15th Triennial World Congress, Barcelona, Spain

DISTURBED FAULT DETECTION AND ISOLATION PROBLEMS FOR LINEAR STATE MODELS IN A NOISY ENVIRONMENT

Gianfranco Parlangeli * **Maria Elena Valcher** **

* *Dipartimento di Ingegneria dell'Innovazione, Università di Lecce - strada per Monteroni, 73100 Lecce, Italy*
** *Dipartimento di Elettronica ed Informatica, Università di Padova - via Gradenigo 6A, 35131 Padova, Italy*

Abstract: In this paper we afford the fault detection and isolation problem in the context of linear discrete-time state-space models whose state equation is affected both by faults and by disturbances. Model as well as measurement errors, described as zero-mean white gaussian noises, are also assumed to additively act both on the state and on the output equations.
Upon introducing several deterministic and stochastic goals, which constitute the mathematical formalization of very natural and practical requirements, necessary and sufficient conditions for the existence of an observer-based fault detector and isolator, which achieves the aforementioned goals, are finally derived.

Keywords: Observer-based fault detector and isolator, residual, estimation error, Moore-Penrose inverse, error covariance matrix.

1. INTRODUCTION

Since the earliest seventies, the research in fault detection and isolation (FDI) has been developed by means of different approaches: detection filters, parity space checks, parameter estimation techniques, etc.. (see (Frank *et al.*, 2000*a*; Frank *et al.*, 2000*b*) for recent and complete surveys). In this paper, we resort to an observer-based approach. Of course, the FDI quality highly depends on how realistic the system representation is. To this end it is important to keep into account all signals affecting the system dynamics. In the recent past the FDI problem has either been afforded in a purely deterministic context (Commault, 1999; Liu and Si, 1997), by keeping track of the disturbances, but ignoring possible noises, or in a purely stochastic environment, disregarding possible disturbances (Keller, 1999).

The paper aims at merging and further developing some of the ideas presented in (Commault, 1999; Keller, 1999). Indeed, we consider as causes affecting the system dynamics (besides the known control inputs) both the deterministic disturbances and the model and measurement errors, described as zero-mean white gaussian noises. Within this setting, we derive necessary and sufficient conditions for the existence of both a good state estimation filter and of a residual generator that ensures an efficient fault isolation.

A comparison with other techniques or results appeared on this topic cannot be performed here, due to the page constraints. The interest reader is referred to (Parlangeli and Valcher, 2001).

2. STATEMENT OF THE PROBLEM

Consider the discrete time linear system

$$\begin{aligned} \mathbf{x}(t+1) &= A\mathbf{x}(t) + B\mathbf{u}(t) + B_d\mathbf{d}(t) + F\mathbf{n}(t) + \mathbf{w}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) + \mathbf{v}(t), \qquad t \geq 0, \end{aligned}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the *state*, $\mathbf{u}(t) \in \mathbb{R}^m$ the *input*, $\mathbf{d}(t) \in \mathbb{R}^q$ the (unknown but deterministic) *disturbance*, $\mathbf{n}(t) \in \mathbb{R}^f$ the *fault intensity* and $\mathbf{y}(t) \in \mathbb{R}^p$ the *measured output*. $\mathbf{w}(t) \in \mathbb{R}^n$ and $\mathbf{v}(t) \in \mathbb{R}^p$ are zero-mean white noise processes, which are assumed to have the covariance matrix:

$$\mathbf{E}\left\{ \begin{bmatrix} \mathbf{w}(t) \\ \mathbf{v}(t) \end{bmatrix} \begin{bmatrix} \mathbf{w}^T(s) & \mathbf{v}^T(s) \end{bmatrix} \right\} = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix} \delta(t-s),$$

with $W = W^T \geq 0, V = V^T > 0$. A, B, B_d, F and C are real matrices of suitable dimensions. $\mathbf{u}$ and $\mathbf{y}$ are measurable, meanwhile the fault intensity $\mathbf{n}(t)$ is unknown and zero-valued when the system is correctly functioning. The initial state $\mathbf{x}(0)$ is uncorrelated to the noise processes $\mathbf{v}$ and $\mathbf{w}$. We assume w.l.o.g. that B_d is of full column rank.

We aim to design a (observer based) fault isolation filter described (for $t \geq 0$) by the equations

$$\begin{aligned} \hat{\mathbf{x}}(t+1) &= A\hat{\mathbf{x}}(t) + B\mathbf{u}(t) + L[\mathbf{y}(t+1) \\ &\quad - CA\hat{\mathbf{x}}(t) - CB\mathbf{u}(t)] \qquad (1) \\ \mathbf{r}(t) &= R[\mathbf{y}(t) - CA\hat{\mathbf{x}}(t-1) - CB\mathbf{u}(t-1)] \qquad (2) \end{aligned}$$

with $\hat{\mathbf{x}}(t) \in \mathbb{R}^n$ the *state estimate*, $\mathbf{r}(t) \in \mathbb{R}^f$ the *residual vector*, L and R real matrices of suitable sizes, we will design in order to obtain the desired goals of FDI. The *estimation error* $\mathbf{e}(t) := \mathbf{x}(t) - \hat{\mathbf{x}}(t)$ (at the time t) updates as

