15th Triennial World Congress, Barcelona, Spain

MODEL ORIENTATION AND WELL CONDITIONING OF SYSTEM MODELS: SYSTEM AND CONTROL ISSUES

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Abstract: Early stages modelling of processes involves issues of classification of variables into inputs, outputs and internal variables, referred to as Model Orientation Problem (MOP) which may be addressed on state space implicit, or matrix pencil descriptions. Defining orientation is equivalent to producing state space models of the regular or singular type. Studying the conditions, under which such models may be derived, as well as the structural properties of the resulting oriented problems, is one of the issues considered here. Oriented models of the S(A,B,C,D) type have transfer functions which may not be well behaving as far as properties of input, output regularity and nondegeneracy. Defining subsystems by reduction of the number of inputs, outputs such that the resulting transfer function is well behaved as far as preservation of degeneracy, is a problem considered in the context of systems having physical input, output variables, which have to be preserved. *Copyright © 2001 IFAC*

Keywords: Linear Systems, Structural Methodologies, Invariants, Modelling, Selection of Inputs, Outputs.

1. INTRODUCTION

An integral part of overall modelling problem for systems is the definition of process variables and their subsequent classification into control variables (inputs), command variables (outputs) and other internal variables. Heuristics linked to the specific domain of applications, or methodologies such as graph analysis, etc. may be used for handling issues of nonredundancy in representations and classification of variables. A natural system description that makes no distinction as far as the role of process variables and their dependence, or independence is for the linear case the matrix pencil model (first order differential descriptions), or the general polynomial, or autoregressive model. Here, the implicit, or matrix pencil models are examined, which characterise all process variables and consider their classification into inputs, outputs and internal variables. This is referred to as **Model Orientation Problem** (MOP) and its solutions are systems of the standard state space type. Investigating the conditions under which MOP is solvable, as well as characterisation of structural properties of solutions, when solutions exist is one of the main topics considered here. Solutions to MOP are linear systems of the

S(A,B,C,D) type and are not always suitable for control design since they may be characterised by input, output structure redundancy and they may be degenerate. Defining subsystems by reducing the number of inputs, outputs such that the reduced systems S(A,B´,C´,D´) are well conditioned, as far as nondegeneracy, input, output regularity etc. is a problem referred to as **Well Conditioning Problem** (Karcanias and Vafiadis, 2001a) (WCP). This problem is considered here in the case where the inputs, outputs are physical variables and thus input, output reduction implies selection of subsets of such variables. The problems considered here are integral parts of early design of processes (Karcanias, 1994; Rijnsdorp, 1991) and are considered in the context of linear systems. A full treatment of the MOP and proof of the results is in (Karcanias and Vafiadis, 2001a).

2. STATEMENT OF THE PROBLEM

Physical modelling may be used for large families of systems. If all important variables are included and there is no effort to guarantee their minimality, and their classification into inputs, internal variables is made (Willems, 1989), the emerging descriptions are referred to as **implicit** (Aplevich, 1991) and in the case of first order differential descriptions they correspond to the **matrix pencil**, or generalised autonomous description (Karcanias and Hayton, 1982; Kuijper and Schumacher, 1990):

$$S(F,G):\ \ F\,p\,\underline{x} = G\,\underline{x},\ \ F,G \in \Re^{t\times u} \qquad (1)$$

where p is the differentiation, or shift operator and $\underline{x}$ is the vector of all problem variables. The study of such descriptions relies on the structure of $sF-G$. For control, it is important to classify the variables in $\underline{x}$ into internal variables, or states $\underline{x}$, assignable, or control variables $\underline{u}$, and measurement, or dependent variables $\underline{y}$. This is expressed in terms of transformation $\underline{x} = Q\,\underline{\tilde{x}}$, where $\underline{\tilde{x}} = [\underline{x}^t, \underline{u}^t, \underline{y}^t]^t$ and $Q \in \Re^{u\times u}$, $|Q| \neq 0$. Q will be called an **orientation transformation** (OT). For first order linear descriptions the most general form of oriented models is the **general singular** (GS) description:

$$S_{gs}\!:\ E\,p\underline{x} = A\,\underline{x} + B\,\underline{u},\ \underline{y} = C\,\underline{x} + D\,\underline{u} \qquad (2)$$

where $E,A \in \Re^{s\times n}, B \in \Re^{s\times p}, C \in \Re^{m\times n}, D \in \Re^{m\times p}$, $t = m+s$, $u = n+p+m$, and in general $s \geq n$.

