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A POLYNOMIAL ALGORITHM FOR THE MPS PARAMETERIZATION UNDER UNCERTAINTY

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Abstract. The aim of this paper is to study the supply planning of assembly systems under lead times uncertainty. The target is to minimize the sum of the average holding cost for the components and the setup cost, while keeping a desired service level. A further analysis of the Periodic Ordering Quantity policy (POQ) is given. The decision variables are the planned lead times of components and the periodicity of the POQ policy. A mathematical method, which gives the optimal values for these parameters, is given. *Copyright © 2002 IFAC*

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1. INTRODUCTION

The paper deals with the problem of Master Production Schedule (MPS) parameterization under lead time uncertainty. A particular case is studied where the demand level is considered constant. For this case the policies without nervousness (Dolgui and Ould-Louly 2002) can be adopted. In fact, the planning change is due to the cumulative demand uncertainty. When the demand is constant there is no need to change, and the MPS can be frozen on the total planning horizon. The problem with constant demand appears, for example, in mass production.

MRP begins with the end-product need date and uses the lead time to calculate the components release date. Clearly, the calculation doesn't take into account the actual lead time because it isn't known at this moment. The calculation uses a forecast parameter called planned lead time. Melnyk and Piper (1981) proposed a forecast method for the planned lead time which is issued from the methods used for random demand:

Planned lead time =
= lead time forecast + safety lead time
= lead time mean + k lead time standard deviation.

However the Wemmerlov's study (1986) shows that the errors of forecast increase the inventories and, in the same time, decrease the customer service level. Molinder (1997) studies this problem. Instead of forecasting, he proposes simulated annealing to find good safety stock and safety lead time. His results show that high planned lead time gives excessive inventory, and small planned lead time gives shortages and delays.

Whybark and Williams (1976) found the use of safety lead time more efficient than safety stock. Grasso and

Tayor (1984) give an opposite conclusion, their simulations find more prudent safety stocks.

As shown, there are a lot of simulation studies dealing with this problem, and there are a lot of studies dealing with the formulation problem, but in our knowledge, there is no exact mathematical method giving the optimal planned lead times.

A particular supply planning problem under lead time uncertainties concerns assembly systems. In fact, for assembly systems, several types of components are needed to produce one finished product. So, the inventories of the different types of components become dependent. Then the supply planning becomes more difficult.

There are some contributions which propose mathematical solutions, but their studies are limited to a single-period problem and do not take into account the dependence between periods. For example the Chu *et al.* (1993) model gives optimal values of the planned times in assembly systems, but only for the single-period problem. The mathematical formulation of the multi-period problems under lead times uncertainty is more difficult.

In the multi-period, orders may cross, that is, they may not be received in the same sequence in which they are placed (He *et al.*, 1998). Some contributions assume that orders do not cross, then they solve, under this assumption, the single-item problem (Graves *et al.*, 1993).

Another problem for assembly systems is the dependence between inventories. Wilhelm and Som (1998) studied this problem and showed that a renewal process can describe end-item inventory level evolution, but they did not study the dependence between component's inventories.

Song *et al*, (2000) also used the renewal theory to analyze supply planning problem for assembly systems. The lead times of the components are random. They proposed several simple heuristic policies to compute how much to order and when to order each component part.

The dependence between inventories is also studied by Gurnani *et al.* (1996), but their assembly system was only with two components, and the considered lead time distribution was simple.

In all these previous models for assembly systems, the decision variables are real, and random variables with continuous distributions.

The paper study the MPS parameterization problem for assembly systems, in which several types of components are needed to produce one type of finished products. The components lead times are random. The objective is to find the optimal values of the following MPS parameters: 1) planned lead times, and 2) sizes of the lots. A Markov model that gives the measure of the average cost is used, and the optimal values of these parameters for the periodic order quantity (POQ) policy are obtained.

2. PROBLEM STATEMENT

The unit holding cost h_i of component i per period, the setup cost c and the desired service level $1-\varepsilon$ are known. The distribution of component i's lead time L_i is also known, and its upper value is equal to u_i. The discrete random variable L_i^k is the lead time of the components i, ordered at the beginning of period k.

The demand D of finished products per period is constant, and a_i components i are needed to assemble the finished product. The component lot-sizes are determined by using the periodic order quantity (POQ) method, with a periodicity of p periods. The orders of components are released at the beginning of the periods $kp+1$, $k=0,1,2,\ldots$etc, and there is no order release in the periods $kp+r$, $r=2,3,\ldots,p$. Then, the supply orders Q_i of components are constant $Q_i=a_iDp$ (p is a decision variable).