$$\begin{aligned} \mathbf{e}(t+1) &= (A - LCA)\mathbf{e}(t) + (B_d - LCB_d)\mathbf{d}(t) \qquad (3) \\ &\quad + (F - LCF)\mathbf{n}(t) + (I - LC)\mathbf{w}(t) - L\mathbf{v}(t+1), \end{aligned}$$

meanwhile the residual vector at time t is

$$\begin{aligned} \mathbf{r}(t) &= RCA\ \mathbf{e}(t-1) + RCB_d\ \mathbf{d}(t-1) \qquad (4) \\ &\quad + RCF\ \mathbf{n}(t-1) + RC\ \mathbf{w}(t-1) + R\ \mathbf{v}(t) \end{aligned}$$

The goals we aim at reaching are the following:

G1) the state estimator must be unbiased when no faults occur, which amounts to saying that the expected value $\mathbf{E}\{\mathbf{e}(t)\}$ of the error vector must be identically zero when $\mathbf{E}\{\mathbf{e}(0)\} = 0$ and $\mathbf{n}(t) = 0$ $\forall t \geq 0$, independently of the disturbance $\mathbf{d}(t)$;

G2) the residual vector evolution must be independent of the disturbance $\mathbf{d}(t)$;

G3) the transfer function from the fault signal to the residual must have a diagonal structure;

G4) the state estimator must be asymptotically stable (or, even better, dead-beat), which amounts to saying that all the eigenvalues of the state estimator must be included in the closed unitary disc (must be located in zero). Indeed, 1) a "good" state observer must provide an expected estimate $\mathbf{E}\{\hat{\mathbf{x}}(t)\}$ whose (expected) error $\mathbf{E}\{\mathbf{e}(t)\}$ asymptotically goes to zero (goes to zero in a finite number of steps, respectively) independently of the value of the initial estimate; 2) in order to make the fault isolation possible, we want that the free evolution component in the residual expression asymptotically (in a finite number of steps) goes to zero;

G5) finally, among all possible observers and residual generators that accomplish the previous tasks, we look for those whose estimation error has minimum square norm.

3. PROBLEM SOLUTION

As far as the first goal is concerned, it is immediately seen that the only way to decouple the estimation error signal from the disturbance is to select the "observer gain matrix" L so that

$$B_d - LCB_d = 0. \qquad (5)$$

Since B_d is of full column rank, equation (5) is solvable (Kitanidis, 1987) if and only if CB_d is of full column rank, too. When so, each gain matrix L which satisfies (5) can be expressed as

$$L = B_d(CB_d)^{\#} + \bar{L}\ \alpha(I - CB_d(CB_d)^{\#}), \qquad (6)$$

where $(CB_d)^{\#}$ denotes the Moore-Penrose inverse of CB_d, while α is a (full row rank) matrix of size $(p-q)\times p$ such that $\Theta_d := \alpha(I - CB_d(CB_d)^{\#})$ is of full row rank (and hence its rows span the whole vector space $(\mathrm{Im}(CB_d))^{\perp}$). Finally, $\bar{L} \in \mathbb{R}^{n\times(p-q)}$ is a free parameter. In the following, *we will assume rank* $(CB_d) = q$, our **first constraint**.

Let us address, now, our second objective. In order to decouple the residual signal from the disturbance, upon having decoupled the estimation error from the disturbance, it is sufficient to impose

$$RCB_d = 0. \qquad (7)$$

The set of all matrices R which fulfill (7) can be parametrized as follows

$$R = \bar{R}\ \alpha(I - CB_d(CB_d)^{\#}) = \bar{R}\Theta_d \qquad (8)$$

where $\bar{R} \in \mathbb{R}^{f\times(p-q)}$ is a free parameter and Θ_d is the same matrix we previously introduced.

Before giving our subsequent goals G3÷G5 a detailed mathematical formulation, it is first convenient to introduce the following definition.

Definition 3.1 *Let (A, G, C) be a triple of matrices representing a strictly proper state-space model, namely $A \in \mathbb{R}^{n\times n}, G \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$, for suitable n, m and $p \in \mathbb{N}$. We define as* detectability indices of the triple (A, G, C) *the m positive (possibly infinite) values*

$$\rho_i(A, G, C) := \begin{cases} \min\{k :\ CA^{k-1}G\mathbf{e_i} \neq 0, k \geq 1\} \\ \qquad\qquad \text{if } \exists\ k : CA^{k-1}G\mathbf{e_i} \neq 0, \\ +\infty, \qquad \text{otherwise,} \end{cases}$$

$i = 1, \ldots, m$, where $\mathbf{e_i}$ denotes the ith canonical vector (in $\mathbb{R}^m$). For the sake of brevity, we will

set $\rho_i := \rho_i(A, G, C)$ *and refer to it as to the* i*th detectability index of* (A, G, C) *or the detectability index of the* i*th column of* G *w.r.t.* A *and* C*. Correspondingly, the* detectability matrix of the triple (A, G, C) *is defined as the* $p \times m$ *real matrix*

$$D(A, G, C) := [CA^{\rho_1 - 1} G\mathbf{e}_1 | \dots | CA^{\rho_m - 1} G\mathbf{e}_m]$$

and it can be expressed as

$$D(A, G, C) = C\Psi(A, G, C), \qquad (9)$$

where $\Psi(A, G, C) := [A^{\rho_1 - 1} G\mathbf{e_1} | \dots | A^{\rho_m - 1} G\mathbf{e_m}]$.

