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OBSERVER-BASED FAULT DETECTION SCHEMES FOR LINEAR UNCERTAIN SYSTEMS

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Abstract: In this paper, fault detection problems for linear uncertain systems are studied. Instead of designing fault detection systems from the viewpoint of increasing the system robustness against unknown inputs and the sensitivity to the faults, an approach is proposed, which allows us to design fault detection systems in such a way that the missed detection rate is minimized for a given false alarm rate.

Keywords: Fault detection; Observers; Uncertainty; Robustness; Optimization

1. INTRODUCTION

In this paper, problems related to the design of observer based fault detection (FD) systems for linear dynamic systems with model uncertainties are studied. The plant model under consideration is described by

$$\dot{x} = \bar{A}x + \bar{B}u + \bar{E}_f f + \bar{E}_d d \quad (1)$$

$$y = Cx + Du + F_d d + F_f f \quad (2)$$

where $x \in R^n$, $u \in R^{k_u}$ and $y \in R^m$ denote the state, input and output vectors of the plant, $f \in R^{k_f}, d \in R^{k_d}$ the fault and unknown input vectors respectively. Without loss of generality, we assume d is L_2-norm bounded, $\|d\|_2 \leq \delta_d$, u is an L_2-signal and $\bar{A} = A + \Delta A, \bar{B} = B + \Delta B, \bar{E}_f = E_f + \Delta E_f, \bar{E}_d = E_d + \Delta E_d$. $A, B, C, D, E_f, E_d, F_d, F_f$ are known system matrices with appropriate dimensions and $\Delta A, \Delta B, \Delta E_f, \Delta E_d$ represent model uncertainties satisfying

$$\left[\, \Delta A \;\; \Delta B \;\; \Delta E_f \;\; \Delta E_d \,\right] = E\Sigma \left[\, F_A \;\; F_B \;\; F_{E_f} \;\; F_{E_d} \,\right]$$
$$\Sigma(t)^T \Sigma(t) \leq \delta I, \delta > 0$$

We will use below the notation $G_u(s) = C(sI - A)^{-1}B + D$ and drop, due to the space limitation, time variable t and complex variable s of Laplace-transformation if it does not cause confusion.

A typical FD system consists of a residual generator and a residual evaluation stage including an evaluation function and a threshold (Gertler, 1998; Chen and Patton, 1999; Frank and Ding, 1997). For the purpose of residual generation, observer-based fault detection systems of the following form

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - \hat{y}), \hat{y} = C\hat{x} + Du \quad (3)$$

$$r(s) = R(s)(y(s) - \hat{y}(s)) \quad (4)$$

are considered, where r is the residual vector and the design parameters are the observer gain L and post-filter $R(s) \in RH_\infty$. It is well-known that (3)-(4) can also be expressed in terms of

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$$r(s) = R(s)\left(\hat{M}_u(s)y(s) - \hat{N}_u(s)u(s)\right) \qquad (5)$$

where $(\hat{M}_u(s), \hat{N}_u(s))$ is a left coprime factorization of $G_u(s)$, and the design of a residual generator is equivalent to the selection of a post-filter (Frank and Ding, 1994).

In this contribution, the H_2-norm of residual vector $r(s)$ is used as the residual evaluation function which evaluates the energy change in r. It is worth to remark that in practice a modified form of the H_2-norm is used for the purpose of residual evaluation. The length of the evaluation window, both in the time and frequency domains, is limited instead of infinitive.

