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DYNAMICS OF A RELAY BASED FREQUENCY RESPONSE ESTIMATOR

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Abstract: In this paper the existence and stability of limit cycles in a relay based estimation experiment are discussed. The relay test is used to estimate the frequency at which a given transfer function achieves a desired magnitude. The existence and stability of limit cycles in such experiment are derived via Poincaré map analysis. Numerical examples point out the properties and limitations of the estimator.

Keywords: Relay feedback; Poincaré maps; stability analysis; limit cycle.

1. INTRODUCTION

Relay feedback systems are a special class of non-linear systems that have proven to be very useful for process identification and on-line controller tuning (Åström and Hägglund, 1995). Analysis of relay feedback systems has early been performed in the frequency domain by describing function analysis, which yields approximate results. Exact conditions have been obtained by time-domain analysis and operator theory, which also provides a deeper understanding on the problem (Johansson *et al.*, 1999; Varigonda and Georgiou, 2001; Bliman and Krasnosel'skii, 1997; Gonçalves *et al.*, 2001).

Recently, a relay feedback experiment has been proposed to estimate the frequency at which a given transfer function achieves a defined magnitude (de Arruda and Barros, 2000*b*). Also, by varying this magnitude it is possible to obtain several points of the transfer function's frequency response. In de Arruda and Barros (2000*b*), this relay experiment was applied to the estimation of the Loop Transfer Function and the Sensitivity Function in a closed loop system. In de Arruda and Barros (2001*b*), a more complete study of the relay feedback experiment is presented, and stability tests are proposed to obtain the values of the transfer function's magnitude where the relay feedback is stable. Finally, in de Arruda and Barros (2001*a*), the use of this relay feedback experiment is studied for the estimation of the $\mathcal{H}_\infty$ norm of a class of transfer functions. Application to PI and PID controllers redesign are reported in de Arruda and Barros (2000*a*), and de Arruda and Barros (2001*c*).

In this paper, time-domain analysis is applied to this experiment exploring its peculiarities. In Section 2 the relay feedback experiment is presented and its main features are described. The time-domain theory for relay feedback systems is summarized in Section 3. The existence of limit cycles in this relay feedback experiment is studied in Section 4, where the time-domain analysis is applied to this case. Two examples are given in Section 5, illustrating the capabilities and open issues in this methodology. Finally, conclusions are given in Section 6.

2. PROBLEM STATEMENT

Let $H(s)$ be a stable transfer function with state-space representation

$$\frac{d}{dt}\bar{x} = A_h\bar{x} + B_h\bar{u}$$
$$y = C_h\bar{x} + D_h\bar{u} \tag{1}$$

such that

$$H(s) = C_h (sI - A_h)^{-1} B_h + D_h \ . \tag{2}$$

In order to estimate the frequency at which $H(j\omega)$ achieves a defined magnitude, define the transfer function $F(s)$ as

$$F(s) = \frac{H(s) - r}{H(s) + r} \ , \tag{3}$$

with r a real positive scalar parameter, and construct a feedback loop with an integrator as shown in Fig. 1. A state space representation for $F(s)$ (from y_i to y_o) is

$$\frac{d}{dt}\bar{x} = A_f\bar{x} + B_f y_i$$
$$y_o = C_f\bar{x} + D_f y_i \tag{4}$$

with

$$A_f = \left(A_h - \frac{B_h C_h}{r}\right) \ , \ B_f = \frac{B_h}{r} \ ,$$
$$C_f = 2\left(1 - \frac{D_h}{r}\right) C_h \text{ and } D_f = \left(\frac{2D_h}{r} - 1\right) \ , \tag{5}$$

so that

$$F(s) = C_f (sI - A_f)^{-1} B_f + D_f \ . \tag{6}$$

With the feedback structure shown in Fig. 1, several points of the frequency response of H can be determined by varying the parameter r, as shown by the following proposition (de Arruda and Barros, 2001*b*):

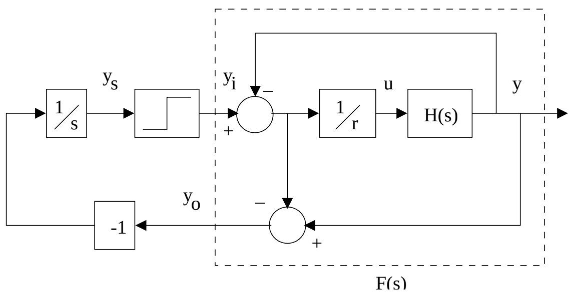

Fig. 1. Structure of $F(s)$.