In the case where $s = n$, S_{gs} will be called **singular** and if $s = n$ and $|E| \neq 0$, then the description will be called **regular** and it is equivalent to the standard state space description:

$$S:\ p\,\underline{x} = A\,\underline{x} + B\,\underline{u},\ \underline{y} = C\,\underline{x} + D\,\underline{u} \qquad (3)$$

Defining an OT Q such that S(F,G) is reduced to S_{gs} or S forms is called **model orientation problem** (MOP) and it is considered here. Part of this study is determining the conditions under which S(F,G) may be reduced to the GS, singular or regular descriptions. Defining oriented models from matrix pencil models is a form of **oriented realisation**. The main concern is the determination of the algebraic structure of the derived oriented model and to show how such a structure evolves from that of the pencil.

Regular descriptions S(A,B,C,D) defined as solutions of MOP may not have good properties as far as control design; in fact, they may be having degenerate transfer functions and be characterised by input, output structure redundancy. Defining subsystems of S by reduction of the input, output structure such that the reduced system S(A,B´,C´,D´) has desirable properties is referred to as **input-output structure reduction problem** (I-ORP) and includes problems such as the squaring down (Karcanias and Giannakopoulos, 1989). When the resulting model has physical input, output variables and it is desired to preserve them, then the I-ORP takes the special form, where only a a set of existing inputs and a b set of existing outputs is used, which leads to an $S_{a,b} = S(A, B_a, C_b, D_{a,b})$ subsystem with corresponding transfer function $H_{a,b}(s)$. Defining a, b sets such that the resulting $S_{a,b}$, $H_{a,b}(s)$ is well structured as far as certain properties is a problem referred to as **well conditioning by input-output reduction** (WCP) and it is also considered here. Note that in a transfer function matrix setup, WCP is equivalent to defining submatrices of $H(s)$ by eliminating certain columns and rows and which have desirable properties.

3. THE MODEL ORIENTATION PROBLEM: CHARACTERISATION OF SOLUTIONS

Considering the matrix pencil description S(F,G) of eqn (1) with an associated matrix pencil $sF-G$ of $t\times u$ dimensions and for which we assume the following Kronecker invariants (Gantmacher, 1959):

$$D_f = \{(s - l_i)^{t_i}, i \in \underset{\sim}{q} : \sum t_i = n_f \} \quad (4)$$

$$D_\infty = \{\hat{s}^{q_i} : q_1 \geq .. \geq q_m \geq 1,\ q_{m+1} = .. = q_{m+d} = 1, \sum_{i=1}^{m} q_i = n_\infty \}$$

$$I_r = \{h_i : h_1 \geq .. \geq h_l \geq 1,\ h_{l+1} = .. = h_{l+g} = 0, \sum h_i = n_r \}$$

$$I_c = \{e_i : e_1 \geq .. \geq e_v \geq 1,\ e_{v+1} = .. = e_{v+h} = 0, \sum e_i = n_c \}$$

where D_f, D_∞ denote the set of finite (fed), infinite elementary divisors (ied), I_r is the set of row minimal indices (rmi) and I_c the set of column minimal indices (cmi). Note that d is the number of linear ied, g is the number of zero rmi and h the number of zero cmi. The above set of invariants completely characterise the Kronecker canonical form $sF_k - G_k$ of $sF - G$ which is defined under strict equivalence, that is there exists a pair $(R,Q): R \in \Re^{t \times t}$, $|R| \neq 0, Q \in \Re^{u \times u}, |Q| \neq 0$ such that (Gantmacher, 1959) $sF_k - G_k = R(sF - G)Q$. The problem considered here is the characterisation of the types of oriented models which may be derived from S(F,G), as well as finding such solutions. It is first noted that:

Remark (1): If $R \in \Re^{t \times t}, |R| \neq 0$, then the space of solutions (smooth and distributions) of S(F,G) and S(RF,RG) are the same.

The above suggests that left transformations do not affect the solution space and thus may be used to simplify the original description S(F,G).

Proposition (1): If $sF - G$ has g zero rmi, then there exists $R \in \Re^{t \times t}, |R| \neq 0$ such that:

$$R[F, G] = \begin{bmatrix} F', G' \\ \hline 0_g \end{bmatrix} \quad (5)$$

and the solutions of S(F,G) and S(F´,G´) are identical.

An S(F,G) system with zero rmi will be called **reducible**; otherwise, will be called **irreducible**. Proposition (1) implies that it can always be assumed to be an irreducible form. The existence of solutions to MOP is examined next.