The last step to a successful fault detection is the establishment of a logic decision unit. In this contribution, we consider a simple but mostly used logic:

$$\text{If } \|r\|_2 > J_{th} \text{ (threshold)} \Longrightarrow \text{alarm} \qquad (6)$$
$$\text{If } \|r\|_2 \leq J_{th} \text{ (threshold)} \Longrightarrow \text{no fault} \qquad (7)$$

A widely accepted way to deal with model based FDI problems is to solve them in the context of robust control theory (Gertler, 1998; Chen and Patton, 1999; Frank and Ding, 1997). So are the concepts robustness and sensitivity, perhaps the most important topics in the field of model based FDI. What is the real idea behind these two concepts? They are indeed the "translation" of two essential requirements on a fault diagnosis system: *false alarm rate and missed detection rate.* False alarms are caused by unknown input vector and model uncertainties. In order to reduce them, thresholds are introduced, which lead in turn to missed detection. In fact, the most difficult task of designing a fault detection system is to find out a suitable trade-off between the false alarm rate and the missed detection rate.

It is evident that setting J_{th} according to $J_{th} = \sup_{\Delta A, \Delta B, \Delta E_d, d, f=0} \|r\|_2$ prevents false alarms. This way of handling FD design problem seems elegant but has two practical problems:

- the performance that the false alarm rate equals zero is achieved at the cost of missed detection rate. This problem may become more serious if $\|d\|_2$ and the model uncertainties rarely reach their maximum;
- how to design the FD system such that for a given false alarm rate the missed detection rate is minimized.

The main objective of this paper is to develop a new approach to the design of fault detection systems from the viewpoint of achieving a suitable trade-off between the false alarm rate and missed detection rate. Motivated by the above discussion, we shall try to solve the following two problems

- Establishment of a trade-off relationship between the false alarm rate and the missed detection rate;
- Direct design of FD systems to minimize the missed detection rate for a given false alarm rate.

2. OUTLINE OF BASIC IDEAS AND PROBLEM FORMULATION

In this section, we are going to outline the basic ideas of our study, major solution steps and problems to be solved for designing an FD system satisfying the requirements mentioned above.

2.1 System dynamics

For our purpose, the system dynamics will first be studied. Depending on the control law used, the dynamics of the overall system (plant + residual generator) can be written in two different ways.

Case I: $u(t)$ is realized independent of the observer used. Since u is known, introducing $e = x - \hat{x}$ yields, after a straightforward calculation,

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} \bar{A} & O \\ \Delta A & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} \bar{B} \\ \Delta B \end{bmatrix} u + \begin{bmatrix} \bar{E}_d \\ \bar{E}_d - LF_d \end{bmatrix} d + \begin{bmatrix} \bar{E}_f \\ \bar{E}_f - LF_f \end{bmatrix} f$$
$$r = R(s)\left(Ce + F_d d + F_f f\right) \qquad (8)$$

Case II: $u(t)$ is based on the observer used, for instance: $u(t) = -K\hat{x} + w_{ref}$. Then, we have

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} \bar{A} - \bar{B}K & -\bar{B}K \\ \Delta A + \Delta BK & A - LC - \Delta BK \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix} + \begin{bmatrix} \bar{B} \\ \Delta B \end{bmatrix} w_{ref} + \begin{bmatrix} \bar{E}_d \\ \bar{E}_d - LF_d \end{bmatrix} d + \begin{bmatrix} \bar{E}_f \\ \bar{E}_f - LF_f \end{bmatrix} f$$
$$r = R(s)\left(Ce + F_d d + F_f f\right) \qquad (9)$$

Note that in Case I, since the observer gain L has no influence on the system dynamics (1)-(2), the overall system is stable iff the plant is stable and (A, C) is detectable. The task of the fault detection system design can then be formulated as finding L and $R(s) \in RH_\infty$ such that the residual generator (observer) (3) is stable and r satisfies the desired system performance. Different from it, in Case II, it is evident that an integrated design of the controller and residual generator (observer) is needed, since both K and L have influences on the whole system dynamics (Murad and Gu, 1996). Thus, the design tasks consist of a) stabilization of the overall system (9); b) satisfying desired control performance; c) satisfying of FD performance by selecting K, L and $R(s)$. Due to the paper space limitation and also for the sake of simplicity, in this contribution we'll focus our attention to Case I. A brief remark on Case II will be given in the concluding remarks.