Proposition 1. Consider the closed loop system shown in Fig 1. Assume that the transfer function $F(s)$ given by Eq. 3 is stable. Then if a limit cycle is present, the frequency of the oscillation, ω_o, is such that

$$|H(j\omega_o)| \approx r \ . \tag{7}$$

Proof. Describing function analysis implies that at ω_o, the frequency of the limit cycle, $\angle F(j\omega_o, r) \approx -90°$. Then,

$$F(j\omega_o, r) = \frac{H(j\omega_o) - r}{H(j\omega_o) + r} \approx -kj \ ,$$

for some real unknown $k > 0$. But,

$$H(j\omega_o) \approx r\frac{1 - kj}{1 + kj} \ ,$$

so that

$$|H(j\omega_o)| \approx r.$$

■

Remark 2. The describing function analysis assumes that only a sinusoidal signal of frequency ω_o is propagating through the transfer function $F(s)$. For the case where that is not a good approximation, there will be an error associated with this assumption. In de Arruda and Barros (2001*b*), Discrete Fourier transform is used to estimate the magnitude of $H(j\omega)$ at frequency ω_o.

Remark 3. A standard relay test can be used to obtain a safe range for the parameter r such that $F(s)$ is stable (see de Arruda and Barros (2001*b*)).

The problem posed in this paper is to evaluate the existence and stability of the limit cycles predicted by Eq. (7), in order to have a more complete understanding of the estimation experiment induced by proposition 1.

3. RELAY FEEDBACK SYSTEMS

Consider a linear system represented in state space form,

$$\dot{x} = Ax + Bu \ ,$$
$$y = Cx \ , \tag{8}$$

connected in feedback with a relay

$$u = rel(y) = \begin{cases} -1 \ , \ y > 0 \text{ and} \\ 1 \ , \ y < 0 \ . \end{cases} \tag{9}$$

where $x \in \mathcal{R}^n$. A limit cycle γ is the limit set of the nontrivial periodic orbit defined by the solution of the nonlinear system,

$$\dot{x} = Ax + B \cdot rel(Cx) = \mathbf{f}(x) \ . \tag{10}$$

The limit cycle is *symmetric* if the periodic solution of (10) satisfies $x(t + h^*) = -x(t)$, where $2h^*$ is the period of the oscillation. It is called *unimodal* if the relay switches twice, and only twice, per oscillation period. From Eq. (9), the switching surface is defined as

$$\mathcal{S}_n = \{x \in \mathcal{R}^n : Cx = 0\} \ . \tag{11}$$

Note that $\mathcal{S}$ is a hyperplane containing the origin that divides the state space into the two distinct regions

$$R^- = \{x \in \mathcal{R}^n : Cx > 0\} \ ,$$
$$R^+ = \{x \in \mathcal{R}^n : Cx < 0\} \ , \tag{12}$$

which are governed respectively by

$$\dot{x} = Ax - B \ , \text{ and } \dot{x} = Ax + B \ . \tag{13}$$

The following propositions which state necessary and sufficient conditions for the existence of limit cycles in the relay feedback system, and sufficient conditions for their asymptotic stability or instability (see Varigonda and Georgiou (2001)).

