15th Triennial World Congress, Barcelona, Spain

FUZZY ROBUST TRACKING FOR THE CHEN'S CHAOTIC ATTRACTOR [1]

B. Castillo-Toledo, J. Jalomo and E. N. Sánchez

CINVESTAV-IPN, Unidad Guadalajara
A. P. 31-438, Pza. La Luna, Col. Verde Valle,
Guadalajara, Jalisco, México. C. P. 44550
email: toledo@gdl.cinvestav.mx

Abstract: In this paper, the problem of forcing the Chen's chaotic attractor to track a sinusoidal and a chaotic reference signals in the presence of uncertainties in the system's parameter values is addressed by combining the theory of robust regulation and the exact Takagi-Sugeno fuzzy model for the Chen's chaotic attractor. On the basis of designing a robust controller for each linear subsystem, it is shown that the aggregated controller assures robust tracking in the presence of variations on the parameters of each linear subsystems and in the membership functions as well.

Keywords: Robust output regulation, Takagi-Sugeno Fuzzy model, Chen Chaotic System

1. INTRODUCTION

In this work, we consider a dynamical system described by

$$\dot{x} = f(x,u,w,\theta) \tag{1}$$
$$\dot{w} = s(w) \tag{2}$$
$$e = h(x,w,\theta) \tag{3}$$

where $\theta \in R^{v}$ is a parameter vector, $u(t) \in R^{m}$ is the input signal, $x(t) \in R^{n}$ is the state of the system, $w(t) \in R^{q}$ represents the state of an external signal generator (exosystem), described by (2), which provides the reference and/or the perturbation signal. Equation (3) describes the output tracking error $e(t) \in R^{m}$ defined as the difference between the system output and the reference signal. The linear approximation of the system (1)-(3) around a specific point $(\bar{x},\bar{w},\bar{u})$ is given by

$$\dot{x} = A_{\theta}x + B_{\theta}u + P_{\theta}w \tag{4}$$
$$\dot{w} = Sw \tag{5}$$
$$e = C_{\theta}x - R_{\theta}w \tag{6}$$

where

$$A_{\theta} = \frac{\partial f(x,w,u,\theta)}{\partial x}\Big|_{((\bar{x},\bar{w},\bar{u})}; S = \frac{\partial s(w)}{\partial w}\Big|_{(\bar{w})};$$
$$B_{\theta} = \frac{\partial f(x,w,u,\theta)}{\partial u}\Big|_{(\bar{x},\bar{w},\bar{u})}; C_{\theta} = \frac{\partial h(x,w,\theta)}{\partial x}\Big|_{(\bar{x},\bar{w})};$$
$$P_{\theta} = \frac{\partial f(x,w,u,\theta)}{\partial w}\Big|_{(\bar{x},\bar{w},\bar{u})}; R_{\theta} = \frac{\partial h(x,w,\theta)}{\partial w}\Big|_{(\bar{x},\bar{w})}.$$

The subindex θ indicates the explicit dependence of each matrix with respect to the parameter vector. In the following, M_0 will denote the value of matrix M for the nominal values of the parameter vector θ.

For such a system, an interesting problem is that of controlling it to track, at least asymptotically, a desired reference signal, preserving at the same time some suitable stability property of the closed-loop scheme. Among the different approaches studied, the so-called regulator theory has provided a frame to accomplish such objectives. The regulator problem for system (1)-(3) consists in finding a state or error feedback controller such that the equilibrium point of the closed system with no external signals is asymptotically stable, and the tracking error goes to zero when the system is under the influence of the exosystem. This problem has been studied intensively both in the linear case (Francis, 1977), and recently in the nonlinear setting (Isidori and Byrnes, 1990; Huang and Rugh, 1990), by showing that the nonlinear regulator problem is solvable by means of the solution of a

