\zeta^{-1},\zeta,\eta^{-1},\eta) \in \mathbf{R}_s^{q\times q}$ $[\zeta^{-1}, \eta^{-1}, \zeta, \eta]$ is said to be a dissipation rate of $\Sigma = (\mathbf{Z},\mathbf{R}^q,\mathcal{B})$ for the supply rate $Q_\Phi(w)$, if $\sum_{t=-\infty}^{t=\infty} Q_\Phi(w)(t) = \sum_{t=-\infty}^{t=\infty} Q_\Delta(w)(t)$, and $Q_\Delta(w)(t) \geq 0$ for all $t \in \mathbf{Z}$ and for all $w \in \mathcal{B}\bigcap l_2^q$. □

As for dissipation rates, in the case of $\mathcal{B} = (\mathbf{R}^q)^Z$, it is easy to see that $\sum_{t=-\infty}^{t=\infty} Q_\Phi(w)(t) = \sum_{t=-\infty}^{t=\infty} Q_\Delta(w)(t)$ for all $w \in l_2^q$ is equivalent to saying

$$\partial\Phi(\lambda^{-1},\lambda) = \partial\Delta(\lambda^{-1},\lambda) \qquad (6)$$

for all nonzero $\lambda \in \mathbf{C}$.

The relation between a supply rate, a storage function, and a dissipation rate on the dipolynomial ring can be formalized as follows.

Theorem 4.1. Let $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta) \in \mathbf{R}_s^{q\times q}$ $[\zeta^{-1}, \eta^{-1}, \zeta, \eta]$ denote a supply rate. Assume that the dynamical system $\Sigma = (\mathbf{Z},\mathbf{R}^q,\mathcal{B})$ has an image representation $w = W(\sigma^{-1},\sigma)l$, where $W(\xi^{-1},\xi) \in \mathbf{R}^{q\times d}[\xi^{-1},\xi]$ is full column rank for all nonzero $\xi \in \mathbf{C}$. Then, the following four conditions are equivalent.

1). For all $w \in l_2^q\bigcap\mathcal{B}$, $\sum_{t=-\infty}^{\infty} Q_\Phi(w)(t) \geq 0$.
2). $W(e^{j\omega},e^{-j\omega})^T \partial\Phi(e^{-j\omega},e^{j\omega}) W(e^{-j\omega},e^{j\omega}) \geq 0$ for all $\omega \in [0,2\pi)$.
3). $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta)$ and $\Sigma = (\mathbf{Z},\mathbf{R}^q,\mathcal{B})$ admit a storage function.
4). $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta)$ and $\Sigma = (\mathbf{Z},\mathbf{R}^q,\mathcal{B})$ admit a dissipation rate.

Moreover, for the supply rate $Q_\Phi(w)$ induced by $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta)$ there is a one-one relation between storage functions $Q_\Psi(w)$ induced by $\Psi(\zeta^{-1},\zeta,\eta^{-1},\eta)$ and dissipation rates $Q_\Delta(w)$ induced by $\Delta(\zeta^{-1},\zeta,\eta^{-1},\eta)$, which is described by

$$Q_\Psi(w)(t+1) - Q_\Psi(w)(t) = Q_\Phi(w)(t) - Q_\Delta(w)(t) \qquad (7)$$

for all time $t \in \mathbf{Z}$ and $w \in \mathcal{B}$ or equivalently,

$$(\zeta\eta - 1)W(\zeta^{-1},\zeta)^T \Psi(\zeta^{-1},\zeta,\eta^{-1},\eta) W(\eta^{-1},\eta) = W(\zeta^{-1},\zeta)^T \Phi(\zeta^{-1},\zeta,\eta^{-1},\eta) W(\eta^{-1},\eta) - W(\zeta^{-1},\zeta)^T \Delta(\zeta^{-1},\zeta,\eta^{-1},\eta) W(\eta^{-1},\eta). \quad (8)$$

□

The outline of the proof:

Before going to the proof, we must show that some lemmas, propositions, and theorems proven in (Kaneko and Fujii, 2000) also holds in the case of $\mathbf{R}_s^{q\times q}[\zeta^{-1},\eta^{-1},\zeta,\eta]$.