Proposition 4. Consider the linear system given by (8) connected in feedback with the relay given by (9). There exists a symmetric unimodal limit cycle with period $2h^*$ in the relay feedback system if and only if the following applies:

$$(i)\ f(h^*) \triangleq C(e^{Ah^*}+I)^{-1}\int_0^{h^*} e^{As}dsB = 0, \quad (14)$$

$$(ii)\ y(t) = C(e^{At}x^* - \int_0^t e^{As}dsB) > 0\ , \quad (15)$$

$\forall t \in (0, h^*)$, where

$$x^* = \left(e^{Ah^*}+I\right)^{-1}\int_0^{h^*} e^{As}dsB\ , \quad (16)$$

is the initial condition $x(0) = x^* \in \mathcal{S}$ that yields the periodic solution.

Proposition 5. The limit cycle in Proposition 4 is asymptotically stable if the Jacobian matrix

$$W = \left[I - \frac{wC}{Cw}\right]e^{Ah^*}\ , \quad (17)$$

with $w = e^{Ah^*}(Ax^* - B)$ has all its eigenvalues inside the open unit disc. It is unstable if W has at least one eigenvalue outside the unit disc.

In this paper the application of these analytical tools to the feedback system presented in Figure 1 is studied. For this particular experiment, the dynamic matrix A of the system including the integrator is singular. By exploiting the peculiarities of the feedback structure in Fig. 1 it is possible to derive closed expressions for equations 14, 15 and 16.

4. THE RELAY FEEDBACK FREQUENCY RESPONSE EXPERIMENT CASE

The complete state space representation including the integrator is obtained in the following. The output of the integrator is chosen as a state variable, x_{n+1}, such that

$$\dot{x}_{n+1} = y_o = C_f\bar{x} + D_f y_i \quad (18)$$

Therefore, with

$$x = \begin{bmatrix} \bar{x} \\ x_{n+1} \end{bmatrix}\ , \quad (19)$$

the complete state space representation becomes

$$\dot{x} = \begin{bmatrix} A_f & 0_{n\times 1} \\ C_f & 0 \end{bmatrix} x + \begin{bmatrix} B_f \\ D_f \end{bmatrix} u\ ,$$
$$y_s = \begin{bmatrix} 0 & \ldots & 0 & 1 \end{bmatrix} x\ , \quad (20)$$

or

$$\dot{x} = Ax + Bu\ ,$$
$$y_s = Cx\ , \quad (21)$$

with $y_s = x_{n+1}$ and $u = y_i$ (see Fig. 1). In this paper it is assumed that matrix A_f is nonsingular.

4.1 *Periodic Solution*

Proposition 6. The solution to the state space equation (21) for $u(t) = -1$, $0 < t < h^*$ is given by

$$x(t) = \begin{bmatrix} \bar{x}(t) \\ x_{n+1}(t) \end{bmatrix}\ ,$$

with

$$\bar{x}(t) = e^{A_f t}\bar{x}(0) - \left(e^{A_f t} - I\right)A_f^{-1}B_f, \quad (22)$$

and

$$x_{n+1}(t) = C_f A_f^{-1}\{(e^{A_f t} - I)\bar{x}(0) - [(\ e^{A_f t} - I)A_f^{-1} - It]B_f\} + x_{n+1}(0) - D_f t\ . \quad (23)$$

Proof. If $u(t) = -1$, the solution of equation (21) is given by

$$x(t) = \exp\left(\begin{bmatrix} A_f & 0_{n\times 1} \\ C_f & 0 \end{bmatrix} t\right)x(0) + \int_0^t \exp\left(\begin{bmatrix} A_f & 0_{n\times 1} \\ C_f & 0 \end{bmatrix} t\right)\bar{u}(t-s)\,dsB$$

Notice that

$$\exp\left(\begin{bmatrix} A_f & 0_{n\times 1} \\ C_f & 0 \end{bmatrix} t\right) = \begin{bmatrix} e^{A_f t} & 0_{n\times 1} \\ C_f\left(e^{A_f t} - I\right)A_f^{-1} & 1 \end{bmatrix}\ ,$$

therefore,

$$x(t) = \begin{bmatrix} e^{A_f t} & 0_{n\times 1} \\ C_f\left(e^{A_f t} - I\right)A_f^{-1} & 1 \end{bmatrix} x(0) - \begin{bmatrix} \int_0^t e^{A_f s}ds & 0_{n\times 1} \\ \int_0^t C_f\left(e^{A_f s} - I\right)A_f^{-1}ds & \int_0^t ds \end{bmatrix} B$$