[1] Work supported by Mexican Consejo Nacional de Ciencia y Tecnología under grant 26358-A

partial differential equations, named Francis-Isidori-Byrnes (FIB) equations. On the other hand, for nonlinear systems, it has been shown that the inclusion of an internal model in the controller structure was also necessary and sufficient for having robust regulation, i.e., the capability of the controller for maintaining the output tracking error within certain predefined bounds while ensuring the stability of the closed-loop system, despite the presence of parameter perturbations (Hepburn and Wonham, 1984). Following these ideas, in Isidori (1995), Dellipriscoli *et al* (1995) and Byrnes *et al* (1997), an error feedback controller which relies on the existence of an internal model is presented. This internal model represents an inclusion of the exosystem dynamics into an observable one, which allows to generate, as in the linear case, all the possible steady state inputs for the admissible values of the system parameters. A remarkable feature is that the controller is constructed on the basis of the linear approximation of the nonlinear system and, in the case when the immersion is linear, the controller becomes fully linear. However, since the solvability of this robust solution relies on the existence of a solution of both the FIB equations and the existence of an internal model, for which no solution is guaranteed a priori, then for many complex physical systems, this may become a drawback. Another additional problem is that a rigorous mathematical model may not be available, but only some local mathematical behavior could be obtained. For this situation, the Tagaki-Sugeno **(TS)** formulation provides a fuzzy model which could describe the dynamics of complex systems under appropriate selection of linear subsystems for each predefined condition of the dynamics of the system. Using these ideas, many works have appeared in the literature for dealing with the control of nonlinear systems, since linear feedback control techniques can be utilized to stabilize the nonlinear dynamics. In this case, the stabilization properties are based on the existence of a common Lyapunov function for each linear subsystem (T. Taniguchi and Wang, 1999; Wang, 1997).

To precise these ideas, suppose that it is possible to describe locally the input-output behavior of system (1)-(3) by a **TS** fuzzy dynamic model described by the following r rules:

$$
\begin{array}{l}
Plant\,rule\,i: \\
IF \\
z_1(t)\,is\,F_{1i}\,and.....and\,z_p(t)\,is\,F_{pi} \\
THEN \\
\sum_i : \left\{ \begin{array}{l} \dot{x}(t) = A_{\theta i}x(t) + B_{\theta i}u(t) + P_{\theta i}w(t) \\ \dot{w}(t) = S_i w(t) \\ e_i(t) = C_{\theta i}x(t) - R_{\theta i}w(t), i = 1,...,r \end{array} \right.
\end{array}
$$

where $z_1(t),...,z_p(t)$ are measurable premise variables, F_{ji} are the corresponding fuzzy sets and the linear subsystems are obtained from some knowledge of the dynamics on the process.

For a given triplet $(x(t),u(t),w(t))$, the composite fuzzy model is obtained by using a singleton fuzzifier, product inference and center of gravity defuzzifier, and is then given by

$$
\begin{aligned}
\dot{x}(t) = \sum_{i=1}^{r}\mu_i A_{\theta i}x(t) + \sum_{i=1}^{r}\mu_i B_{\theta i}u(t) \\
+ \sum_{i=1}^{r}\mu_i P_{\theta i}w(t) \qquad (7)
\end{aligned}
$$

$$\dot{w}(t) = \sum_{i=1}^{r}\mu_i S_i w(t) \qquad (8)$$

$$e(t) = \sum_{i=1}^{r}\mu_i \left[C_{\theta i}x(t) - R_{\theta i}w(t)\right] \qquad (9)$$

where μ_i is the normalized weight for each rule calculated from the membership functions for z_j in F_{ji} and satisfying $\mu_i = \mu_i[z(t)] \geq 0$ and $\sum_{i=1}^{r}\mu_i[z(t)] = 1$, $z(t) = \left[z_1(t),...,z_p(t)\right]^T$.