First, we consider the following lemma. The corresponding proof in the case of $\mathbf{R}_s^{q\times q}[\zeta,\eta]$ was shown in Lemma 3.1 in (Kaneko and Fujii, 2000).

Lemma 4.1. Let $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta) = \sum_{k,l=-m}^{n} \Phi_{k,l}\zeta^k\eta^l \in \mathbf{R}_s^{q\times q}[\zeta^{-1},\eta^{-1},\zeta,\eta]$. Then, the following three conditions are equivalent.

1. $\sum_{t=-\infty}^{\infty} Q_\Phi(w)(t) = 0$ for all $w \in l_2^q$.
2. $\partial\Phi(\xi^{-1},\xi) = 0$.
3. There exists a $\Psi(\zeta^{-1},\zeta,\eta^{-1},\eta) \in \mathbf{R}_s^{q\times q}[\zeta^{-1}, \eta^{-1}, \zeta, \eta]$ such that

$$(\zeta\eta - 1)\Psi(\zeta^{-1},\zeta,\eta^{-1},\eta) = \Phi(\zeta^{-1},\zeta,\eta^{-1},\eta).(9)$$

Proof of Lemma 4.1: At first, define $\Phi'(\zeta^{-1}, \eta^{-1}, \zeta, \eta) := (\zeta\eta)^m\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta)$. Clearly, this is an element of $\mathbf{R}_s^{q\times q}[\zeta,\eta]$, so we use the notation $\Phi(\zeta,\eta)$ instead of $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta)$.

($1 \Leftrightarrow 2$). By applying arbitrary $w \in l_2^q$ to $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta)$ and $\Phi'(\zeta,\eta)$ and summing up from $-\infty$ to ∞, we ontain $\{\sum_{-\infty}^{\infty} Q_\Phi(w)(t) = 0, {}^\forall w \in l_2^q\} \Leftrightarrow \{\sum_{-\infty}^{\infty} Q_{\Phi'}(w)(t) = 0, {}^\forall w \in l_2^q\}$. It follows from Lemma 3.1 in (Kaneko and Fujii, 2000) that $\left\{\sum_{-\infty}^{\infty} Q_{\Phi'}(w)(t) = 0, {}^\forall w \in l_2^q\right\} \Leftrightarrow \left\{\Phi'(\xi^{-1},\xi) = 0\right\}$. Moreover, it is easy to see that $\left\{\Phi'(\xi^{-1},\xi) = 0\right\} \Leftrightarrow \left\{\partial\Phi(\xi^{-1},\xi) = 0\right\}$. From these equivalent relations, we obtain $\{\sum_{t=-\infty}^{\infty} Q_\Phi(w)(t) = 0, {}^\forall w \in l_2^q\} \Leftrightarrow \{\partial\Phi(\xi^{-1},\xi) = 0\}$.

($2 \Rightarrow 3$). Again, consider $\Phi'(\zeta^{-1},\eta^{-1},\zeta,\eta) \in \mathbf{R}_s^{q\times q}[\zeta^{-1},\eta^{-1},\zeta,\eta]$ defined the previous proof. It follows from the proof of Lemma 3.1 in (Kaneko and Fujii, 2000) that there exists a $\Psi'(\zeta,\eta) \in \mathbf{R}_s^{q\times q}[\zeta^{-1},\eta^{-1},\zeta,\eta]$ such that $(\zeta\eta - 1)\Psi'(\zeta,\eta) = \Phi'(\zeta,\eta)$. Define $\Psi(\zeta^{-1},\zeta,\eta^{-1},\eta) := \Psi'(\zeta,\eta)(\zeta\eta)^{-m}$. This is one of two-variable dipolynomial matrices satisfying the condition 3.

($3 \Rightarrow 1$). Applying an arbitrary $w \in l_2^q$ to Eq.(9) and summing up from ∞ to ∞ allows us to conclude that the condition 1 holds . This completes the proof of Lemma 4.1. □

The following lemma corresponds to Proposition 3.1 in (Kaneko and Fujii, 2000).