We are, now, in a position to formally introduce our **second and third constraints**. If $\Psi_f := \Psi(A, F, C)$, we assume that:

$$\text{rk}(C\Psi_f) = f \qquad (10)$$

$$\text{rk}([C\Psi_f | \, CB_d]) = \text{rk}(C\Psi_f) + \text{rk}(CB_d). \quad (11)$$

Condition (10) is necessary and sufficient for the solution of the FDI problem in its traditional form, namely for giving a diagonal form to the fault-to-residual transfer matrix. Condition (11) formalizes the fact that the contributions of the fault and of the disturbances on the residual must be linearly independent in order to make it possible, at the same time, the fault isolation and the complete decoupling from the disturbances.

As a first step toward our third goal, we need to derive the transfer matrix from the fault to the residual. By resorting to (3) and (4), we get

$$W_{f \to r}(z) = z^{-1} RCF + z^{-1} RCA[zI - A + LCA]^{-1} \cdot (F - LCF) = RC(zI - A + ALC)^{-1} F.$$

Therefore, once we introduce the expressions (6) and (8) derived for R and L, we get

$$W_{f \to r}(z) = \bar{R}\bar{C}(zI - \bar{A} + A\bar{L}\bar{C})^{-1} F, \quad (12)$$

where $\bar{A} := AK$, $\bar{C} := \Theta_d C$ and $K := I - B_d(CB_d)^{\#} C$. Notice that $\bar{R}$ and $\bar{L}$ are the free parameters we are remained to fix in order to reach our remaining goal and that $\bar{C} = \alpha CK$.

We can now explain the reason for our constraint $\text{rk}(C\Psi_f) = f$. The FDI problem solution, as clearly stated, for instance, in (Commault, 1999) and (Liu and Si, 1997), requires the transfer matrix from the fault signal to the residual vector to be diagonal, which is just our objective G3. If we consider the expression of $W_{f \to r}(z)$ just derived, and resort to a suitably revised version of the results presented in (Commault, 1999; Liu and Si, 1997) (in fact, our transfer matrix is slightly different), we can state that a necessary condition for the existence of real matrices $\bar{R}$ and $\bar{L}$ such that $W_{f \to r}(z)$ is a (square) proper rational matrix, endowed with a diagonal structure and nonzero diagonal entries, is that the detectability matrix of the triple $(\bar{A}, F, \bar{C})$, we will denote, according to (9), as $\bar{C}\bar{\Psi}_f$, is of full column rank. We are in a position, now, to link this condition to our second constraint (10).

Proposition 3.2 *Under the previous assumptions, and, in particular, if* $\text{rk}([C\Psi_f \mid CB_d]) = \text{rk}(C\Psi_f) + \text{rk}(CB_d)$*, we have*

$$\text{rk}(C\Psi_f) = f \quad \Leftrightarrow \quad \text{rk}(\bar{C}\bar{\Psi}_f) = f.$$

Also, when so, the detectability indices $\rho_1, \dots, \rho_f$ *of* (A, F, C) *coincide with the detectability indices* $\bar{\rho}_1, \dots, \bar{\rho}_f$ *of* $(\bar{A}, F, \bar{C})$ *and* $\bar{\Psi}_f = \Psi(\bar{A}, F, \bar{C}) = \Psi(A, F, C) = \Psi_f$*.*

Proof Assume, first, (10) and, w.l.o.g., that the columns of F are ordered according to their detectability indices so that, $\rho_i \leq \rho_{i+1}, \forall i$. Accordingly, we can block partition the matrix F as follows (see (Keller, 1999)), where F_i is the block consisting of all columns of F having detectability index i w.r.t. A and C:

$$F = [\, F_1 \quad F_2 \quad F_3 \quad \dots \quad F_s \,]. \qquad (13)$$

This implies that the detectability matrix of the triple (A, F, C) takes the form

$$C\Psi_f = C\,[\, F_1 \quad AF_2 \quad A^2 F_3 \quad \dots \quad A^{s-1} F_s \,].$$