The finished product demands are satisfied at the end of each period and unsatisfied demands are backordered and have to be satisfied during next periods.

As the lead times are uncertain, the orders have to be released before the need instant (the planned lead times). But as the ordered quantities are the same, then the planned lead times are equivalent to initial inventories. So, the aim is to find the optimal values of the initial inventories a_iDx_i, where x_i, $i=1,2,\ldots$ n, are the planned lead times, and the optimal values of the parameter p of the POQ policy.

Given that the maximal value of component i's lead time is equal to u_i, only the orders made in the previous u_i-1 periods may not be arrived yet. The orders made before already arrived. The number $N_i^{p,m}$ of component i expected deliveries at the end of the period $m=kp+r$ is easy to calculate.

Let

$$L_i^{m+1-j},\ j=r,\ r+p,\ r+2p,\ldots,\ r+\frac{u_i-1-r}{p}p,$$

be the lead times of the orders made at the beginning of the periods $kp+1$, $(k-1)p+1,\ldots,(k-\frac{u_i-1-r}{p})p+1$. The order made in the period $m+1-j$ is delivered after the end of the period m when $L_i^{m+1-j} > j$, i.e. if $1_{L_i^{m+1-j} > j}$ is equal to 1. Then, the random variables

$$N_i^{p,kp+r} = \sum_{j=0}^{\frac{u_i-1-r}{p}} 1_{L_i^{(k-j)p+1} > jp+r}, \quad i = 1, \cdots, n, \quad (1)$$

give the global state of the previous orders. In fact, $N_i^{p,m}$ is the number of component i's orders that are not arrived yet (at the end of the period m, they are still waited for).

3. PERFORMANCE MEASURE

Proposition 1.

(i) There is shortage at the end of the period $m=kp+r$ when the following condition is true

$$R_k(X, p, N^{p,kp+r}) > 0,$$

where

$$R_k(X, p, N^{p,kp+r}) = \max_{i=1,\cdots,n} (pN_i^{p,kp+r} + r - p - x_i)^+ .$$

(ii) The cost of the period $m=kp+r$ is:

$$C_k(X, p, N_1^{p,m}, \cdots, N_n^{p,m}) = c \times 1_{r=1} + D\sum_{i=1}^{n} h_i a_i (x_i + p - r - pN_i^{p,kp+r}) + DH \max_{i=1,\cdots,n} (pN_i^{p,kp+r} + r - p - x_i)^+ ,$$

where

$$X = (x_1, \cdots, x_n) \text{ and } H = \sum_{i=1}^{n} h_i a_i .$$

The value $(Z)^+$ is equal to max{Z, 0}.

Proof.

There is a shortage at the end of period m when the amount of components, arrived in the inventory since the first period, are not sufficient to satisfy the cumulative demand $D(kp+r)$ of the finished product.

The amount of components i needed by this cumulative demand is equal to $a_i D(kp+r)$. The number of orders made since the beginning until the end of the period $(kp+r)$ is equal to k+1, and the number of the delivered orders is equal to (k+1-$N_i^{p,kp+r}$).

There is a shortage if there is a component i for which the initial inventory $a_i D x_i$ plus the delivered amount $Q_i(k+1- N_i^{p,kp+r})$ is smaller than the cumulative needed amount $a_i D(kp+r)$. So, there is a shortage at the end of period m, if there is a component i satisfying $(pN_i^{p,kp+r} + r - p - x_i) > 0$. So there is shortage if $R_k(X, p, N^{p,kp+r}) > 0$.

In addition, the number of satisfied demands is equal to [$kp+r$- $R_k(X, p, N^{p,kp+r})$]. The inventory S_i^m of component i at the end of period m is equal to the initial inventory $a_i D x_i$ plus the delivered amount $Q_i(k+1- N_i^{p,kp+r})$ without the quantity used for the satisfied demand:

$$S_i^m = a_i D(x_i + p - r - pN_i^{p,m}) + a_i D \max_{j=1,\cdots,n} (pN_j^{p,m} + r - p - x_j)^+ . \quad (2)$$

Given that there is a setup cost c if r=1, the sum of the holding cost of the components and the setup cost at the end of period m is equal to:

$$C_m(X, p, N_1^{p,m}, \cdots, N_n^{p,m}) = c \times 1_{r=1} + D\sum_{i=1}^{n} h_i a_i (x_i + p - r - pN_i^{p,kp+r}) + DH \max_{i=1,\cdots,n} (pN_i^{p,kp+r} + r - p - x_i)^+ .$$

This cost is a random variable. To express the cost on an infinite horizon, a Markov chain is proposed for which a state $Z \in \{0,1\}^{u_i-1}$ is a binary vector that describes the orders made in the previous u_i-1 periods.