For this system, we introduce the *Fuzzy Robust Regulator Problem* **(FRRP)** which consists on finding a set of triplets $(K_i, G_{i1}, G_{i2}), i = 1,..,r$ such that, for all admissible parameter values in a suitable neighborhood $\mathscr{P}$ of the nominal ones, the following conditions hold:

FRS) The equilibrium point $(x,\zeta) = (0,0)$ of the system

$$
\begin{aligned}
\dot{x} &= \sum_{i=1}^{r}\mu_i A_{\theta i}x(t) + \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j B_{\theta i}\tilde{H}_j\zeta(t) \\
\dot{\zeta} &= \sum_{i=1}^{r}\sum_{j=1}^{r}\mu_i\mu_j G_i C_{\theta j}x(t) + \sum_{i=1}^{r}\mu_i F_i\zeta(t)
\end{aligned}
$$

is asymptotically stable.

FRR) The solution of the closed-loop system satisfies that

$$\lim_{t\to\infty} e(t) = 0.$$

In Xiao (2000), a combination of the regulation theory and the Tagaki-Sugeno fuzzy modelisation for designing a controller was presented for both state and error feedback. Along the same lines, in Castillo-Toledo (2001), the use of the robust regulation theory for solving the FRRP is proposed. The basic idea is to take the rules for the dynamic regulator as

$$
\begin{array}{l}
Controller\,rule\,i: \\
IF \\
z_1(t)\,is\,F_{1i}\,and.....and\,z_p(t)\,is\,F_{pi} \\
THEN \\
\dot{\zeta}(t) = F_i\zeta(t) + G_i e \\
u(t) = \tilde{H}_i\zeta(t),
\end{array}
$$

so that he final controller is given by

$$\dot{\zeta} = \sum_{i=1}^{r}\mu_i F_i\zeta + \sum_{i=1}^{r}\mu_i G_i e \qquad (10)$$

$$u = \sum_{i=1}^{r}\mu_i\tilde{H}_i\zeta. \qquad (11)$$

The following result gives conditions for the solution of the FRRP.

Theorem 1. Assume the following conditions hold:

FH1) The pairs (A_{0i}, B_{0i}) are stabilizable

FH2) The pairs $\begin{pmatrix} A_{0i} & -B_{0i}H_i \\ 0 & \Phi_i \end{pmatrix}, \begin{pmatrix} C_{0i} & 0 \end{pmatrix}$ with $H_i = diag(H_j)$ and $H_j = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix}$; are detectable

FH3) For all $\theta \in \mathscr{P}$,

$$\text{rank} \begin{pmatrix} A_{\theta i} - \lambda I & B_{\theta i} \\ C_{\theta i} & 0 \end{pmatrix} = n + p$$

for each λ which is an eigenvalue of S.

FH4) There exist triplets (K_i, G_{1i}, G_{2i}) and a matrix P such that

$$M_{ii}^T P + P M_{ii} < 0$$

for $i = 1, \ldots, r$ and

$$\left(\frac{M_{ij} + M_{ji}}{2}\right)^T P + P \left(\frac{M_{ij} + M_{ji}}{2}\right) < 0$$

for $i < j \le r$, where $M_{ij} = \begin{pmatrix} A_{0i} & B_{0i}\tilde{H}_j \\ G_i C_{0j} & F_i \end{pmatrix}$,

then the **FRRP** is solvable.

Proof: We note first that thanks to FH3, there exist mappings $x_{ss} = \Pi_\theta w$ and $u_{ss} = \Gamma_\theta w$ satisfying the equations

$$\begin{aligned} \Pi_\theta S_i &= A_{\theta i}\Pi_\theta + B_{\theta i}\Gamma_\theta + P_\theta \\ 0 &= C_{\theta i}\Pi_\theta - R_{\theta i} \end{aligned}$$

and this, together with assumptions FH1 and FH2 guarantees that, for each subsystem, the robust regulation problem is solvable, namely, there exists a controller (10) with