Notice that, by (10), all blocks $CA^{i-1} F_i$ are devoid of zero columns and all detectability indices have finite values. We want to evaluate the matrix $\bar{C}\bar{\Psi}_f$. The definition of detectability indices allows to immediately get the following set of identities:

$$\begin{aligned}
\bar{C}F &= \Theta_d C\,[\, F_1 \;\; 0 \;\; \dots \;\; 0 \,] = \bar{C}\,[\, F_1 \;\; 0 \;\; \dots \;\; 0 \,] \\
\bar{C}\bar{A}F &= \Theta_d CAK\,[\, F_1 \quad F_2 \quad F_3 \; \dots \; F_s \,] \\
&= \Theta_d\,[\, CAKF_1 \quad CAF_2 \; 0 \; \dots \; 0 \,] \\
&= \bar{C}\,[\, AKF_1 \quad AF_2 \; 0 \; \dots \; 0 \,] \\
&\dots \\
\bar{C}\bar{A}^{s-1} F &= \bar{C}\,[\, * \quad \dots \quad * \quad A^{s-1} F_s \,].
\end{aligned}$$

These identities prove that $\bar{\rho}_i \leq \rho_i$, $\forall i$. In order to prove that the indices ordinately coincide, we show that none of the columns of $\bar{C}A^{i-1} F_i = \Theta_d CA^{i-1} F_i$ is zero. Suppose, by contradiction, $\Theta_d CA^{i-1} F_i \mathbf{e}_j = 0$ for some canonical vector $\mathbf{e}_j$. Then the jth column of $CA^{i-1} F_i$ would be an element of $\ker(\Theta_d) = \text{Im}(CB_d)$ and hence $CA^{i-1} F_i \mathbf{e}_j = CB_d \mathbf{v}^i$, $\exists \mathbf{v}^i$. Due to (11), however, the above equality cannot be satisfied except for $CA^{i-1} F_i \mathbf{e}_j = 0$. This contradicts the rank assumption (10). As a further consequence of the previous reasonings, once having proven that (A, F, C) and $(\bar{A}, F, \bar{C})$ have the same detectability indices, from the previous identities one also gets $\bar{C}\bar{\Psi}_f = \bar{C}\Psi_f$. Finally, we are remained to prove that $\text{rk}\,(\bar{C}\Psi_f) = \text{rk}\,(C\Psi_f) = f$. If not, it would follow that $\bar{C}\Psi_f \mathbf{u} = \Theta_d C\Psi_f \mathbf{u} = 0$, $\exists \mathbf{u} \neq 0$, namely $C\Psi_f \mathbf{u} = CB_d \mathbf{v}$, $\exists\ \mathbf{v}$. This, by the same reasoning previously adopted, would contradict (11). The reverse is proven along the same lines. ∎

The assumption $\text{rk}(\bar{C}\bar{\Psi}_f) = \text{rk}(C\Psi_f) = f$ ensures the solvability of the fault identification problem in the form presented in (Commault, 1999), i.e. the existence of $\bar{R}$ and $\bar{L}$ such that

$$W(z) := \bar{R}\bar{C}(zI - \bar{A} + \bar{L}\bar{C})^{-1}F$$

is a nonsingular diagonal transfer matrix. The fault-to-residual matrix (12) takes a slightly different form. Nonetheless it can be proven that, due to the specific relationships existing between A, $\bar{A}$ and $\bar{C}$, condition (10) is also sufficient for the existence of matrices $\bar{R}$ and $\bar{L}$ such that $W_{f\to r}(z)$ is nonsingular diagonal. We will not address here the general case, but just focus on a specific (by the way, the most efficient) solution to the fault identification problem, previously proposed in (Keller, 1999). It consists of imposing to $W_{f\to r}$ the diagonal structure

$$W_{f\to r}(z) = \begin{bmatrix} z^{-\rho_1} & & \\ & \ddots & \\ & & z^{-\rho_m} \end{bmatrix},$$

ρ_i denoting, as usual, $\rho_i(A, F, C) = \rho_i(\bar{A}, F, \bar{C})$. In order to solve the fault isolation problem by resorting to this same approach, we observe that

$$W_{f\to r}(z) = \sum_{t\geq 0} \bar{R}\bar{C}(\bar{A} - A\bar{L}\bar{C})^t F z^{-t-1}.$$

Let us assume, now, as in the proof of Proposition 3.2 and w.l.o.g. that F is block partitioned as in (13). The previous relation then becomes

$$W_{f\to r}(z) = \sum_{t\geq 0} \bar{R}\bar{C}(\bar{A} - A\bar{L}\bar{C})^t \left[F_1 z^{-t-1} | \dots | A^{s-1}F_s z^{-t-s}\right]$$

If $f_i = \dim F_i$, and we keep in mind the expression of the detectability matrix of the triple $(\bar{A}, F, \bar{C})$, we can rewrite the previous identity as:

$$W_{f\to r}(z) = \sum_{t\geq 0} \bar{R}\bar{C}(\bar{A} - A\bar{L}\bar{C})^t \Psi_f \begin{bmatrix} z^{-t-1}I_{f_1} & & \\ & \ddots & \\ & & z^{-t-s}I_{f_s} \end{bmatrix}$$

$$= \bar{R}\bar{C}(zI - \bar{A} + A\bar{L}\bar{C})^{-1}\Psi_f \begin{bmatrix} I_{f_1} & & \\ & \ddots & \\ & & z^{-s+1}I_{f_s} \end{bmatrix}.$$