Considering the fact that in Case I for any $R(s), L$ we are able to find a $R^*(s)$ such that for a given L_o the relation $r(R^*, L_o) = r(R, L)$ holds, we shall only consider the design of a post filter $R(s)$ for a given (suitable) L in the following of this paper. In the above relation, $r(R, L)$ denotes the residual signal generated by the residual generator with observer gain L and post-filter $R(s)$.

For the sake of simplicity, we write (8) in the following form

$$r = G_{ru}u + G_{rd}d + G_{rf}f \tag{10}$$

with G_{ru}, G_{rd} and G_{rf} denoting the transfer function matrices from u, d and f to r.

2.2 Establishment of an adaptive threshold

Remember the objective of our study, we first introduce δ_d, δ_Δ :

$$\delta_d \leq \sup_d \|d\|_2 = \Delta_d, \delta_\Delta \leq \delta$$

and define the threshold as follows

$$J_{th} = \sup_{\Sigma^T\Sigma \leq \delta_\Delta I, \|d\|_2 \leq \delta_d, f=0} \|r\|_2 \tag{11}$$

Since for any d, $\|d\|_2 > \delta_d$ or $\Delta A, \Delta B, \Delta E_d$ which lead to $\Sigma^T\Sigma > \delta_\Delta I$, we may have $\|r\|_2 > J_{th}$ for $f = 0$, i.e. it leads to a false alarm. Thus, the false alarm rate may be non-zero and its size depends on the frequency that $\|d\|_2 > \delta_d$ or $\Sigma^T\Sigma > \delta_\Delta I$. For our purpose on the one side and to simplify the problem formulation on the other side, we introduce the following index

$$\begin{aligned} I_{FAR} &= 1 - \frac{\sup_{\Sigma^T\Sigma \leq \delta_\Delta I, \|d\|_2 \leq \delta_d, f=0} \|r\|_2}{\sup_{\Sigma^T\Sigma \leq \delta I, \|d\|_2 \leq \triangle_d, f=0} \|r\|_2} \\ &= 1 - \frac{J_{th}}{J_{th,\max}} \end{aligned} \tag{12}$$

with $J_{th,\max}$ denoting the maximal possible threshold which covers all possible d and model uncertainties. It is clear that I_{FAR} measures the frequency of false alarms. If the threshold J_{th} is set to be

$$J_{th} = J_{th,\max} = \sup_{\Sigma^T\Sigma \leq \delta I, \|d\|_2 \leq \triangle_d, f=0} \|r\|_2$$

then $I_{FAR} = 0$, which means, as expected, no false alarm. On the other side, if J_{th} is set to be zero, then $I_{FAR} = 1$, which means that there exists definitively at least a false alarm. It is also evident that the smaller δ_d or δ_Δ are defined, the larger I_{FAR} may become, which leads to a higher false alarm frequency.

It follows from (10) that, due to the existence of model uncertainties, the threshold to be established should be a function of d, u and model uncertainties. We denote it with

$$J_{th} = \gamma^d_{\delta_\Delta}\delta_d + \gamma^u_{\delta_\Delta}\|u\|_2 \tag{13}$$

where both $\gamma^d_{\delta_\Delta}$ and $\gamma^u_{\delta_\Delta}$ are some constants depending on δ_Δ. Note that the threshold consists of two terms and the term $\gamma^u_{\delta_\Delta}\|u\|_2$ depends on the input signals and can be on-line calculated. Such kind of threshold is called adaptive threshold.

As shown in (13), the main task of establishing a threshold is to determine $\gamma^d_{\delta_\Delta}$ and $\gamma^u_{\delta_\Delta}$.