$$F_i = \begin{pmatrix} A_{0i} + B_{0i}K_i - G_{i1}C_{0_i} & 0 \\ -G_{i2}C_{0i} & \Phi_i \end{pmatrix}; G_i = \begin{pmatrix} G_{i1} \\ G_{i2} \end{pmatrix}$$
$$\tilde{H}_i = \begin{pmatrix} K_i, & H_i \end{pmatrix}.$$

such that

$$\lim_{t\to\infty} e_i(t) = 0.$$

Now, for the overall stability part when $w = 0$, the closed-loop of (10–11) with the system (7–9) at the nominal values of the parameter vector can be written as

$$\dot{x}(t) = \sum_{i=1}^r \mu_i A_{0i} x(t) + \sum_{i=1}^r \mu_i B_{0i} \left(\sum_{j=1}^r \mu_j \tilde{H}_j \zeta\right)$$
$$\dot{\zeta}(t) = \sum_{i=1}^r \mu_i F_i \zeta(t) + \sum_{i=1}^r \mu_i G_i \left(\sum_{j=1}^r \mu_j C_{\theta j} x(t)\right)$$

which, defining $x_e(t) = \begin{pmatrix} x(t) & \zeta(t) \end{pmatrix}^T$ may be rewritten as

$$\begin{aligned} \dot{x}_e = & \sum_{i=1}^r \mu_i^2 \begin{pmatrix} A_{0i} & B_{0i}\tilde{H}_i \\ G_i C_{0i} & F_i \end{pmatrix} \begin{pmatrix} x(t) \\ \zeta(t) \end{pmatrix} + \\ & \sum_{i=1}^{r-1} \sum_{j=i+1}^r \mu_i \mu_j \left\{ \begin{pmatrix} A_{0i} & B_{0i}\tilde{H}_j \\ G_i C_{0j} & F_i \end{pmatrix} + \right. \\ & \left. + \begin{pmatrix} A_{0j} & B_{0j}\tilde{H}_i \\ G_j C_{0i} & F_j \end{pmatrix} \right\} x_e \\ = & \left\{ \sum_{i=1}^r \mu_i^2 M_{ii} + 2 \sum_{i=1}^{r-1} \sum_{i=i+1}^r \mu_i \mu_j \left[\frac{M_{ij} + M_{ji}}{2}\right] \right\} x_e. \end{aligned}$$

where M_{ij} defined as above. Now, taking the Lyapunov function

$$V(x, \zeta) = x_e^T P x_e$$

we have that

$$\begin{aligned} \dot{V} = & \dot{x}_e^T P x_e + x_e^T P \dot{x}_e \\ = & x_e^T \left\{ \sum_{i=1}^r \mu_i^2 M_{ii} + 2 \sum_{i=1}^{r-1} \sum_{i=i+1}^r \mu_i \mu_j \left[\frac{M_{ij} + M_{ji}}{2}\right] \right\}^T P \\ & + x_e^T P \left\{ \sum_{i=1}^r \mu_i^2 M_{ii} + 2 \sum_{i=1}^{r-1} \sum_{i=i+1}^r \mu_i \mu_j \left[\frac{M_{ij} + M_{ji}}{2}\right] \right\} x_e \end{aligned}$$

From this, we obtain

$$\begin{aligned} \dot{V} = & x_e^T \left\{ \sum_{i=1}^r \mu_i^2 (M_{ii}^T P + P M_{ii}) \right\} x_e + \\ & 2 \sum_{i=1}^{r-1} \sum_{i=i+1}^r \mu_i \mu_j x_e^T \left\{ \left(\frac{M_{ij} + M_{ji}}{2}\right)^T P \right. \\ & \left. + P \left(\frac{M_{ij} + M_{ji}}{2}\right) \right\} x_e \end{aligned}$$

Now, if assumption FH4 holds, then $\dot{V}(x, \zeta)$ is definite negative, and therefore $x(t)$ and $\zeta(t)$ converge globally asymptotically to zero. Since the property of stability in the first approximation is not destroyed by small parameter variations, the controller in question stabilizes any plant so long the parameter vector θ stays on some open neighborhood $\mathscr{P}$.