The average cost on the infinite horizon and the service level (1 - average number of shortages) are:

$$C(X, p) = \frac{1}{p} \sum_{r=1}^{p} E[C^r(X, p)], \quad (3)$$

$$SL(X, p) = 1 - R(X, p) = 1 - \frac{1}{p} \sum_{r=1}^{p} \Pr[R^r(X, p) > 0], \quad (4)$$

where $E(Z)$ is the expected value of Z,

$$C^r(X, p) = c\, 1_{r=1} + D\sum_{i=1}^{n} h_i a_i (x_i + p - r - pN_i^{p,n_i p+r}) + DH \max_{i=1,\cdots,n} (pN_i^{p,n_i p+r} + r - p - x_i)^+ , \quad (5)$$

$$R^r(X,p) = \max_{i=1,\cdots,n} (pN_i^{p,n_i p+r} + r - p - x_i)^+ , \quad (6)$$

$$n_i = \frac{u_i - 1 - r + p}{p} . \quad (7)$$

To simplify these equations, let h_i be the holding cost of the amount a_iD, instead of the unit holding cost.

Theorem 1.
Explicit forms for the average cost (3) and the service level (4) are:

$$C(X,p) = \frac{c}{p} + \frac{p-1}{2} H + \sum_{i=1}^{n} h_i [x_i - E(N_i^p)]$$
$$+ H \sum_{k \geq 0} [1 - \frac{1}{p} \sum_{r=1}^{p} \prod_{i=1}^{n} F_i^{p,r} (\frac{x+k-r+p}{p})] , \quad (8)$$

$$SL(X,p) = \frac{1}{p} \sum_{r=1}^{p} \prod_{i=1}^{n} F_i^{p,r} (\frac{x_i - r + p}{p}) , \quad (9)$$

where

$$F_i^{p,r}(x) = \Pr(N_i^{p,r} \leq x) \text{ and } N_i^p = \sum_{r=1}^{p} N_i^{p,r} .$$

Proof.
In fact,

$$E[C^r(X,p)] = c \times 1_{r=1} + H(p-r)$$
$$+ \sum_{i=1}^{n} h_i [x_i - pE(N_i^{p,n_i p+r})] + H\, E[R^r(X,p)] , \quad (10)$$

but,

$$E[R^r(X,p)] = \sum_{k \geq 0} [R^r(X,p) > k]$$
$$= \sum_{k \geq 0} [1 - \prod_{i=1}^{n} \Pr((pN_i^{p,n_i p+r} + r - p - x_i)^+ \leq k)]$$
$$= \sum_{k \geq 0} [1 - \prod_{i=1}^{n} F_i^{p,r} (\frac{x+k-r+p}{p})] . \quad (11)$$

So it is easy to obtain (9) from (4) and (11):

$$SL(X,p) = 1 - \frac{1}{p} \sum_{r=1}^{p} \Pr[R^r(X,p) > 0]$$
$$= \frac{1}{p} \sum_{r=1}^{p} \prod_{i=1}^{n} F_i^{p,r} (\frac{x_i - r + p}{p}) . \quad (12)$$

Moreover, using the equation (11), the equation (10) becomes:

$$E[C^r(X,p)] = c \times 1_{r=1} + H(p-r)$$
$$+ \sum_{i=1}^{n} h_i [x_i - pE(N_i^{p,n_i p+r})]$$
$$+ \sum_{k \geq 0} [1 - \prod_{i=1}^{n} F_i^{p,r} (\frac{x+k-r+p}{p})] . \quad (13)$$

Using (13) in (3), the equation (8) is obtained.