Lemma (1) (Karcanias, 1990)**:** Consider the irreducible system S(F,G) with Kronecker invariants as described in (Gantmacher, 1959). There always exist a strict equivalence pair (R,Q) such that:

$$R(sF - G)Q = sF^* - G^* \quad (6a)$$

where: $sF^* - G^* =$ (6b)

$$\left[\begin{array}{cccc|c|c|c} sI - A_f & & 0 & & 0 & & \\ & sH_\infty - I & & & & 0 & 0 \\ & 0 & sI - A_c & & -B_c & & \\ & & & sI - A_r & 0 & 0_h & \\ & 0 & & C_r & 0 & & \\ \hline & & 0 & & & & I_d \end{array}\right] =$$

$$= \left[\begin{array}{c|c|c} sE' - A' & -B' & 0 \\ \hline 0 & 0 & I_d \end{array}\right] \quad (6c)$$

where A_f is $n_f \times n_f$ and it is characterised by D_f, $sH_\infty - I$ is $n_\infty \times n_\infty$ and $sH_\infty - I = = block - diag\{sH_{q_i} - I, q_i > 1\}$, A_c is $n_c \times n_c$ and corresponds to all $e_i > 0$, B_c is $n_c \times v$ and full rank, A_r is $n_r \times n_r$ and corresponds to all $n_j > 0$ and C_r is $l \times n_r$ and has full rank.

Theorem (1): Consider the irreducible system $S(F,G)$: $F\, p\underline{x} = G\underline{x}$, $F, G \in \Re^{t \times u}$ where the associated pencil $sF - G$ is assumed to have a general structure. There always exists a $R \in \Re^{t \times t}, |R| \neq 0$, and a transformation $Q \in \Re^{u \times u}$, $|\mathrm{Q}| \neq 0$, such that:

$$\underline{x} = Q\underline{x}' = Q\begin{bmatrix} \underline{x}' \\ \underline{u}' \\ \underline{y}' \end{bmatrix}, \quad R(pF - G)G = pF' - G' \quad (7a)$$

which reduces S(F,G) to the equivalent oriented description $S(F', G')$:

$$S(F', G'): (pF' - G')\underline{x}' = \begin{bmatrix} pE' - A' & -B' & 0 \\ -C' & -D' & I_d \end{bmatrix} \begin{bmatrix} \underline{x}' \\ \underline{u}' \\ \underline{y}' \end{bmatrix} = 0 \quad (7b)$$

where $pE' - A'$ is a nonsquare of dimensions $(n' + l) \times n', n' = n_f + n_\infty + n_r + n_c$, B' is $(n' + l) \times (u + h)$, C' is $d \times n'$ and D' is $d \times (u + h)$.

The above result establishes the existence of a general singular system as the solution to MOP. Furthermore, the construction of such transformation is intimately linked to derivation of Kronecker canonical forms, which is behind the construction of the form $sF^* - G^*$ of (6b). Theorem (1) together with Lemma (1) establish a relationship between the Kronecker structure of $pF - G$ and the nature of solutions of MOP and this is established by the following corollaries. Any solution of MOP will be called an **oriented realisation** of S(F,G).

Corollary (1): Any irreducible system S(F,G) has an oriented realisation $S(E', A'; B', C', D')$ which has the following properties:

(i) $S(E', A'; B', C', D')$ is general singular, iff the set of Kronecker invariants contains nonzero rmi.

(ii) $S(E', A'; B', C', D')$ is singular, iff the set of Kronecker invariants has no rmi and contains nonlinear ied.

(iii) $S(E', A'; B', C', D')$ is regular, iff the set of Kronecker invariants has no rmi and no nonlinear ied.

The presence of nonzero rmi in the pencil $pF - G$ implies that oriented realisations are of the nonsquare, or general singular type. Regarding the original description S(F,G) this has some additional implications on redundancy of the representation. We first note that for general singular representations the dynamic part is described by the pencil $[pE' - A', -B']$, where $pE' - A'$ is nonsquare. A **normal** representation of this pencil (defined in a nonunique manner by column permutations) is the pencil $[pE'' - A'', -B'']$, where $[pE'' - A'']$ is square. Clearly, normal representations may be extended to the S(F´,G´) description of (7b), and this leads to the definition of the normal representation of the general singular description represented as:

$$S(F'', G''): \begin{bmatrix} pE'' - A'' & -B'' & 0 \\ -C'' & -D'' & I_d \end{bmatrix} \begin{bmatrix} \underline{x''} \\ \underline{u''} \\ \underline{y''} \end{bmatrix} = 0 \quad (8)$$

Proposition (1): If $pF - G$ has nonzero rmi, then any oriented realisation S(F´,G´) of S(F,G) has every normal representation S(F´´,G´´) with $pE'' - A''$ singular.