2.3 Design of residual generator

Recall that following detection logic (6)-(7) a fault f can be detected if and only if

$$\|r\|_2 = \|G_{ru}u + G_{rd}d + G_{rf}f\|_2 > J_{th} \tag{14}$$

As a result, we claim that for some $d, \Delta A, \Delta B, \Delta E_d$, ΔE_f a fault f can be detected if and only if

$$\|G_{ru}u + G_{rd}d + G_{rf}f\|_2 > \gamma^d_{\delta_\Delta}\delta_d + \gamma^u_{\delta_\Delta}\|u\|_2 \tag{15}$$

We now introduce two sets

- the set of detectable faults $\Omega_{R,J_{th}}(d, \Delta)$

$$\begin{aligned} &\Omega_{R,J_{th}}(d, \Delta) \\ &= \{f \mid f \neq 0, \|G_{ru}u + G_{rd}d + G_{rf}f\|_2 > J_{th}\} \end{aligned} \tag{16}$$

- the set of undetectable faults $\Omega^\perp_{R,J_{th}}(d, \Delta)$

$$\begin{aligned} &\Omega^\perp_{R,J_{th}}(d, \Delta) \\ &= \{f \mid f \neq 0, \|G_{ru}u + G_{rd}d + G_{rf}f\|_2 \leq J_{th}\} \end{aligned} \tag{17}$$

Since the missed detection rate is proportional to the number of undetectable faults, minimizing the missed detection rate under a given false alarm rate is equivalent to minimizing the dimension of set $\Omega^\perp_{R,J_{th}}$. Note that

$$\Omega = \{f \mid f \neq 0\} = \Omega^\perp_{R,J_{th}}(d, \Delta) \cup \Omega_{R,J_{th}}(d, \Delta)$$

we further have

$$\begin{aligned} &\min_{R(s)\in\mathbf{RH}_\infty} \dim \Omega^\perp_{R,J_{th}}(d, \Delta) \\ \Leftrightarrow &\max_{R(s)\in\mathbf{RH}_\infty} \dim \Omega_{R,J_{th}}(d, \Delta) \end{aligned}$$

Following this, we formulate the problem of designing fault detection systems as finding $R(s) \in \mathbf{RH}_\infty$ such that for all d, $\Delta A, \Delta B, \Delta E_d, \Delta E_f$ the dimension of the set of detectable faults reaches maximum, i.e.

$$\max_{R(s)\in\mathbf{RH}_\infty} \dim \Omega_{R,J_{th}}(d, \Delta) \tag{18}$$

for all $d, \Delta A, \Delta B, \Delta E_d, \Delta E_f$. Solving optimization problem (18) is another main task of this contribution.

3. PROBLEM SOLUTIONS

In this section, solutions for the above-defined two problems will be derived.

3.1 Study of system dynamics

For our purpose, we first study the influence of d and model uncertainties on the system dynamics. It follows from (8) that $G_{rd}d + G_{ru}u$ can be expressed as follows:

$$G_{rd}d + G_{ru}u = R(s)\left[C(sI - A + LC)^{-1}\varphi + F_d d\right]$$
$$\varphi = \Delta A x_d + (\bar{E}_d - LF_d)d + \Delta B u$$
$$x_d = (sI - \bar{A})^{-1}(\bar{E}_d d + \bar{B}u))$$

Note that

$$\Delta A x_d + \Delta E_d d + \Delta B u$$
$$= E\Sigma \begin{bmatrix} F_A & F_B & F_{E_d} \end{bmatrix} \begin{bmatrix} x_d \\ u \\ d \end{bmatrix} = E\Sigma\varphi_d$$

$$\varphi_d = \left(F_A(sI - \bar{A})^{-1}\tilde{B} + \tilde{D}\right)\begin{bmatrix} u \\ d \end{bmatrix} \quad (19)$$
$$\tilde{B} = \begin{bmatrix} \bar{B} & \bar{E}_d \end{bmatrix}, \tilde{D} = \begin{bmatrix} F_B & F_{E_d} \end{bmatrix} \quad (20)$$