Lemma 4.2. Consider $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta) = \sum_{k,l=m}^{n} \Phi_{k,l}\ \zeta^k\eta^l \in \mathbf{R}_s^{q\times q}[\zeta^{-1},\eta^{-1},\zeta,\eta]$. Then, $\partial\Phi(\lambda^{-1}, \lambda) \geq 0^\forall\lambda \in \mathbf{C}$ such that $|\lambda| = 1$ if and only if $\sum_{t=-\infty}^{-\infty} Q_\Phi(w)(t) \geq 0$, ${}^\forall w \in l_2^q$.

Proof of Lemma 4.2: Similarly to the proof of the previous lemma, we define $\Phi'(\zeta,\eta) := (\zeta\eta)^m\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta) \in \mathbf{R}_s^{q\times q}[\zeta,\eta]$. It is easy to see that $\partial\Phi(\xi^{-1},\xi) = \Phi'(\xi^{-1},\xi)$ and $\sum_{t=-\infty}^{-\infty} Q_\Phi$ (w) (t) $= \sum_{t=-\infty}^{-\infty} Q_{\Phi'}(w)(t)$ for all $w \in l_2^q$. It follows from Proposition 3.1 that $\{\Phi'(\xi^{-1},\xi) \geq 0$ for all $\xi \in \mathbf{C}$ such that $|\xi| = 1\}$ is equivalent to $\{\sum_{t=-\infty}^{-\infty} Q_{\Phi'}(w)(t)$ for all $w \in l_2^q\}$, so the equivalence condition in this lemma also holds. This completes the proof of Lemma 4.2. □

The following lemma corresponds to Proposition 3.2 in (Kaneko and Fujii, 2000), which is related to lossless systems.

Lemma 4.3. Let $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta) \in \mathbf{R}_s^{q\times q}[\zeta^{-1}, \eta^{-1}, \zeta, \eta]$ denote a supply rate. Assume that the dynamical system $\Sigma = (\mathbf{Z},\mathbf{R}^q,\mathcal{B})$ has an image representation $w = W(\sigma^{-1},\sigma)l$, where $W(\xi^{-1},\xi)$ $\in \mathbf{R}^{q\times p}[\xi^{-1},\xi]$ is full column rank for all nonzero $\xi \in \mathbf{C}$. Then, the following three conditions are equivalent.

1). For all $w \in l_2^q \bigcap \mathcal{B}$, $\sum_{t=-\infty}^{\infty} Q_\Phi(w)(t) = 0$.
2). $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta)$ and $\Sigma = (\mathbf{Z},\mathbf{R}^q,\mathcal{B})$ admit a storage function, i.e. there exists a $\Psi(\zeta^{-1},\zeta,\eta^{-1},\eta) \in \mathbf{R}_s^{q\times q}[\zeta^{-1},\eta^{-1},\zeta,\eta]$ such that $Q_\Psi(w)(t+1) - Q_\Psi(w)(t) = Q_\Phi(w)(t)$ for all time $t \in \mathbf{Z}$ and $w \in \mathcal{B}$.
3). $W(e^{j\omega},e^{-j\omega})^T\partial\Phi(e^{-j\omega},e^{j\omega})W(e^{-j\omega},e^{j\omega}) = 0$ for all $\omega \in [0,2\pi)$.

Proof of Lemma 4.3:

($1 \Rightarrow 3$): By connecting the supply rate induced and the dynamical system we define $\bar{\Phi}(\zeta^{-1},\eta^{-1}, \zeta, \eta) := W(\zeta^{-1},\zeta)^T\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta)W(\eta^{-1},\eta) \in \mathbf{R}_s^{q\times q}$ $[\zeta^{-1}, \eta^{-1}, \zeta,\eta]$. It follows from Lemma 4.2 that the condition 3 is equivalent to $\sum_{t=-\infty}^{\infty}$ $Q_{\bar{\Phi}}(w)$ (t) $= 0$ for all $l \in l_2^p$. Assume that the condition 1 holds. Since $w = W(\sigma^{-1},\sigma)d \in \mathcal{B}$ is in l_2^q for all $d \in l_2^p$, applying an arbitrary $\bar{d} \in l_2^p$ to $\sum_{t=-\infty}^{t=\infty} Q_{\bar{\Phi}}(l)(t)$ implies applying $\bar{w} = W(\sigma^{-1},\sigma)\bar{l} \in l_2^q \bigcap \mathcal{B}$ to $\sum_{t=-\infty}^{t=\infty} Q_{\bar{\Phi}}(w)(t)$. It follows from this observation that the condition 1 implies $\sum_{t=-\infty}^{t=\infty} Q_{\bar{\Phi}}(l)(t) = 0$ for all $l \in l_2^p$.

($3 \Rightarrow 2$): Assume that the condition 3 holds. By regarding $e^{j\omega}$ as ξ, it follows from Lemma 4.1 that there exists a $\Psi'(\zeta^{-1},\eta^{-1},\zeta,\ \eta) \in \mathbf{R}_s^{p\times p}[\zeta^{-1},\eta^{-1}, \zeta,\ \eta]$ such that $(\zeta\eta - 1)\bar{\Psi}(\zeta^{-1},\zeta,\eta^{-1},\eta) = \bar{\Phi}(\zeta^{-1}, \eta^{-1},\ \zeta,\ \eta)$. By using this dipolynomial matrix, we define $\Psi(\zeta^{-1},\zeta,\eta^{-1},\eta) := W^\dagger(\zeta^{-1},\zeta)^T\ \bar{\Psi}(\zeta^{-1}, \eta^{-1},\ \zeta,\eta)W^\dagger(\eta^{-1},\eta)$, where $W^\dagger(\xi^{-1},\xi) {\in} \mathbf{R}^{p\times q}[\xi, \xi]$ is full row rank for all nonzero $\xi \in \mathbf{C}$ and satisfies that $W^\dagger(\xi^{-1},\xi)W(\xi^{-1},\xi) = I_p$. By premultiplying and postmultiplying $(\zeta\eta - 1)\Psi(\zeta^{-1},\zeta,\eta^{-1},\eta)$ by $W(\zeta^{-1},\zeta)^T$ and $W(\eta^{-1},\eta)$, respectively, we can obtain

$$W(\zeta^{-1},\zeta)^T((\zeta\eta-1)\Psi(\zeta^{-1},\zeta,\eta^{-1},\eta))W(\eta^{-1},\eta)$$
$$= W(\zeta^{-1},\zeta)^T W^\dagger(\zeta^{-1},\zeta)^T$$

$$\times((\zeta\eta-1)\bar{\Psi}(\zeta^{-1},\zeta,\eta^{-1},\eta))$$
$$\times W^{\dagger}(\eta^{-1},\eta)W(\eta^{-1},\eta)$$
$$=\bar{\Phi}(\zeta^{-1},\zeta,\eta^{-1},\eta)$$
$$=W^{T}(\zeta^{-1},\zeta)\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta)W(\eta^{-1},\eta).$$

Applying an arbitrary $l\in(\mathbf{R}^{p})^{Z}$ to this equation yields the condition 2.

($2\Rightarrow 1$): This is an immediate consequence of summing up the equation in the condition 2 from $-\infty$ to ∞ along an arbitrary $w\in\mathcal{B}\cap l_2^q$. This completes the proof of Lemma 4.3. □

Now we go back to the proof of Theorem 4.1.

(1⇒4): Similarly to the proof of the previous lemma, we define $\bar{\Phi}(\zeta^{-1},\zeta,\eta^{-1},\eta):=M(\zeta^{-1},\zeta)^{T}$ $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta)M(\eta^{-1},\eta)\in\mathbf{R}_s^{q\times q}[\zeta^{-1},\eta^{-1},\zeta,\eta]$. It follows from Lemma 4.2 that the condition 4 ⇔ $\sum_{t=-\infty}^{t=\infty}Q_{\bar{\Phi}}(l)(t)\geq 0$ for all $l\in l_2^p$ By using similar discussion used in the proof of 1) ⇔ 3) of Lemma 4.3, we can observe that the conditoin 1 implies $\sum_{t=-\infty}^{t=\infty}Q_{\bar{\Phi}}(d)(t)\geq 0$ for all $l\in l_2^p$.