Finally, by remarking that some of the f_i's could be zero, we get a more straightforward expression, in terms of detectability indices,

$$W_{f\to r}(z) = \bar{R}\bar{C}(zI - \bar{A} + A\bar{L}\bar{C})^{-1}\Psi_f \begin{bmatrix} z^{-\rho_1+1} & & \\ & \ddots & \\ & & z^{-\rho_m+1} \end{bmatrix}.$$

In order to reach the desired structure for $W_{f\to r}$, we need to find $\bar{R}$ and $\bar{L}$ such that $\bar{R}\bar{C}(zI - \bar{A} + A\bar{L}\bar{C})^{-1}\Psi_f = z^{-1}I_f$. This can be achieved by imposing

$$\bar{R}\bar{C}\Psi_f = I_f \tag{14}$$

$$(\bar{A} - A\bar{L}\bar{C})\Psi_f = A(K - \bar{L}\bar{C})\Psi_f = 0. \tag{15}$$

By the assumption (10), equations (14) and (15) are both solvable, and their solutions can be parametrized as follows:

$$\bar{R} = (\bar{C}\Psi_f)^{\#} + \hat{R}\beta(I - \bar{C}\Psi_f(\bar{C}\Psi_f)^{\#})$$

$$\bar{L} = (K\Psi_f + W_A)(\bar{C}\Psi_f)^{\#} + \hat{L}\beta(I - \bar{C}\Psi_f(\bar{C}\Psi_f)^{\#})$$

where β is a (fixed) matrix of size $(p-q-f)\times(p-q)$ such that $\beta(I - \bar{C}\Psi_f(\bar{C}\Psi_f)^{\#})$ is a $(p-q-f)\times f$ (full row rank) matrix whose rows generate the vector space $\left(\text{Im}(\bar{C}\Psi_f)\right)^{\perp}$, W_A is an arbitrary matrix whose columns belongs to $\ker(A)$, $\hat{R}$ and $\hat{L}$ are free matrix parameters.

It is worthwhile, at this point, to briefly summarize the present expressions of the two parameter matrices R and L. Indeed, we have:

$$L = L_0 + \left[W_A(\bar{C}\Psi_f)^{\#} + \hat{L}\Theta_f\right]\Theta_d \tag{16}$$

$$R = [(\bar{C}\Psi_f)^{\#} + \hat{R}\Theta_f]\Theta_d,$$

where $\Theta_f := \beta(I - \bar{C}\Psi_f(\bar{C}\Psi_f)^{\#})$ and $L_0 := B_d(CB_d)^{\#} + K\Psi_f(\bar{C}\Psi_f)^{\#}\Theta_d$.

Since, in the following, the freedom degree $\hat{R}$ will not be further exploited (G4 and G5 only involve the estimation error), we will set $R = (\bar{C}\Psi_f)^{\#}\Theta_d$.

As a fourth step, we address the asymptotic stability (or, eventually, the dead beat) problem for the obtained FDI (discrete time) filter. The matrix responsible for the free evolution of the error vector is $A - LCA$ (see (3)). However, since we have partially exploited the arbitrariety of L in order to fulfill the previous conditions, we have to keep into account the formula derived, up to now, for L. Such a formula involves two free parameters, i.e. the matrix W_A, whose columns belong to $\ker(A)$ (a steady assumption from now on), and $\hat{L}$ which is completely free.

In order to get rid of one of the two parameters, and get a nicer expression for the pair whose detectability (reconstructibility) we have to test, we will resort to the well-known property that, given two square matrices A and Q, the spectrum of AQ is the same as the spectrum of QA.

Proposition 3.3 *Set $\hat{A} := (I_n - L_0C)A$ and $\hat{C} := \Theta_f\Theta_d CA = \Theta_f\bar{C}A$. There exist matrices W_A and $\hat{L}$ such that the matrix $A - LCA$, with L given by (16), is asymptotically stable (nilpotent) if and only if the pair $(\hat{A}, \hat{C})$ is detectable (reconstructible). Even more,*

- *if the pair $(W_A, \hat{L})$ makes the matrix $A - LCA$ asymptotically stable (nilpotent) then $\hat{A} - \hat{L}\hat{C}$ is asymptotically stable (nilpotent);*
- *if $\hat{A} - \hat{L}\hat{C}$ is asymptotically stable (nilpotent) then for every W_A the pair $(W_A, \hat{L})$ makes $A - LCA$ asymptotically stable (nilpotent).*

Proof Notice, first, that $A - LCA$ has the same spectrum of $A - ALC$. Upon replacing in $A - ALC$ the expression of L, as given in (16), we get:

$$A - ALC = A - A\Big\{L_0 + \Big(W_A(\bar{C}\Psi_f)^{\#} + \hat{L}\Theta_f\Big)\Theta_d\Big\} C$$
$$= A\left(I_n - L_0C - \hat{L}\Theta_f\Theta_dC\right),$$

where we exploited the fact that $AW_A = 0$. Finally, once we notice that $A(I_n - L_0C - \hat{L}\Theta_f\Theta_dC)$ has the same spectrum as the matrix $(I_n - L_0C - \hat{L}\Theta_f\Theta_dC)A = \hat{A} - \hat{L}\hat{C}$, the proof is complete. ■