We can re-write $G_{rd}d$ $+G_{ru}u$ as

$$G_{rd}d + G_{ru}u =$$
$$R(s)\left[C(sI - A + LC)^{-1}(\bar{E}_{\bar{d}} - L\bar{F}_{\bar{d}}) + \bar{F}_{\bar{d}}\right]\bar{d}$$
$$\bar{E}_{\bar{d}} = \begin{bmatrix} E & E_d \end{bmatrix}, \bar{F}_{\bar{d}} = \begin{bmatrix} O & F_d \end{bmatrix}, \bar{d} = \begin{bmatrix} \Sigma\varphi_d \\ d \end{bmatrix} \quad (21)$$

Denote

$$\bar{G}_{r\bar{d}} = C(sI - A + LC)^{-1}\bar{E}_{\bar{d}} + \bar{F}_{\bar{d}} \quad (22)$$

and note that $G_{rf}f$ can also be, without loss of generality, expressed by

$$G_{rf}f = R(s)\bar{G}_{rf}f$$

we finally have

$$r = R(s)\left(\bar{G}_{r\bar{d}}\bar{d} + \bar{G}_{rf}f\right) \quad (23)$$

3.2 Calculation of the adaptive threshold

Following (11) and (23), we have

$$J_{th} = \sup_{\Sigma^T\Sigma \le \delta_\Delta I, \|d\|_2 \le \delta_d} \left\|R\bar{G}_{r\bar{d}}\bar{d}\right\|_2$$
$$= \left\|R\bar{G}_{r\bar{d}}\right\|_\infty \sup_{\Sigma^T\Sigma \le \delta_\Delta I, \|d\|_2 \le \delta_d} \left\|\bar{d}_2\right\|_2$$

We now consider $\left\|\bar{d}_2\right\|_2$ for $\Sigma^T\Sigma \le \delta_\Delta I, \|d\|_2 \le \delta_d$. Since

$$\sup_{\Sigma^T\Sigma \le \delta_\Delta I, \|d\|_2 \le \delta_d} \left\|\bar{d}_2\right\|$$
$$= \sup_{\Sigma^T\Sigma \le \delta_\Delta I, \|d\|_2 \le \delta_d} \|\varphi_d\|_2 \delta_\Delta + \delta_d$$

the key problem becomes the calculation of $\sup_{\Sigma^T\Sigma \le \delta_\Delta I, \|d\|_2 \le \delta_d} \|\varphi_d\|_2$. To this end, we have the following lemma known from the LMI technique (Boyd and Feron, 1994).

Lemma 1. Given a uncertain LTI system

$$\dot{x} = (A + \Delta A)x + (B + \Delta B)w$$
$$z = Cx + Dw, x(0) = 0$$
$$\Delta A = E_1\Sigma F_1, \Delta B = E_2\Sigma F_2, \Sigma^T\Sigma \le I$$

and $\gamma > 0$, if there exist $\varepsilon_1 > 0, \varepsilon_2 > 0$ and a positive-definite matrix P such that the following LMI

$$\begin{bmatrix} Q & PB & C^T & PE_1 & PE_2 \\ B^T P & -\gamma^2 I + \varepsilon_2 F_2^T F_2 & D^T & 0 & 0 \\ C & D & -I & 0 & 0 \\ E_1^T P & 0 & 0 & -\varepsilon_1 I & 0 \\ E_2^T P & 0 & 0 & 0 & -\varepsilon_2 I \end{bmatrix} < 0$$
$$Q = PA + A^T P + \varepsilon_1 F_1^T F_1$$

holds for all the model uncertainties ΔA and ΔB, then the system is asymptotically stable and the H_∞-norm of transfer function G_{zw} satisfies

$$\|G_{zw}\|_\infty < \gamma$$

Note the definition of φ_d given by (19), it becomes evident that

$$\sup_{\Sigma^T\Sigma \le \delta_\Delta I, \|d\|_2 \le \delta_d} \|\varphi_d\|_2 \le \left\|G_{\varphi_d\bar{d}}\right\|_\infty (\delta_d + \|u\|_2)$$
$$G_{\varphi_d\bar{d}} = F_A(sI - A - \Delta A)^{-1}\tilde{B} + \tilde{D}$$