which gives (22) and (23). ■

4.2 *Existence of Limit cycles*

Proposition 7. Consider the relay feedback system in Fig. 1 with $\{A_f, B_f, C_f, D_f\}$ being the state space representation of $F(s)$. Then, there exists a symmetric unimodal limit cycle with period $2h^*$ if and only if the following applies:

$$(i)\ \bar{f}(h^*) \triangleq C_f A_f^{-1}[(e^{A_f h^*}+I)^{-1}(e^{A_f h^*} - I) A_f^{-1} - I\frac{h^*}{2}]B_f + D_f\frac{h^*}{2} = 0\ , \quad (24)$$

$$(ii)\ y(t) = C_f A_f^{-1}\{(e^{A_f t} - I)\bar{x}^* - [(e^{A_f t} - I)A_f^{-1} - It]B_f\} - D_f t > 0\ \forall t \in (0; h^*)\ , \quad (25)$$

with

$$x^* = \begin{bmatrix} \left(e^{A_f h^*}+I\right)^{-1}\left(e^{A_f h^*} - I\right)A_f^{-1}B_f \\ 0 \end{bmatrix} \quad (26)$$

as the initial condition $x(0) = x^* = \left[(\bar{x}^*)^T \;\; x^*_{n+1} \right]^T$ that yields the periodic solution ($x^* \in \mathcal{S}_{n+1} = \{x \in \mathcal{R}^{n+1} : Cx = 0\}$).

Proof. As

$$\int_0^{h^*} e^{As} ds = \begin{bmatrix} \left(e^{A_f t} - I\right)^{-1} A_f^{-1} & 0_{n\times 1} \\ C\left(e^{A_f h^*} - I - A_f h^*\right) A^{-2} & h^* \end{bmatrix}, \quad (27)$$

and

$$(e^{Ah^*} + I)^{-1} = \begin{bmatrix} e^{A_f t} + I & 0_{n\times 1} \\ C_f\left(e^{A_f t} - I\right) A_f^{-1} & 2 \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} \left(e^{A_f t} + I\right)^{-1} & 0_{n\times 1} \\ -C_f\left(e^{A_f t} + I\right)^{-1}\left(e^{A_f t} - I\right) A_f^{-1}/2 & 1/2 \end{bmatrix}. \quad (28)$$

Then, (24) and (26) follows by replacing (27) and (28) in (14) and (16), and also from the fact that at the switching surface $x^*_{n+1} = 0$. Eq. (25) follows from the fact that there must exist no other switchings at $0 < t < h^*$. ■

4.3 *Poincaré Map*

Proposition 8. The limit cycle obtained with the relay feedback system in Fig. 1 is asymptotically stable if

$$\bar{W} = e^{A_f h^*} - \frac{\bar{w} C_f A_f^{-1}\left(e^{A_f h^*} - I\right)}{C_f A_f^{-1}(\bar{w} + B_f) - D_f}, \quad (29)$$

with $\bar{w} = e^{A_f h^*}(A_f \bar{x}^* - B_f)$, has all its eigenvalues inside the open unit disc. It is unstable if at least one eigenvalue of $\bar{W}$ is outside the unit disc.

Proof. From proposition 5 and the system's equation,

$$w = \begin{bmatrix} \bar{w} \\ CA_f^{-1}(\bar{w} + B_f) - D_f \end{bmatrix},$$

and

$$W = \left[\begin{array}{c|c} \bar{W} & -\frac{\bar{w}}{C_f A_f^{-1}(\bar{w}+B_f) - D_f} \\ \hline 0_{n\times 1} & 0 \end{array}\right]$$

where $C_f A_f^{-1}(\bar{w} + B_f) - D_f = \dot{y}_o(h^*) \neq 0$ by assumption. Clearly W has a null eigenvalue, which is within the unit cycle. The remaining eigenvalues are obtained from the matrix $\bar{W}$. Hence, according to proposition 5, if $\bar{W}$ has all its eigenvalues inside the open unit disc, the limit cycle is stable. If $\bar{W}$ has any of its eigenvalues outside the unit disc, the limit cycle is unstable. ■