Remark The above proposition states a quite important fact, namely that of the two free parameters W_A and $\hat{L}$, only the latter is relevant as far as the spectrum assignment of the matrix $A - LCA$ is concerned, meanwhile W_A takes no effective role. As a consequence, we will choose $\hat{L}$ in order to attribute, when possible, the desired spectrum to $A - LCA$, and we will exploit the remaining parameter W_A for achieving our final goal G5. As a first step, we need to know when the stability problem is solvable, which amounts to provide necessary and sufficient conditions for the detectability (the reconstructibility) of the pair $(\hat{A}, \hat{C})$. We have the following result.

Proposition 3.4 *Under the previous assumptions, the pair $(\hat{A}, \hat{C})$ is detectable (reconstructible) if and only if*

$$\text{rk}\begin{bmatrix} \lambda I_n - A & B_d & \Psi_f \\ C & 0 & 0 \end{bmatrix} < n+q+f \Rightarrow \begin{matrix} |\lambda| < 1 \\ (\lambda = 0). \end{matrix} \quad (17)$$

Proof The detectability property of the pair $(\hat{A}, \hat{C})$ can be equivalently restated by saying that

$$\text{rk}\begin{bmatrix} \lambda I_n - (I_n - L_0C)A \\ \Theta_f\Theta_dCA \end{bmatrix} < n \quad \Rightarrow \quad |\lambda| < 1.$$

This amounts to saying that if $\exists\ \mathbf{v} \neq \emptyset$ such that

$$\begin{bmatrix} \lambda I_n - (I_n - L_0C)A \\ \Theta_f\Theta_dCA \end{bmatrix} \mathbf{v} = 0, \quad (18)$$

then $|\lambda| < 1$. Since, $\Theta_dC = \bar{C}$ and

$$I_n - L_0C = I_n - B_d(CB_d)^{\#}C - K\Psi_f(\bar{C}\Psi_f)^{\#}\Theta_dC$$
$$= K(I_n - \Psi_f(\bar{C}\Psi_f)^{\#}\bar{C}),$$

condition (18) can also be rewritten as

$$\begin{cases} \lambda\mathbf{v} = K(I_n - \Psi_f(\bar{C}\Psi_f)^{\#}\bar{C})A\mathbf{v} \\ 0 = \Theta_f\bar{C}A\mathbf{v}. \end{cases} \quad (19)$$

We consider, first, the second identity in (19), which can be equivalently restated as

$$\bar{C}A\ \mathbf{v} = \bar{C}\Psi_f\ \mathbf{u}, \qquad \text{for some } \mathbf{u} \in \mathbb{R}^f. \quad (20)$$

Condition (20), then, can be rewritten by making use of a family of equivalent conditions. In fact,

$$0 = \bar{C}(A\ \mathbf{v} - \Psi_f\ \mathbf{u}) = \Theta_dC(A\ \mathbf{v} - \Psi_f\ \mathbf{u}) \quad \Leftrightarrow$$
$$\Leftrightarrow\ \exists\ \mathbf{t} \in \mathbb{R}^q \text{s.t. } C(A\ \mathbf{v} - \Psi_f\ \mathbf{u}) = CB_d\ \mathbf{t}\ \Leftrightarrow$$
$$\Leftrightarrow \exists \mathbf{t} \in \mathbb{R}^q, \mathbf{c} \in \ker\ (C): (A\ \mathbf{v} - \Psi_f\ \mathbf{u}) = B_d\mathbf{t} + \mathbf{c}. \quad (21)$$

On the other hand, by replacing (20) within the first of the two identities in (19) one gets

$$\lambda\mathbf{v} = K(A\ \mathbf{v} - \Psi_f\mathbf{u}). \quad (22)$$

So, (19) can be equivalently rewritten as:

$$\exists\mathbf{t} \in \mathbb{R}^q, \mathbf{c} \in \ker\ (C): \begin{cases} \lambda\mathbf{v} = K(A\ \mathbf{v} - \Psi_f\mathbf{u}) \\ (A\mathbf{v} - \Psi_f\ \mathbf{u}) = B_d\mathbf{t} + \mathbf{c}. \end{cases}$$

Also, by using (21) in the first identity of the above formula, one obtains

$$\lambda\ \mathbf{v} = \left[I_n - B_d(CB_d)^{\#}C\right] B_d\ \mathbf{t} + K\mathbf{c} = \mathbf{c}\ , \quad (23)$$

which finally leads to state that the pair $(\hat{A}, \hat{C})$ is detectable if and only if

$$\left.\begin{matrix} \lambda\mathbf{v} = \mathbf{c}, \quad \mathbf{v} \neq 0,\ \mathbf{c} \in \ker(C) \\ 0 = A\ \mathbf{v} - \Psi_f\ \mathbf{u} - B_d\ \mathbf{t} - \mathbf{c} \end{matrix}\right\} \Rightarrow |\lambda| < 1.$$