Now, interatively using Lemma 1 for searching for the inf . of γ_1 satisfying (24),

$$\left\|F_A(sI - A - \Delta A)^{-1}\tilde{B} + \tilde{D}\right\|_\infty < \gamma_1 \quad (24)$$

gives $\sup_{\Sigma^T\Sigma \le \delta_\Delta I, \|d\|_2 \le \delta_d} \|\varphi_d\|_2 \le \gamma_1(\delta_d + \|u\|_2)$. Finally, we can calculate the threshold as follows:

$$\gamma_{\delta_\Delta}^d = \left\|R\bar{G}_{r\bar{d}}\right\|_\infty (1 + \gamma_1\delta_\Delta), \gamma_{\delta_\Delta}^u = \left\|R\bar{G}_{r\bar{d}}\right\|_\infty \gamma_1\delta_\Delta$$
$$J_{th} = \gamma_{\delta_\Delta}^d \delta_d + \gamma_{\delta_\Delta}^u \|u\|_2 = \left\|R\bar{G}_{r\bar{d}}\right\|_\infty \alpha$$
$$\alpha = (1 + \gamma_1\delta_\Delta)\delta_d + \gamma_1\delta_\Delta \|u\|_2$$

Note that α is a variable which only depends on the system parameters and is independent of the residual generator design.

3.3 Design of the residual generator

In this sub-section, we outline the basic idea and present an approach to the solution of optimization problem (18). We begin with the so-called co-inner-outer factorization (CIOF) of transfer function matrix $\bar{G}_{r\bar{d}}, \bar{G}_{r\bar{d}} = G_{do}(s)G_{di}(s)$, where $G_{di}(s)$ is the co-inner matrix of $\bar{G}_{r\bar{d}}$ satisfying $G_{di}(j\omega)G_{di}^{\top}(-j\omega) = I$, $G_{do}(s)$ is the co-outer and RH_∞-left-invertible, i.e. there exists a RH_∞-transfer function matrix $G_{do}^{-1}(s)$ such that $G_{do}^{-1}(s)G_{do}(s) = I$ (Zhou *et al.*, 1995). Setting $R(s) = Q(s)G_{do}^{-1}(s)$ yields,

$$\begin{aligned}\|r\|_2 - J_{th} &= \|R\left(\bar{G}_{r\bar{d}}\bar{d} + \bar{G}_{rf}f\right)\|_2 - \left\|R\bar{G}_{r\bar{d}}\right\|_\infty \alpha \\ &= \|QG_{do}^{-1}\left(\bar{G}_{r\bar{d}}\bar{d} + \bar{G}_{rf}f\right)\|_2 - \|QG_{di}\|_\infty \alpha\end{aligned}$$

where $Q(s) \in \mathbf{RH}_\infty$ stands for an arbitrarily selectable matrix of an appropriate dimension. Note that

$$\begin{gathered}\|QG_{di}\|_\infty = \|\left(QG_{di}\right)^*\|_\infty = \|Q\|_\infty \\ \|QG_{do}^{-1}\left(\bar{G}_{r\bar{d}}\bar{d} + \bar{G}_{rf}f\right)\|_2 \leq \\ \|Q\|_\infty \|G_{do}^{-1}\left(\bar{G}_{r\bar{d}}\bar{d} + \bar{G}_{rf}f\right)\|_2\end{gathered}$$

It turns out: for all $Q(s) \in \mathbf{RH}_\infty$

$$\begin{aligned}\|r\|_2 - J_{th} &= \|QG_{do}^{-1}\left(\bar{G}_{r\bar{d}}\bar{d} + \bar{G}_{rf}f\right)\|_2 - \|Q\|_\infty \alpha \\ &\leq \|Q\|_\infty \left(\|G_{do}^{-1}\left(\bar{G}_{r\bar{d}}\bar{d} + \bar{G}_{rf}f\right)\|_2 - \alpha\right)\end{aligned}$$