5. EXAMPLES

5.1 *Second Order Transfer Function*

Consider the second order transfer function

$$H(s) = \frac{1}{(s+1)^2}.$$

The relay experiment shown in Fig. (1) is performed for different values of parameter r. The frequency of the oscillation, ω_o is measured and the magnitude of $H(j\omega_o)$ is computed using the N-point DFT technique, as reported in de Arruda and Barros (2001*b*), denoted here by r_{DFT}. The results are shown in Fig. 2 for $r = \left[\, 0.2 \;\; 0.3 \;\; 0.4 \;\; 0.5 \;\; 0.6 \;\; 0.7 \;\; 0.8 \;\; 0.9 \,\right]$.

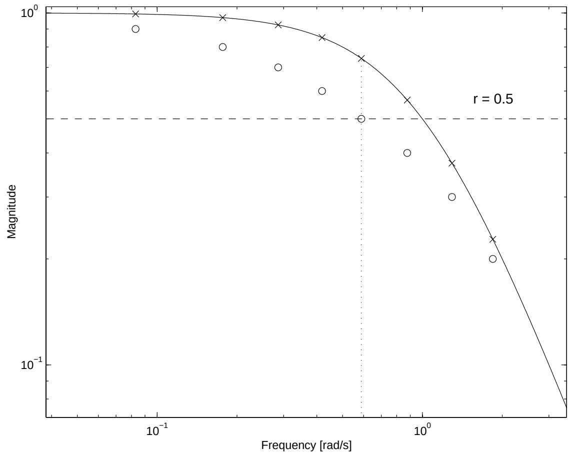

Fig. 2. Magnitude Bode Plot: $|H(j\omega)|$ (−), r versus ω_o (o) and r_{DFT} versus ω_o ($\times$).

Take for instance the case $r = 0.5$. Then from the experiment, $\omega_o = 0.5888$ and $r_{DFT} = 0.7399$ computed with $N = 1000$ samples taken over one period. $H(s)$ has state space representation

$$A_h = \begin{bmatrix} -2 & -1 \\ 1 & 0 \end{bmatrix}, \; B_h = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$C_h = \begin{bmatrix} 0 & 1 \end{bmatrix} \text{ and } D_h = 0,$$

while $F(s)$ is

$$A_f = \begin{bmatrix} -2 & -3 \\ 1 & 0 \end{bmatrix}, \; B_f = \begin{bmatrix} 2 \\ 0 \end{bmatrix},$$
$$C_f = \begin{bmatrix} 0 & 2 \end{bmatrix} \text{ and } D_f = -1.$$

Using Proposition 7, the unique limit cycle has a period of $2h^* = 10.6694$. Note that $\pi/h^* = 0.5889 \cong \omega_o$, and that $\left|H\left(j\frac{\pi}{h^*}\right)\right| = 0.7425 \cong r_{DFT}$.

The stability of this limit cycle is confirmed from Proposition 5, where $\bar{w} = [0.0070 \;\; -0.0130]^T$ and

$$\bar{W} = \begin{bmatrix} 0.0127 & 0.0194 \\ -0.0235 & -0.0491 \end{bmatrix},$$

which has all eigenvalues inside the open unit disc ($eig\ \bar{W} = (0.0041; -0.0406)$). Similar results apply for the remaining values of r. Note that, from the root locus of $H(s)$, $F(s)$ is stable for $r > 0$.

5.2 *High Order Dynamics*

Consider the transfer function

$$H(s) = \frac{1}{(s+0.1)(s^2 + 2\cdot 0.2s + 1)}. \quad (30)$$

The magnitude Bode plot of $H(j\omega)$ is shown in Fig. 3. Note that for r between 2.3984 and 2.7138, $|H(j\omega_o)| = r$ is satisfied for several frequencies for each r. Results for some values of r are shown in the figure, and discussed in the sequel.