For the regulation part, this follows immediately from the fact that each controller is robust for each subsystem, and therefore guarantees zero tracking error, i.e. $\lim_{t\to\infty} e_i(t) = 0$, and then the total error is

$$\begin{aligned} \lim_{t\to\infty} e(t) = & \lim_{t\to\infty} \sum_{i=1}^r \mu_i \left[e_i(t)\right] \\ = & \sum_{i=1}^r \mu_i \left[\lim_{t\to\infty} e_i(t)\right] = 0. \end{aligned}$$

2. APPLICATION TO THE CHEN'S CHAOTIC ATTRACTOR

2.1 *Periodic reference tracking*

The approach presented so far is used in this work to pursue the trajectory control for the well-known Chen chaotic attractor described by

$$\dot{x}_1 = a(x_1 - x_2)$$
$$\dot{x}_2 = (c-a)x_1 - x_1x_3 + cx_2 + u$$
$$\dot{x}_3 = x_1x_2 - bx_3$$

This is a nonlinear system exhibiting chaotic behavior for some specific values of the parameters a, b, c. A remarkable feature of this chaotic system is that it may be exactly described by a TS Fuzzy Model (Chen and Ueta, 1999), so the use of the TS fuzzy model is a valid alternative for synthesize a controller.

Under this premise, we use the fuzzy regulator approach to derive a nonlinear control law which ensures robust output regulation when applied to the original system. For simulations purposes, we take the values of $a = 35, b = 3$ and $c = 28$, and perform all calculations needed for the controller. We verify that the conditions **FH5** are satisfied for the system and by using Using the Matlab LMI Toolbox, a matrix P was found.

The simulations results of the controller applied to the fuzzy system are shown in Figures 1a) trough 1d). Figure 1a) shows the output tracking error response for variations up to 25% in the values of a and b, and -75% in the value of c.

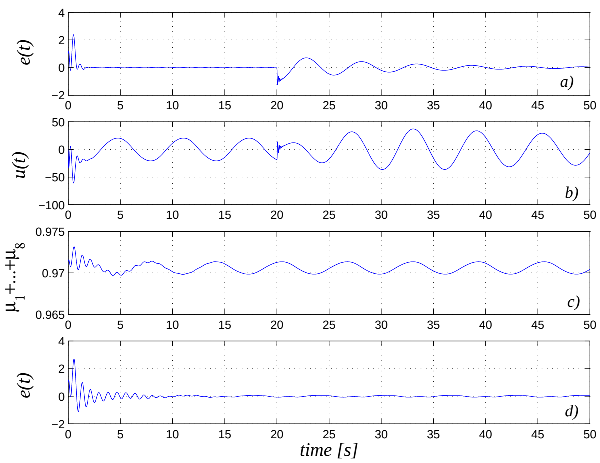

Fig. 1. Output tracking error

As we may observe, the controller is able to take the output tracking error asymptotically to zero despite the variations in the parameters. We have also introduced some variations in the membership functions in order to violate the condition $\sum_{i=1}^{8} \mu_i(t) = 1$ as shown in Figure 1c). We may observe that the output tracking error in Figure 1d) goes asymptotically to zero even under this circumstances, showing the robustness of the controller to these perturbations as well.

2.2 *A Chaotic reference tracking*

In this case we take as the reference generator, the Lorenz chaotic system, whose behavior is described by

$$\dot{w}_1 = \alpha(w_2 - w_1)$$
$$\dot{w}_2 = \gamma w_1 - w_2 - w_1w_3$$
$$\dot{w}_3 = w_1w_2 - \beta w_3.$$

With $\alpha = 10$, $\beta = 8/3$, $\gamma = 28$, this system produces a chaotic dynamics. As in the case of the Chen's attractor, it is possible to show that this system may be exactly described by a Takagi-Sugeno fuzzy model. We then apply the technique mentioned so far to obtain a controller that allows to track the reference signals given by w_1, w_2, and w_3. It is interesting to note that the linearization of the system at the equilibrium point $(0,0,0)$ exhibits an unstable eigenvalue and therefore the classical nonlinear regulation theory can no longer be applied. However, since the calculations for the Takagi-Sugeno scheme needs only the linear submodels, then it is still possible to design a Fuzzy Robust Regulator.