($4\Rightarrow 3$): From the spectral factorization of $\partial\bar{\Phi}(\xi^{-1},\xi)\in\mathbf{R}^{p\times p}[\xi]$ there exists a $D(\xi)\in\mathbf{R}^{*\times p}[\xi]$ such that $\partial\bar{\Phi}(\xi^{-1},\xi)=D(\xi^{-1})^{T}D(\xi)$ (cf.(Popov, 1973)). Now, define new two-variable dipolynomial matrix $\Delta(\zeta^{-1},\zeta,\eta^{-1},\eta):=M^{\dagger}(\zeta^{-1},\zeta)^{T}D(\zeta)^{T}D(\eta)M^{\dagger}(\eta^{-1},\eta)$. It is obvious that $Q_{\Delta}(w)(t)=\|D(\sigma)M^{\dagger}(\sigma^{-1},\sigma)w(t)\|^{2}\geq 0$ for all $w\in\mathcal{B}$ and $t\in\mathbf{Z}$. Moreover, calculating $\sum_{t=-\infty}^{t=\infty}Q_{\Delta}(w)(t)$ for an arbitrary $w\in l_2^p\bigcap\mathcal{B}$ yields

$$\sum_{t=-\infty}^{t=\infty}Q_{\Delta}(w)(t)=\sum_{t=-\infty}^{t=\infty}\|D(\sigma)M^{\dagger}(\sigma^{-1},\sigma)w(t)\|^{2}$$
$$=\sum_{t=-\infty}^{t=\infty}Q_{\bar{\Phi}}(M^{\dagger}(\sigma^{-1},\sigma)w)(t)=\sum_{t=-\infty}^{t=\infty}Q_{\bar{\Phi}}(d)(t)$$
$$=\sum_{t=-\infty}^{t=\infty}Q_{\Phi}(M(\sigma^{-1},\sigma)d)(t)=\sum_{t=-\infty}^{t=\infty}Q_{\Phi}(w)(t).$$

where the second equality of the above calculation can be obtained by Lemma 4.3. Thus, $\Delta(\zeta,\eta)$ induces one of the dissipation rates for $\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta)$ and $(\mathbf{Z},\mathbf{R}^{q},\mathcal{B})$.

(3⇒2): Assume that there exists a dissipation rate induced by $\Delta(\zeta^{-1},\zeta,\eta^{-1},\eta)\in\mathbf{R}^{q\times q}[\zeta^{-1},\zeta,\eta^{-1},\eta]$. Since for all $w\in l_2^q\bigcap\mathcal{B}$ $\sum_{t=-\infty}^{t=\infty}Q_{\Phi}(w)(t)=\sum_{t=-\infty}^{t=\infty}Q_{\Delta}(w)(t)$ holds, it follows from Lemma 4.3 that there exists $\Psi(\zeta^{-1},\zeta,\eta^{-1},\eta)\in\mathbf{R}_s^{q\times q}[\zeta^{-1},\eta^{-1},\zeta,\eta]$ such that $Q_{\Psi}(w)(t+1)-Q_{\Psi}(w)(t)=Q_{\Phi}(w)(t)-Q_{\Delta}(w)(t)$ for all $w\in\mathcal{B}$ and $t\in\mathbf{Z}$. From $Q_{\Delta}(w)(t)\geq 0$, $Q_{\Psi}(w)(t+1)-Q_{\Psi}(w)(t)\leq Q_{\Phi}(w)(t)$ for all $w\in\mathcal{B}$ and $t\in\mathbf{Z}$. This means an existence of the storage function.