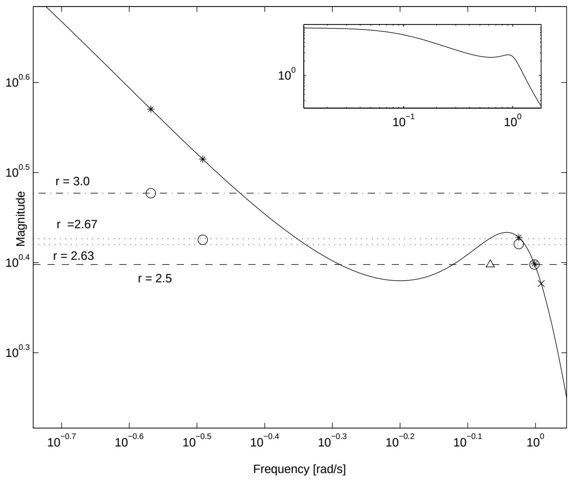

Fig. 3. Magnitude Bode plot for $H(s)$: $|H(j\omega)|$ (−), r versus ω_o (o), r_{DFT} versus ω_o (∗) and critical point (×).

Consider the case $r = 2.5$, such that the line $r = 2.5$ intercepts $|H(j\omega)|$ at three different frequencies. The state space representation for $F(s)$ is given by

$$A_f = \begin{bmatrix} -0.5 & -1.04 & -0.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \; B_f = \begin{bmatrix} 0.4 \\ 0 \\ 0 \end{bmatrix},$$

$$C_f = \begin{bmatrix} 0 & 0 & 2 \end{bmatrix} \text{ and } D_f = -1 \, .$$

According to Proposition 7, this relay feedback system has two distinct limit cycles at $h^* = \begin{bmatrix} 3.1549 & 3.6631 \end{bmatrix}$, instead of three as expected. At $h_1^* = 3.1549$, $\bar{w}_1 = [-1.2412 \; 10.2659 \; 0.9081]^T$, and

$$\bar{W}_1 = \begin{bmatrix} -0.9815 & -0.1748 & -0.0049 \\ 1.5963 & 1.0204 & 1.6791 \\ 1.1083 & 0.1302 & 0.1673 \end{bmatrix},$$

which has the eigenvalues (-0.9794;0.9384;0.2472). Thus, at $\omega_1^* = \pi/h_1^* = 0.9958 \;\; rad/s$, the limit cycle is stable. This corresponds to the circle in Fig. 3. Starting from the null initial conditions, the output signals after the transient are shown in Fig 4.

For $h_2^* = 3.6631$, $\bar{w}_2 = [-0.8207 \; 0.9921 \; 0.6675]^T$, and

$$\bar{W}_2 = \begin{bmatrix} -2.3396 & -0.8728 & -1.1884 \\ 1.5167 & 0.8955 & 1.6328 \\ 2.1443 & 0.6399 & 1.0780 \end{bmatrix},$$

which has the eigenvalues (-1.2376;0.4358±j0.2531). In this case, the limit cycle at $\omega_2^* = \pi/h_2^* = 0.8576 \;\; rad/s$ is unstable. This corresponds to the triangle in Fig. 3. The instability behavior of the output of the process at this operating condition is illustrated in Fig. 5, where the experiment is performed with initial condition x^*, obtained from Proposition 7.

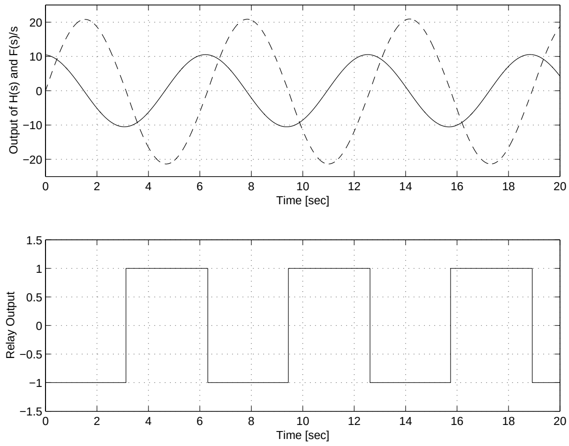

Fig. 4. Stable limit cycle. In the upper graphic, $y(t)$ (−) and $y_s(t)$ (−−).