We can rewrite the previous condition in matrix terms as follows:

$$\begin{bmatrix} \lambda I_n - A & B_d & \Psi_f \\ \lambda C & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{v} \\ \mathbf{t} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ \exists \begin{bmatrix} \mathbf{v} \\ \mathbf{t} \\ \mathbf{u} \end{bmatrix}, \mathbf{v} \neq 0 \quad (24)$$

$\Rightarrow\ |\lambda| < 1$. We easily notice that 1) the case $\lambda = 0$ is of no interest for our analysis, while, for $\lambda \neq 0$, condition $\lambda\mathbf{v} \in \ker(C)$ holds if and only if $\mathbf{v} \in \ker(C)$; 2) as an immediate consequence of (11), no nonzero vector $[\,0 \quad \mathbf{t}^T \quad \mathbf{u}^T\,]^T$ exists in

$$\ker\begin{bmatrix} \lambda I_n - A & B_d & \Psi_f \\ \lambda C & 0 & 0 \end{bmatrix}.$$

This ensures that "(24) $\Rightarrow |\lambda| < 1$" is equivalent to saying that condition

$$\begin{bmatrix} \lambda I_n - A & B_d & \Psi_f \\ C & 0 & 0 \end{bmatrix}\begin{bmatrix} \mathbf{v} \\ \mathbf{t} \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \exists \begin{bmatrix} \mathbf{v} \\ \mathbf{t} \\ \mathbf{u} \end{bmatrix} \neq 0 \quad (25)$$

implies $|\lambda| < 1$, and hence can be rewritten, in compact form, as

$$\text{rk}\begin{bmatrix} \lambda I_n - A & B_d & \Psi_f \\ C & 0 & 0 \end{bmatrix} < n+f+q \Rightarrow |\lambda| < 1.$$

This concludes the proof of the detectability part. The reconstructibility follows the same lines. ■

We assume, now, that $\hat{L}$ has been suitably chosen in order to endow the matrix $\hat{A} - \hat{L}\hat{C}$ (and hence $A - LCA$) with a given spectrum. So, we are remained with the final goal of choosing the parameter W_A in order to minimize the square norm of the estimation error. As is well-known, this is equivalent to the problem of minimizing the trace of the error covariance matrix. The observer gain matrix L is now

$$L = \left(L_0 + \hat{L}\Theta_f\Theta_d\right) + W_A(\bar{C}\Psi_f)^{\#}\Theta_d,$$

where all matrices appearing in the formula are now set, except for W_A. Since all the columns of W_A belong to $\ker(A)$, if Θ_A is a full column rank matrix, such that $\text{Im}(\Theta_A) = \ker(A)$, then

$W_A = \Theta_A Y$, where Y is now a free parameter. So

$$L = \left(L_0 + \hat{L}\Theta_f\Theta_d\right) + \Theta_A Y(\bar{C}\Psi_f)^{\#}\Theta_d. \quad (26)$$

By exploiting the estimation error updating formula (3), the assumptions on $\mathbf{w}$, $\mathbf{v}$, and $\mathbf{x}(0)$, we get for the error covariance matrix

$$P(t) := \mathbf{E}\left\{\left(\mathbf{e}(t) - \mathbf{E}\{\mathbf{e}(t)\}\right)\left(\mathbf{e}(t) - \mathbf{E}\{\mathbf{e}(t)\}\right)^T\right\}$$

the updating equation

$$\begin{aligned} P\,(t+1) &= (A-LCA)P(t)(A-LCA)^T + \\ &+ (I-LC)W(I-LC)^T + LVL^T. \end{aligned} \quad (27)$$

By replacing now (26) in (27), we obtain

$$\begin{aligned} P(t+1) =& P_0(t) - \left[\Theta_A Y(\bar{C}\Psi_f)^{\#} + \hat{L}\Theta_f\right]\Theta_d\Delta(t) \\ &- \left\{\Delta^T(t)\Theta_d^T - \left[\Theta_A Y(\bar{C}\Psi_f)^{\#} + \hat{L}\Theta_f\right]Q(t)\right\} \cdot \\ &\cdot \left[\left((\bar{C}\Psi_f)^{\#}\right)^T Y^T\Theta_A^T + \Theta_f^T\hat{L}^T\right] \end{aligned}$$

where we set

$$\begin{aligned} P_0(t) &:= (I_n - L_0C)(AP(t)A^T + W)(I_n - L_0C)^T \\ &\quad + L_0VL_0^T \\ \Delta(t) &:= C(AP(t)A^T + W)(I_n - L_0C)^T - VL_0^T \\ Q(t) &:= \Theta_d(V + C(AP(t)A^T + W)C^T)\Theta_d^T. \end{aligned}$$