In addition, it is easy to see that the components x_i of the optimum X must satisfy $0 \leq x_i \leq u_i - 1$. Then, the initial optimization problem is as follows:

$$\text{Min } C(X,p) , \quad (14)$$

subject to:

$$\frac{1}{p} \sum_{r=1}^{p} \prod_{i=1}^{n} F_i^{p,r} (\frac{x_i - r + p}{p}) \geq 1 - \varepsilon , \quad (15)$$

$$0 \leq x_i \leq u_i - 1, \text{ i = 1, 2, …, } n, \quad (16)$$

$$0 \leq p \leq u_i - 1 . \quad (17)$$

To simplify the optimization, the following constraints (18) are considered instead of (15):

$$F^{p,r}(\frac{x_i - r + p}{p}) \geq (1-\varepsilon)^{\frac{1}{n}} ,$$
$$r = 1, \ldots, p ,\ i = 1, \ldots, n . \quad (18)$$

Even after this reduction, the optimization remains still difficult because the objective function is not linear and the decision variables, x_i, i=1,2,…,n, and p are integer.

4. OPTIMIZATION

In this section, the problem is solved under the assumption that holding costs of the quantities $Q_i=a_iD$ per period are the same, and the lead time L_i of the different components have the same distribution probability. Then, the costs h_i, i=1…n, can be noted by h, and the distributions $F_i^{p,r}$, i=1…n, can be noted by $F^{p,r}$.

The obtained problem will be noted by $\boldsymbol{\Pi}$ and an exact method which solves this problem in polynomial time will be given. The following notations are used.

E_k is the set of vectors $\boldsymbol{X}$, each vector's component is an integer variable satisfying: $x_i = x_j$, for $i \leq k$ and $j \leq k$.

P_k^1 and P_k^2 are applications from E_k to E_{k+1} which are defined as follows:

$$P_k^1(X) = Y \Leftrightarrow \begin{cases} y_i = x_1, & if \quad i \leq k+1, \\ y_i = x_i, & otherwise, \end{cases}$$

$$P_k^2(X) = Y \Leftrightarrow \begin{cases} y_i = x_{k+1}, & if \quad i \leq k+1, \\ y_i = x_i, & otherwise. \end{cases}$$

Then two vectors, $P_k^1(X)$ and $P_k^2(X)$, of the set E_{k+1} are associated to every element $\boldsymbol{X}$ of E_k.

Theorem 2.
The following equation is true under the assumptions of the problem $\boldsymbol{\Pi}$: $\forall\, k \in \{1, 2, \cdots, n-1\}$,

$$\underset{X \in E_k}{Min}\ C(X, p) = \underset{X \in E_{k+1}}{Min}\ C(X, p). \qquad (19)$$

Proof.
Note that $E_{k+1} \subset E_k$. Then it is sufficient to show that: $\forall\, k \in \{1, 2, \cdots, n-1\}$,

$$\underset{X \in E_{k+1}}{Min}\ C(X, p) \leq \underset{X \in E_k}{Min}\ C(X, p). \qquad (20)$$

For each vector $\boldsymbol{X}$ of E_k, there is a vector of E_{k+1} which gives a smaller cost. Precisely, at least one of the two vectors, $P_k^1(X)$ and $P_k^2(X)$, gives a cost less than the cost $C(X,p)$: $\forall X \in E_k$,

$$Min\{C\,[P_k^1(X), p], C\,[P_k^2(X), p]\} \leq C(X, p). \qquad (21)$$

In order to prove the theorem, it is sufficient to show the inequality (21).

Let's suppose that (21) is false, then the two following equations are obtained:

$$C(X, p) < C\,[P_k^1(X), p]\,, \qquad (22)$$

$$C(X, p) < C\,[P_k^2(X), p]\,. \qquad (23)$$

The equation (22) can easily be rewritten as:

$$h(x_1 - x_{k+1}) > \frac{H}{p} \sum_{j \geq 0} \sum_{r=1}^{p} [(F^{p,r}(\tfrac{x_1 + j - r + p}{p}) - F^{p,r}(\tfrac{x_{k+1} + j - r + p}{p})] \times F^{p,r^k}(\tfrac{x_1 + j - r + p}{p}) \prod_{i=k+2}^{n} F^{p,r^k}(\tfrac{x_i + j - r + p}{p}). \qquad (24)$$

In the same time, (23) gives:

$$k\,h(x_1 - x_{k+1}) < \frac{H}{p} \sum_{j \geq 0} \sum_{r=1}^{p} [F^{p,r^k}(\tfrac{x_1 + j - r + p}{p}) - F^{p,r^k}(\tfrac{x_{k+1} + j - r + p}{p})] \times \prod_{i=k+1}^{n} F^{p,r}(\frac{x_i + j - r + p}{p}). \qquad (25)$$