The above inequality shows that condition

$$\|G_{do}^{-1}\left(\bar{G}_{r\bar{d}}\bar{d} + \bar{G}_{rf}f\right)\|_2 - \alpha > 0 \qquad (25)$$

is a necessary condition under which fault f becomes detectable. Note that (25) is expressed only in terms of the model parameters $G_{do}, G_{di}, \bar{G}_{rf}$ and f as well as d, moreover no assumption on $R(s)$ has been made by the derivation, thus the following theorem holds.

Theorem 2. Given system (1)-(2) and threshold (13), a fault f can then be detected only if (25) holds.

Following Theorem 2, we know that increasing δ_d, $\delta_\Delta$reduces the false alarm rate on the one side and makes detecting f more difficult and thus increases the missed detection rate on the other side. From this point of view, we say that (25) allows us to establish a relationship between the false alarm rate and the missed detection rate and further, based on it, to make a suitable trade-off between them.

Note that setting $Q(s) = I$ and therefore $R(s) = G_{do}^{-1}(s)$ leads to

$$\|r\|_2 - J_{th} = \|G_{do}^{-1}\left(\bar{G}_{rd}\bar{d} + \bar{G}_{rf}f\right)\|_2 - \alpha$$

This means that (25) is also a sufficient condition for f to be detectable if $R(s)$ is set to be $G_{do}^{-1}(s)$. Using this result we are able to prove the following theorem.

Theorem 3. $R^*(s) = G_{do}^{-1}(s)$ is the optimal solution of optimization problem (18).

Proof. Denote a parameter matrix different from $R^*(s)$ by $\hat{R}(s)$. According to Theorem 3, the following inequality should hold: for all $d, \Delta A, \Delta B, \Delta E_d, \Delta E_f$

$$\dim \Omega_{\hat{R},J_{th}}(d,\Delta) \leq \dim \Omega_{R^*,J_{th}}(d,\Delta)$$

To prove it, we only need to show that for any $d, \Delta A, \Delta B, \Delta E_d, \Delta E_f$ and $f \in \Omega_{\hat{R},J_{th}}(d,\Delta)$ we also have $f \in \Omega_{R^*,J_{th}}(d,\Delta)$. To this end, we rewrite $\hat{R}(s)$ as $\hat{Q}(s)G_{do}^{-1}(s)$ for some $\hat{Q}(s) \in \mathbf{RH}_\infty$. Recall that

$$\begin{aligned}&\|\hat{R}\left(\bar{G}_{rd}\bar{d} + \bar{G}_{rf}f\right)\|_2 - \left\|\hat{R}\bar{G}_{rd}\right\|_\infty \alpha \\ &= \|\hat{Q}(G_{do}^{-1}\bar{G}_{rf}f + G_{di}\bar{d})\|_2 - \|\hat{Q}\|_\infty \alpha \\ &\leq \|\hat{Q}\|_\infty \left(\|G_{do}^{-1}\bar{G}_{rf}f + G_{di}\bar{d}\|_2 - \alpha\right)\end{aligned}$$

It follows from the definition of the set of detectable faults, (16), that $f \in \Omega_{\hat{R},J_{th}}(d,\Delta)$ only if

$$\|\hat{R}(\bar{G}_{rd}\bar{d} + \bar{G}_{rf}f)\|_2 - J_{th} > 0$$

This means in turn

$$\begin{gathered}\|G_{do}^{-1}\bar{G}_{rf}f + G_{di}\bar{d}\|_2 - \alpha \geq \\ \frac{\|\hat{R}(\bar{G}_{rd}\bar{d} + \bar{G}_{rf}f)\|_2 - \alpha}{\|\hat{Q}\|_\infty} > 0\end{gathered}$$

and thus $f \in \Omega_{R^*,J_{th}}(d,\Delta)$. This proves the theorem. ■

We would like to emphasize that the optimal solution $R^*(s)$ is independent of d and the model uncertainties, and thus it ensures that for all possible unknown inputs and model uncertainties the dimension of the set of detectable faults reaches maximum.