In this case, we consider the Chen's attractor with three inputs, namely

$$\dot{x}_1 = a(x_1 - x_2) + u_1$$
$$\dot{x}_2 = (c-a)x_1 - x_1x_3 + cx_2 + u_2$$
$$\dot{x}_3 = x_1x_2 - bx_3 + u_3.$$

The simulation results are presented in Figure 2. We have introduced on $t = 20s$ variations up to 25% in a and b and -75% in c. Figure 2 shows both the reference and output signals. We may observe that the system track the chaotic signals with small errors. Finally, as in the previous case, in Figure 3 the output tracking errors when variations on the membership functions are introduced.

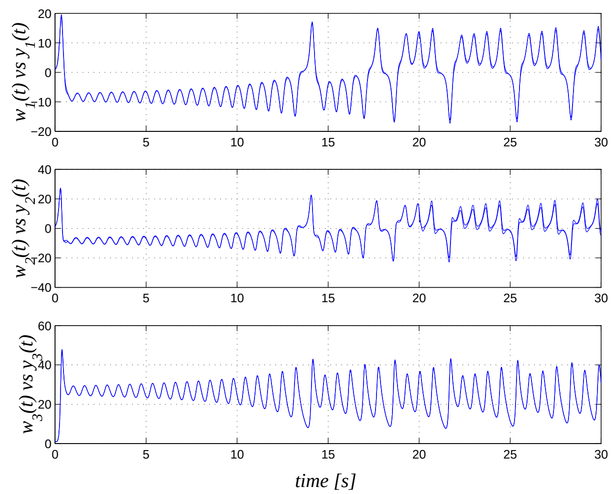

Fig. 2. Output versus reference signals

3. CONCLUSIONS

In this paper we have presented a robust regulation scheme for the Chen's Chaotic Attractor, based on a combination of the theory of robust regulation for linear systems and the Tagaki-Sugeno fuzzy modelisation for nonlinear systems. Taking advantage on the fact that the Chen's model may be exactly described by a TS fuzzy model, we design stabilizing gains for each linear systems associated with a rule in the fuzzy

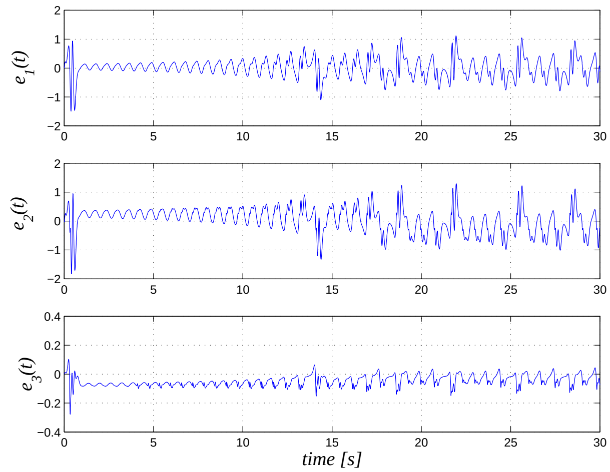

Fig. 3. Output tracking error

model, which guarantees the stability property of the overall model. This allows the design of a regulator for each subsystem which is robust with respect to variations in the parameters associated to each linear subsystem and also in the presence of parameter variations in the membership functions. This scheme allows to ensure asymptotic zero output tracking error for the TS fuzzy model of the Chen's attractor and for the nonlinear system as well.

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