The above property clearly suggests that there is some redundancy in the components of $\underline{x'}$ vector and this is described by the following:

Corollary (2): Let S(F´,G´) be a general singular oriented realisation of S(F,B). Then:

(i) There always exist n_r independent linear relations amongst the coordinates of the original vector $\underline{x}$.

(ii) The space of solutions of S(F´,G´) is given by the set of n_r linear relations and the solutions of the reduced system.

$$S(\tilde{F}, \tilde{G}): \begin{bmatrix} p\tilde{E} - \tilde{A} & -\tilde{B} & 0 \\ -\tilde{C} & -D' & I_d \end{bmatrix} \begin{bmatrix} \underline{\tilde{x}} \\ \underline{u'} \\ \underline{y'} \end{bmatrix} = 0 \quad (9)$$

where $\underline{\tilde{x}}$ is vector of dimension $n_f + n_\infty + n_c$, $\underline{u'}$, $\underline{y'}$ as before and with the associated pencil $p\tilde{F} - \tilde{G}$ having the same Kronecker invariants with $pF - G$ except the set of rmi.

The proof of the above result is constructive and indicates how the set of n_r linear relations is derived from the orientation transformation, as well as a procedure to construct any reduced systems $S(\tilde{F}, \tilde{G})$ that expresses the dynamic solutions. The singular system $S(E, A; B)$ defined by $[p\tilde{E} - \tilde{A}, -\tilde{B}]$ will be called a **reduced realisation** of S(F,G) and its properties are described below.

Corollary (3): For any S(E,A,B) reduced realisation of S(F,G) the following hold true:

(i) The pencil $[p\tilde{E} - \tilde{A}, -\tilde{B}]$ has as Kronecker invariants the set of fed, ied and cmi of S(F,G).

(ii) The number of inputs is given by the number of cmi $v + h$ of S(F,G). Furthermore, h expresses the order of redundancy of the input structure i.e. number of dependent inputs.

(iii) The system S(E,A;B) is controllable, if and only if S(F,G) has no fed and ied. Furthermore, the system is regular if and only S(F,G) has no ied.

The analysis here provides a solution to MOT for general autonomous description S(F,G), a characterisation of the type of resulting oriented realisations and a procedure to construct them based on Kronecker form transformations. So far, no constraint has been imposed on the orientation transformation.

4. WELL CONDITIONING BY INPUT, OUTPUT REDUCTION

The input output structure reduction problem is now considered, which may be thought as a follow up to orientation, or which may be posed independently, if early modelling produces a transfer function H(s), with undesirable features for control design. Thus, consider the regular state space system

$$S(A,B,C,D): \; \dot{\underline{x}} = A\,\underline{x} + B\,\underline{u},\; \underline{y} = C\,\underline{x} + D\,\underline{u} \quad (10)$$

where $A \in \Re^{n\times n}, B \in \Re^{n\times r}, C \in \Re^{q\times n}, D \in \Re^{n\times r}$ with a corresponding transfer function $H(s) = C(sI - A)^{-1}B + D \in \Re^{q\times r}(s)$ and let $r = rank_{\Re(s)}\{H(s)\}$. Clearly $r \le \min(q,r)$ and whenever strict inequality holds, then the system is called **degenerate**, otherwise, it will be called **nondegenerate**. The selection of the output structure may be such that the system is degenerate and this has implications as far as the potential for control design.

Remark (2) (Rosenbrock, 1970): r defines the maximal number of output variables that may be controlled independently (output function controllability criterion). Furthermore, r defines the minimal number of independent inputs required to control r outputs.

It shall be assumed in the following that the input, output structure of the model is regular:

$$rank\left\{\begin{bmatrix} B \\ D \end{bmatrix}\right\} = r,\; rank\,\{[C, D]\} = q \quad (11)$$

Such conditions can always be achieved by simple, input, output redesign. Failure of such conditions to hold implies in certain cases degeneracy (Karcanias and Vafiadis, 2001a) of the resulting model. Here the concern is on the case of strong degeneracy, which is the result of the deeper model structure. The selection of a reduced, input, output, structure model that is nondegenerate is a straightforward problem when general input, output transformations are allowed. Here the case where the input, output sets are physical variables is considered. Given the model H(s), or S(A,B,C,D), which is referred here as a progenitor model, define: A maximal cardinality subset of the input, output sets such that the resulting transfer function is nondegenerate and has the maximal possible rank. The solution to the above problems is referred to as **well conditioning** of the progenitor model. For the system matrix pencil (Rosenbrock, 1970):

$$P(s) = \begin{bmatrix} sI - A & -B \\ -C & -D \end{bmatrix} \in \Re^{(n+q)\times(n+r)}(s) \quad (12)$$

and denote by: $Z_r = N_r\{P(s)\}$, $\dim Z_r = s$, $Z_\ell = N_\ell\{P(s)\}$, $\dim Z_\ell = J$.