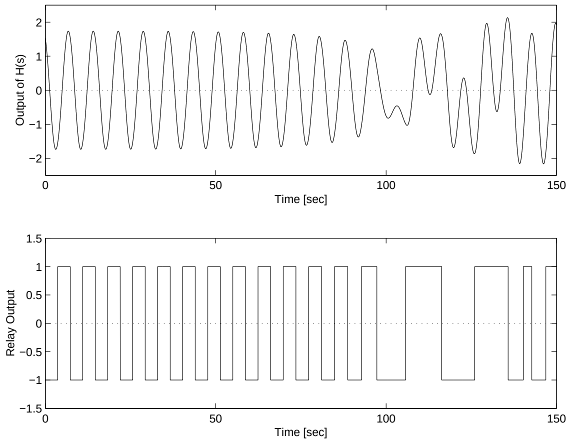

Fig. 5. Unstable limit cycle for $H(s)$.

It should be pointed out that application of Proposition 7 for $2.63 < r < 2.67$ indicates that there exists no unimodal symmetric limit cycles in the relay feedback of Fig. 1. Take for instance $r = 2.65$. Starting from null initial conditions, the output of the relay feedback system after transient is shown in Fig. 6. Indeed, the oscillation does not correspond to a symmetric unimodal limit cycle, even if observed over a longer period. However, this corresponds to a very small range compared to the frequency response of $H(s)$. Furthermore, the frequency response can be estimated by injecting sinusoidal signals and measuring the output, since the frequency range where the gap is defined is known from the experiment. Also note that $F(s)$ is stable for $r > 2.3815$.

6. CONCLUSIONS

The relay experiment discussed in this paper allows, according to the describing function analysis, to obtain the frequency response of a process by varying a parameter in the feedback loop. The describing function analysis does not always yield accurate results, particularly when the magnitude of the transfer function is not small for frequencies above the frequency of

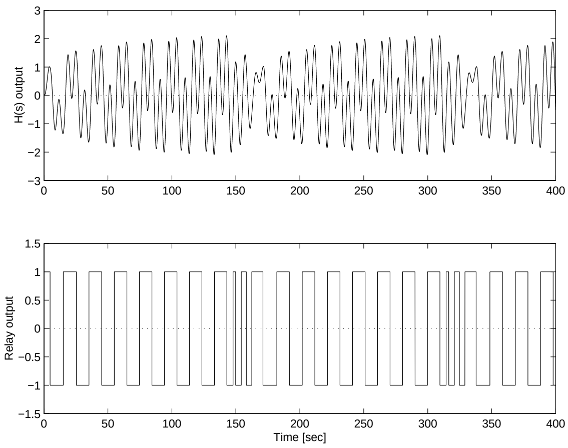

Fig. 6. Behavior of the System for $r = 2.65$.

oscillation. The correct value can be obtained from Fourier analysis of the input-output data.

For the cases where the frequency response presents the same magnitude for more than one frequency, the describing function analysis predicts more than one limit cycle. It is desirable to observe all of them in order to completely evaluate the frequency response. Time-domain analysis of examples points out that there might not be a limit cycle for each corresponding magnitude, and that some may be unstable. Hence, in order to apply this method in a practical setting further improvements are necessary, as suggested.

In this paper time-domain theory for relay feedback systems is applied to the frequency response estimator. Necessary and sufficient conditions for the existence and stability of limit cycles have been derived, providing expressions that can be directly applied to this relay experiment. Numerical examples discussing the divergences between the describing function analysis and the exact analysis have been given. The insight provided by these results on the behavior of the system will guide further improvements in the method aiming at detecting all possible points in the frequency response, which is where future research will be concentrated.

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