(2⇒1): Assume that there exists a storage function induced by $\Psi(\zeta^{-1},\zeta,\eta^{-1},\eta)\in\mathbf{R}^{q\times q}[\zeta^{-1},\zeta,\eta^{-1},\eta]$. Summing up $Q_{\Phi}(w)(t)\geq Q_{\Psi}(w)(t+1)-Q_{\Psi}(w)(t)$ for an arbitrary $w\in l_2^q\bigcap\mathcal{B}$ from $t=-\infty$ to $t=\infty$ yields $\sum_{t=-\infty}^{t=\infty}Q_{\Phi}(w)(t)\geq 0$. Finally, one-one relation between dissipation rates and storage functions is an immediate consequence of Lemma 4.1 and the above discussions. This completes the proof of Theorem 4.1 . □

In the case of $\mathcal{B}=(\mathbf{R}^{q})^{Z}$, Eq.(7) holds for all $w\in(\mathbf{R}^{q})^{Z}$, so it can be also described by

$$(\zeta\eta-1)\Psi(\zeta^{-1},\zeta,\eta^{-1},\eta)=\Phi(\zeta^{-1},\zeta,\eta^{-1},\eta)-\Delta(\zeta^{-1},\zeta,\eta^{-1},\eta).\quad(10)$$

4.2 *The extremal storage functions*

Let $\Phi(\zeta^{-1},\eta^{-1},\zeta,\eta)\in\mathbf{R}^{q\times q}[\zeta^{-1},\eta^{-1},\zeta,\eta]$ induce a supply rate. Suppose that $\Delta(\zeta,\eta)$ is a one of two variable polynomial matrix inducing one of dissipation rates. From the proof of Theorem 4.1, we can see that if $\Delta(\zeta,\eta)=D(\zeta)^{T}D(\eta)\in\mathbf{R}_s^{q\times q}[\zeta,\eta]$ induces a dissipation rate, then $\Delta(\zeta,\eta)(\zeta\eta)^{m}=(D(\zeta)\zeta^{m})^{T}D(\eta)\eta^{m}$ also induces a dissipation rate for all $m\in\mathbf{Z}$. Particularly, this is a two-variable dipolyomial matrix in the case of $m<0$. All of these dissipation rates play the same role in the frequency domain while one of them are shifted from the other of them in the time domain. In addition, the corresponding storage functions are different each other. Let $\Psi_{\Delta}^{(m)}(\zeta^{-1},\eta^{-1},\zeta,\eta)$ denote the correspponding storage functions satisfying

$$(\zeta,\eta-1)\Psi_{\Delta}^{(m)}(\zeta^{-1},\eta^{-1},\zeta,\eta)=\Phi(\zeta^{-1},\eta^{-1},\zeta,\eta)-\Delta(\zeta,\eta)(\zeta\eta)^{m}\quad(11)$$

for all $m\in Z$. Then, we obtain the following theorem relating the relationship among all of $\Psi_{\Delta}^{(m)}(\zeta,\eta)$.

Theorem 4.2. Let $\Psi_{\Delta}^{(m)}(\zeta^{-1},\eta^{-1},\zeta,\eta)$ denote the corresponding storage functions for a dissipation rate induced by $\Delta(\zeta,\eta)$ as stated above. Then,

$$\Psi_{\Delta}^{(n)}(\zeta^{-1},\eta^{-1},\zeta,\eta)\leq\Psi_{\Delta}^{(h)}(\zeta^{-1},\eta^{-1},\zeta,\eta).\quad(12)$$

for all $n\geq h$ $(h,n\in\mathbf{Z})$.□

Out line of the proof of Theorem 4.2:

First, we can write the following two dissipation equiations

$$Q_{\Psi_{\Delta}^{n}}(w)(t+1)-Q_{\Psi_{\Delta}^{n}}(w)(t)=$$

$$Q_\Phi(w)(t) - Q_\Delta(\sigma^n w)(t) \quad (13)$$

and

$$Q_{\Psi^(_\Delta h)}(w)(t+1) - Q_{\Psi^(_\Delta h)}(w)(t) = Q_\Phi(w)(t) - Q_\Delta(\sigma^h w)(t). \quad (14)$$