Note that $\left\|R^*\bar{G}_{rd}\right\|_\infty = 1$. Thus, in case that the optimal post-filter $R^*(s)$ is used the adaptive threshold is given by

$$\begin{aligned}J_{th} &= \gamma_{\delta_\Delta}^d \delta_d + \gamma_{\delta_\Delta}^u \|u\|_2 = \alpha \\ &= (1 + \gamma_1\delta_\Delta)\delta_d + \gamma_1\delta_\Delta \|u\|_2\end{aligned}$$

3.4 Algorithms and on-line calculation

In this sub-section, the main results achieved above will be summarised in form of two algorithms that are used for the FD system design and the calculation of the adaptive threshold.

Algorithm for the design of FD systems

- Do a left coprime factorization of $G_u(s)$ or design an observer (3) (i.e. selecting an L that ensures the stability of the observer)
- Form $\bar{G}_{r\bar{d}}$ according to (21), (22)
- Do a co-inner-outer factorization of transfer function matrix $\bar{G}_{r\bar{d}}$

- Set the optimal solution $R^*(s) = G_{do}^{-1}(s)$.

Algorithm for the calculation of the adaptive threshold

- Form $F_A(sI - A - \Delta A)^{-1}\tilde{B} + \tilde{D}$ according to (20)
- Search for (minimum) γ_1 using the LMI-technique on the basis of Lemma 1
- Set $J_{th} = (1 + \gamma_1\delta_\Delta)\delta_d + \gamma_1\delta_\Delta \|u\|_2$

The needed on-line calculation

- Operation of residual generator (3)-(4) or (5)
- Residual evaluation: calculation of $\|r\|_2$
- Calculation of $\|u\|_2$ for the purpose of the on-line calculation of the adaptive threshold
- Comparison between $\|r\|_2$ and J_{th}

4. CONCLUDING REMARKS

From the viewpoint of a trade-off between the false alarm rate and missed detection rate, an approach to the design of fault detection system for technical processes with model uncertainties has been developed. Core of this approach is

- the derivation of a relationship between the false alarm rate and the size of unknown inputs and model uncertainties;
- the formulation of the design problem, minimizing the missed detection rate under a given false alarm rate, as an optimization problem and
- the derivation of an optimal solution.

We would like to make following remarks on the results achieved, which cannot be discussed in detail due to the limited space.

- As mentioned in Sub-section 2.1, an integrated design of the controller and FDI system is needed if the controller and the FDI system are developed on the basis of a common dynamic system, for instance an observer. In this case, the selection of both parameter matrices, K and L, play a key role, different from the case discussed in this contribution. Considering that the design of control system is of primary interest and the post-filter $R(s)$ has no influence on the dynamics of the control loop (plant + controller in a closed loop), the integrated design can be carried out in two steps: a). design of K and L to ensure the desired control performance. For this purpose, the known robust control theory can be used; b). design of $R(s)$ to achieve the desired FDI performance. To this end, the approach proposed in this paper can be used.
- The derived solution can also be presented in a state-space form, which provides us also with an alternative solution to the design of fault detection filters (Niemann and Stoustrup, 1996; Edelmayer and Keviczy, 1997).
- Although the study carried out in this paper aims at solving the robust fault detection problem, the achieved results can also be used to approach the robust fault isolation problem following two different schemes (Gertler, 1998; Chen and Patton, 1999; Frank and Ding, 1997): a) Reduce the fault isolation problem to a unknown input decoupling problem; b) First solve the fault isolation problem without considering the disturbances, which will result in k_f residual generators, then optimize each residual generator by taking into account the influence of the disturbances on each residual signal.
- The approach has also been successfully used in different laboratory systems.

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