Remark (3) (Karcanias and Vafiadis, 2001a): If $t = rank_{\Re(s)}\{P(s)\}$, $r = rank_{\Re(s)}\{P(s)\}$, then $t = n + r$, $s = \dim N_r\{H(s)\} = r - r$, $J = \dim N_l\{H(s)\} = q - r$.

Thus degeneracy may be studied either on S(A,B,C,D) or H(s). Furthermore, if $q \ge r$ degeneracy can be studied by considering properties of Z_r, or $N_r\{H(s)\}$ and if $q \le r$ by considering properties of Z_ℓ, or $N_\ell\{H(s)\}$ only. In the following it shall be assumed that $q \ge r$ and shall be concerned with the properties of Z_r, which is a rational vector space and it is characterised by a set of Forney dynamical indices (Forney, 1975), the column minimal indices of P(s), $I_c(P) = \{\tilde{e}_1 \ge \tilde{e}_2 \ge \ldots \ge \tilde{e}_s > 0\}$. Note, that input regularity ($rank\,[B^t, D^t] = r$) implies that we have no zero cmi.

Using the (A,B,C,D) parameters, the following sequence of matrices may be defined:

$$M_0 = \begin{bmatrix} B \\ D \end{bmatrix}, \ldots, M_k = \left[\begin{array}{ccccc} A^k B & A^{k-1}B & .. & AB & B \\ \hline CA^{k-1}B & CA^{k-2}B & .. & CB & D \\ CA^{k-2}B & CA^{k-3}B & .. & D & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ CAB & CB & .. & 0 & 0 \\ CB & D & .. & 0 & 0 \\ D & 0 & .. & 0 & 0 \end{array}\right] \quad (13)$$

the significance of which is demonstrated by the following result:

Proposition (2) (Karcanias and Vafiadis, 2001a): The system S(A,B,C,D) with $q \ge r$ has a vector in Z_r with degree k if and only if $N_r\{M_k\} \ne \{0\}$.

Clearly, nondegeneracy may be achieved by making sure that all M_k have no right nullity, or equivalently full rank ($q \ge r$ assumption). Such infinite sets of tests may be reduced to a single test as it is shown below.

Lemma (2) (Karcanias and MacBean, 1981): If $q \geq r$, then the maximal possible value of a cmi of P(s) is:

(i) If $D \neq 0$, $rank(D) = d$, then $\tilde{e}_{\max} = n - q + 2d - 1 = v^*$

(ii) If D = 0, then $\tilde{e}_{\max} = n - q - 1 = v^*$

Theorem (2): The system S(A,B,C,D) with $q \geq r$ is nondegenerate, if and only if $N_r\{M_{v^*}\} = \{0\}$, where v^* is defined as in Lemma (2).

A searching procedure may be initiated based on selecting subsets of the columns of $\left[B^t, D^t\right]^t$ and implementing this on M_{v^*} matrix in the appropriate block manner until the reduced M'_{v^*} has full rank. Simplified tests by using sufficient conditions for nondegeneracy are:

Corollary (4): For the system S(A,B,C,D) with $q \geq r$ the following conditions hold true:

(i) If D has full rank then the system is nondegenerate.

(ii) If $D = 0$ and CB has full rank, then the system is nondegenerate.

The above corollary suggests that a simpler procedure based on making the reduced D', or CB' full rank by selecting the appropriate set of inputs may be used; however, this may lead to systems which have a nondegenerate $H'(s)$, but with rank less than r. Selection procedures are described in (Karcanias and Vafiadis, 2000).

5. CONCLUSIONS

Two problems of the early modelling of processes have been addressed which have a strong systems and control content. The first was the model orientation problem (MOT) which have been considered using general transformations and the second was the well conditioning by input-output reduction (WCP). A complete solution of MOT has been given and the deployed approach has also the potential for the development of algorithms for constructing oriented models using Kronecker theory. The characterisation of conditions for well conditioning has been derived which may lead to a searching procedure for selecting well behaving solutions. Reducing the searching effort by developing appropriate tests is an issue under current investigation.

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