Subtracting Eq.(13) from Eq.(14) yields

$$Q_{\Psi_d}(w)(t+1) - Q_{\Psi_d}(w)(t) = Q_\Delta(\sigma^h w)(t) - Q_\Delta(\sigma^n w)(t). \quad (15)$$

where $\Psi_d(\zeta^{-1}, \eta^{-1}, \zeta, \eta) := \Psi_\Delta^{(n)}(\zeta^{-1}, \eta^{-1}, \zeta, \eta) - \Psi_\Delta^{(h)}(\zeta^{-1}, \eta^{-1}, \zeta, \eta)$. Note that we can choose arbitrary vectors $w(t + N(\Psi_d)^{(-)} + 1), \cdots, w(t + N(\Psi_d)^{(+)} + 1)$. Summing up Eq.(15) from t to $-\infty$ yields

$$Q_{\Psi_d}(w)(t+1) = \sum_{n}^{i=h+1} Q_\Delta(\sigma^i w)(t) \geq 0. \quad (16)$$

Arbitrariness of $w(t + N(\Psi_d)^{(-)} + 1), \cdots, w(t + N(\Psi_d)^{(+)} + 1)$ implies $\Psi_d(\zeta^{-1}, \eta^{-1}, \zeta, \eta) \geq 0$, which completes the proof. □

The above theorem says that the storage function corresponding a dissipation rate $Q_\Delta(w)$ is greater than any storage function corresponding forward shift of $Q_\Delta(w)$.

Next, we can obatin the following thereom relating to the existence of extremal storage functions under the restricted condition.

Theorem 4.3. Let $\Phi(\zeta^{-1}, \eta^{-1}, \zeta, \eta)$ induce a supply rate. Assume that $\partial\Phi(e^{-j\omega}, e^{j\omega}) > 0$ for all $\omega \in [0, 2\pi)$. For any $m \in \mathbf{Z}$, define

$$\Psi_+^{(m)}(\zeta^{-1}, \eta^{-1}, \zeta, \eta) := (\Phi(\zeta^{-1}, \eta^{-1}, \zeta, \eta) - A(\zeta)^T A(\eta)(\zeta\eta)^m)/(\zeta\eta - 1)$$
$$\Psi_-^{(m)}(\zeta^{-1}, \eta^{-1}, \zeta, \eta) := (\Phi(\zeta^{-1}, \eta^{-1}, \zeta, \eta) - H(\zeta)^T H(\eta)(\zeta\eta)^m)/(\zeta\eta - 1)$$

where $A(\xi)$, $H(\xi)$ are polynomial matrices satisfying

$$\partial\Phi(\xi^{-1}, \xi) = A(\xi^{-1})^T A(\xi) = H(\xi^{-1})^T H(\xi)$$

and $\det(H(\xi))$ is Hurwitz, and $\det(A(\xi))$ is anti-Hurwitz. Then, for any storage function$\Psi(\zeta^{-1}, \eta^{-1}, \zeta, \eta)$ satisfying $N(\Psi)^{(-)} \geq N(\Psi_-^{(m)})^{(-)}$,

$$\Psi_-^{(m)}(\zeta^{-1}, \eta^{-1}, \zeta, \eta) \geq \Psi(\zeta^{-1}, \eta^{-1}, \zeta, \eta)$$

holds, and for any storage function$\Psi(\zeta^{-1}, \eta^{-1}, \zeta, \eta)$ satisfying $N(\Psi)^{(+)} \leq N(\Psi_+^{(m)})^{(+)}$

$$\Psi(\zeta^{-1}, \eta^{-1}, \zeta, \eta) \geq \Psi_+^{(m)}(\zeta^{-1}, \eta^{-1}, \zeta, \eta).$$

holds. □

5. CONCLUSION

This paper deals with dissipativeness on the dipolynomial ring. As a result, we have shown that it is also possible to formalize the dissipativeness on the dipolynomial ring in discrete time and the existence of extremal storage functions. The generalization presented in this paper will be effective to extend dissipation theory in discrete time from the theoretical points of view.

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