Using the fact that

$$\alpha^k - \beta^k = (\alpha - \beta) \sum_{l=0}^{k-1} \alpha^l \cdot \beta^{k-1-l}$$

for

$$\alpha = F^{p,r^k}(\tfrac{x_1 + j - r + p}{p}) \text{ and } \beta = F^{p,r^k}(\tfrac{x_{k+1} + j - r + p}{p}),$$

the equation (25) becomes:

$$\frac{k\,h(x_1 - x_{k+1})}{b + nh} < \sum_{j \geq 0} [F(x_1 + j) - F(x_{k+1} + j)] \times \sum_{l=0}^{k-1} F^l(x_1 + j) F^{k-1-l}(x_{k+1} + j) \prod_{i=k+1}^{n} F(x_i + j),$$

that can be rewritten as follows :

$$\frac{k\,h(x_1 - x_{k+1})}{b + nh} < \sum_{l=0}^{k-1} \sum_{j \geq 0} [F(x_1 + j) - F(x_{k+1} + j)]\, F^l(x_1 + j) F^{k-l}(x_{k+1} + j) \prod_{i=k+2}^{n} F(x_i + j). \qquad (26)$$

Using the fact that

$$(\mu - \eta)\,\mu^l\,\eta^{k-l} \leq (\mu - \eta)\,\mu^k$$

for $\mu = F^{p,r}(\tfrac{x_1 + j - r + p}{p})$ and $\eta = F^{p,r}(\tfrac{x_{k+1} + j - r + p}{p})$,

equation (26) becomes:

$$h(x_1 - x_{k+1}) < \frac{H}{p} \sum_{j \geq 0} \sum_{r=1}^{p} [(F^{p,r}(\tfrac{x_1 + j - r + p}{p}) - F^{p,r}(\tfrac{x_{k+1} + j - r + p}{p})] \times F^{p,r^k}(\tfrac{x_1 + j - r + p}{p}) \prod_{i=k+2}^{n} F^{p,r}(\tfrac{x_i + j - r + p}{p}). \qquad (27)$$

Equations (24) and (27) are in contradiction. Then, the equation (21) is true.

Corollary 1.
The problem $\boldsymbol{\Pi}$ has an optimal solution in which the initial inventories have the same value for every type of components.

Proof.
The proof is immediate by transitivity from the equation (19):

$$\underset{X \in E_1}{Min}\, C(X,p) = \underset{X \in E_n}{Min}\, C(X,p) \quad .$$

Then, there is an optimal solution

$$\boldsymbol{X} = (x, ..., x, ..., x) \in E_n.$$

This optimal solution can be calculated using the following theorem.

Theorem 3.
For each value of p satisfying (17), the optimum $X = (x, ..., x, ..., x)$ which minimizes the problem $\boldsymbol{\Pi}$ is obtained when x is the smallest integer satisfying (18).

Proof.
According to corollary 1, the cost $C(\boldsymbol{X},p)$ can be reduced to a two-variable function $C_e(x, p)$:

$$C(x,...,x,...,x,p) = C_e(x,p) = \tfrac{c}{p} + \tfrac{p-1}{2} nh +$$

$$nh[x - E(N^p)] + nh \sum_{k \geq 0} [1 - \tfrac{1}{p} \sum_{r=1}^{p} F^{p,r^n} (\tfrac{x+k+p-r}{p})],$$

where

$$E(N_1^p) = E(N_2^p) = \cdots = E(N_n^p) = E(N^p) \,.$$

Let $G(x,p)$ be the following function:

$$G(x, p) = C_e(x+1, p) - C_e(x, p)$$
$$= \tfrac{nh}{p} \sum_{r=1}^{p} F^{p,r^n} (\tfrac{x+p-r}{p}) \,. \quad (24)$$

$G(x,p)$ is positive function. Then, the value of x which minimizes the cost C_e is the smallest integer satisfying the constraints (16) and (18).

Finally it is ease to compute the optimal solution of problem $\boldsymbol{\Pi}$ in a polynomial time. Then we obtain the optimal planned lead times , x_i, i=1,2,…,n, and the optimal periodicity p of the POQ policy.

5. CONCLUSIONS

The problem of MPS parameterization for assembly systems, when several types of components are needed to assemble one type of finished products, is studied. The optimized parameters are the planned lead time and the size of the lots. The lead times of the components are independent random variables, and the demand of the finished product is constant. The objective is minimizing the sum of the holding cost of components and the setup cost while keeping a desired service level.

The proposed model takes into account the dependence between inventory levels at different periods, and also the dependence between the inventories of the different types of components.

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