Notice that none of the above matrices depends on Y, and $Q(t)$ is positive definite. By imposing, now, that the derivative of the trace of $P(t+1)$ with respect to Y is zero, and by observing that both $\Theta_A^T\Theta_A$ and $(\bar{C}\Psi_f)^{\#}Q(t)\left((\bar{C}\Psi_f)^{\#}\right)^T$ are positive definite, and hence nonsingular, we get

$$\begin{aligned} Y(t) =& \Theta_A^{\#}\left(\Delta^T(t)\Theta_d^T + \hat{L}\Theta_f Q(t)\right)\left((\bar{C}\Psi_f)^{\#}\right)^T \cdot \\ &\cdot \left((\bar{C}\Psi_f)^{\#}Q(t)\left((\bar{C}\Psi_f)^{\#}\right)^T\right)^{-1} \end{aligned}$$

By replacing the expression of Y just given within (26) we finally obtain

$$\begin{aligned} L(t) =& \left(L_0 + \hat{L}\Theta_f\Theta_d\right) + \Theta_A\Theta_A^{\#}\left(\Delta^T(t)\Theta_d^T + \right. \\ &\left. + \hat{L}\Theta_f Q(t)\right)\left((\bar{C}\Psi_f)^{\#}\right)^T \cdot \\ &\cdot \left((\bar{C}\Psi_f)^{\#}Q(t)\left((\bar{C}\Psi_f)^{\#}\right)^T\right)^{-1}(\bar{C}\Psi_f)^{\#}\Theta_d. \end{aligned}$$

Remark We initially started with an apparently time invariant gain L and ended up with a time varying expression for it. This fact does not affect the results of the previous steps. In detail: G1) even with a time varying $L(t)$, the identity $B_d - L(t)CB_d = 0$ is always fulfilled, i.e., the estimation error is always decoupled from the disturbances; G2) this second goal pertained only the parameter R and hence no discussion is required; G3) even though the equations describing the fault detector and isolator become time-varying, the fault to residual relation is still time invariant. So, it makes sense to consider the corresponding transfer matrix and endow it with a diagonal structure; G4) $A-LCA$ only depends on fixed parameters, and hence it is time invariant.

Example Consider a dynamic system with $A = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$, $B = 0$, $B_d = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $F = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Let the covariance matrices be $V = \sigma I_2$ and $W = \sigma_w I_2$, with $\sigma > 0$ and $\sigma_w \geq 0$. The rank constraints for the solvability of the decoupling problem are satisfied, moreover

$$\text{rk}\left[\begin{array}{cc|cc} \lambda - 1 & 1 & 1 & 0 \\ 0 & \lambda & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] = 4 \quad \forall \lambda \in \mathbb{C}.$$

By Propositions 3.3 and 3.4, then, there exists $\hat{L}$ such that $\hat{A} - \hat{L}\hat{C}$ is nilpotent. After some simple calculations we get $K = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $\Theta_d = [0 \quad 1]$, $(CB_d)^{\#} = [1 \quad 0]$, $\bar{C} = \Theta_d C = \Theta_d$, so $L_0 = I$. So, noting that $\hat{A} = (I - L_0C)A = 0$, we can choose $\hat{L} = 0$. Since

$$\begin{aligned} P_0(t) &= L_0VL_0^T = V = \sigma I_2, \\ \Delta(t) &= -VL_0^T = -\sigma I_2, \\ Q(t) &= [0 \quad 1]\left(V + AP(t)A^T + W\right)[0 \quad 1]^T = \sigma + \sigma_w \end{aligned}$$

the observer gain matrix takes the form:

$$L(t) = \begin{bmatrix} 1 & -\frac{\sigma}{2(\sigma+\sigma_w)} \\ 0 & \frac{\sigma+2\sigma_w}{2(\sigma+\sigma_w)} \end{bmatrix}$$

while the residual gain matrix is

$$R = \left[(\bar{C}\Psi_f)^{\#} + \hat{R}\Theta_f\right]\Theta_d = [0 \quad 1]$$

4. REFERENCES

Commault, C. (1999). On the disturbed fault detection and isolation problem. *System & Control Letters* **38**, 73–78.

Frank, P.M., S.X. Ding and B. Koppen-Seliger (2000*a*). Current developments in the theory of fdi. In: *Proceedings of the SAFEPROCESS 2000 Conference*. Budapest. pp. 16–27.

Frank, P.M., S.X. Ding and T. Marcu (2000*b*). Model-based fault diagnosis in technical processes. *Transactions of the Institute of Measurement and Control* **22,1**, 57–101.

Keller, J. Y. (1999). Fault isolation filter design for linear stochastic systems. *Automatica* **35**, 1701–1706.

Kitanidis, Peter K. (1987). Unbiased minimum-variance linear state estimation. *Automatica* **23, no.6**, 775–778.

Liu, B. and J. Si (1997). Fault isolation filter design for linear time-invariant systems. *IEEE Trans. Aut. Contr.* **AC-42, no. 5**, 704–707.

Parlangeli, G. and M.E. Valcher (2001). Disturbed fault detection and isolation problems for linear state models in a noisy environment. *submitted to Automatica*.