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ROBUST OUTPUT FEEDBACK CONTROLLER DESIGN VIA GENETIC ALGORITHM

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Abstract: This paper proposes guaranteed cost design of robust output feedback controller for continuous linear parametric uncertain systems. Proposed algorithms are computationally simple and tightly connected with the Lyapunov stability theory and the LQR optimal state feedback design. The proposed approach allows for prescribing the structure of the output feedback gain matrix (including the decentralized one) by the designer. New design method proposed in this paper, exploit genetic algorithm to design robust controller with guaranteed cost for polytopic linear continuous time systems. Numerical example is given to illustrate the performance of the proposed robust controller.

Keywords: Robust controller, Linear systems, Genetic algorithm

1. INTRODUCTION

In the excellent survey on static output feedback controller design in (Syrmos et al,1997) is stated that the static output feedback problem is one of the most important open questions in control engineering. Simply stated, the problem is as follows: given a dynamic system, find a static output feedback so that the closed loop system has some desirable characteristics. During the last two decades numerous papers dealing with the design of robust output feedback control schemes have been published (Benton and Smith, 1999; El Ghaoui and Balakrishnam, 1994; Imai, 1997; Iwasaki, Skelton and Geromel, 1994; Kose and Jabbari, 1999; Kozák, 1995; Li Yu and Jian Chu, 1999; Xu and Darouch, 1998; Yong Yan Gao and You Xian Sun, 1998; Veselý and Sekaj, 2000; Hei Ke Tam and Lam,1999). Various approaches have been applied to study the two aspects of this stabilization problem, namely the conditions under which a linear system described in the state space can be stabilized via output feedback and the respective procedure for obtaining a stabilizing control law (Kučera and De Souza, 1995; Syrmos et al,1997). In the above papers, the authors basically conclude that despite the availability of many approaches and numerical algorithms the static output feedback problem is still open. This is justified by the fact that up to now there are no testable necessary and sufficient conditions available to test stability of a static output feedback system.

Recently it has been shown that an extremely wide array of robust controller design problems can be reduced to the problem of finding a feasible point under a Biaffine Matrix Inequality (BMI) constraint. The BMI has been introduced in (Goh et al, 1995) as a geometric reformulation of many robust control problems. However it is known that BMI problems are *NP*-hard (Toker and Ozbay , 1995). The main result of (Toker and Ozbay , 1995) shows that it is rather unlikely to find an algorithm for solving general BMI problems and it has also been shown that simultaneous stabiliza-

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tion of N plants via static output feedback is an *NP*- hard problem. The BMI feasibility problem is discussed and a branch and bound global optimization algorithm to find an $\varepsilon-$ global minimum in a finite number of iterations is presented in (Goh et al, 1995). In the above papers, the authors explain why the BMI feasible problem inevitably holds such a central place in the robust control synthesis problem.

The theory of linear matrix inequalities (LMIs) (Boyd et al, 1994) has been used to design robust output feedback controllers in (Benton and Smith, 1999;) , Li Yu and Jian Chu, 1999; Yong Yan Gao and You Xian Sun, 1998). Most of the above works present iterative algorithms in which a set of equations, or set of LMI problems, are repeated until certain convergence criteria are met. In (Yong Yan Gao and You Xian Sun, 1998) a necessary and sufficient condition for simultaneous stabilizability via static output feedback has been obtained and an iterative LMI algorithm has been proposed to obtain the output feedback gain. The authors in (Kose and Jabbari, 1999) study conditions under which the designed output feedback controllers can be divided into two stages and the dynamic output feedback can be obtained. In (Benton and Smith, 1999) a LMI based algorithm has been proposed which does not require iteration of the LMI solution. The goal is to eliminate the need for iteration by an appropriate choice of initializing state feedback matrix. The proposed algorithm can be used to robustly stabilize a polytopic system via static output feedback. The V-K iteration algorithm proposed in (El Ghaoui and Balakrishnam, 1994) is based on an alternative solution of two convex LMI optimization problems obtained by fixing the Lyapunov matrix or the gain controller matrix. This algorithm is guaranteed to converge, but not necessarily, to the global optimum of the problem depending on the starting conditions.

In this paper, the alternative way to BMI and LMI problem of robust static output feedback controller design is given using the genetic algorithm. From the Lyapunov stability theory, the well known necessary and sufficient conditions to stabilize continuous time systems via static output feedback have been used to design a robust controller with guaranteed cost for polytopic systems. The paper is organized as follows. In Section 2 the problem formulation and some preliminary results are brought. The main results are given in Section 3. In Section 4 the obtained theoretical results are applied to an example. We have used the standard notation. A real symmetric positive (negative) definite matrix P is denoted by $P > 0 (P < 0)$. Much of the notation and terminology follows from references (Benton and Smith, 1999; Boyd et all, 1994; Kučera and De Souza, 1995).

2. PRELIMINARIES AND PROBLEM FORMULATION

In the context of robustness analysis and robust controller synthesis for linear time invariant systems the following uncertain model is commonly used

$$\dot{x} = (A + \delta A)x + (B + \delta B)u \qquad (1)$$
$$y = Cx, \quad x(0) = x_0$$

where $x \in R^n, u \in R^m$ and $y \in R^l$ are the state, control and output vector, respectively; A, B and C are known matrices of appropriate dimensions; $\delta A = \{\delta a_{ij}\}, \delta B = \{\delta b_{lk}\}$ are unknown but norm bounded uncertainties. The following types of uncertainty descriptions are often used in the robustness investigations.

- Norm bounded uncertainties, unstructured model
$$\|\delta A\| \leq q_a, \quad \|\delta B\| \leq q_b \qquad (2)$$
where $\|.\|$ represents any matrix norm, and q_a and q_b are nonnegative constants.
- Element bounded uncertainties, structured model
$$|\delta A|_m \leq A_m, \quad |\delta B|_m \leq B_m \qquad (3)$$
where $|.|$ represents the modules of corresponding matrix, $A_m = \{a^m_{ij}\}, B_m = \{b^m_{lk}\}$ are matrices with nonnegative entries and corresponding dimensions, respectively, and
$$a^m_{ij} \geq |\delta a_{ij}|, \quad b^m_{lk} \geq |\delta b_{lk}|$$
- Matrix affine type uncertain structure
$$\delta A = \sum_{i=1}^{p} \epsilon_i A_i, \quad \delta B = \sum_{j=1}^{s} \gamma_j B_j \qquad (4)$$
$$\underline{\epsilon_i} \leq \epsilon_i \leq \overline{\epsilon_i}, \quad \underline{\gamma_j} \leq \gamma_j \leq \overline{\gamma_j}$$
where A_i, B_j are known matrices, ϵ_i, γ_j are unknown parameters. The ϵ_i, γ_j can vary in time arbitrarily fast provided that each element is within given bounds. In general, a polytope description of uncertainties results in less conservative controller designs than other uncertainty characterizations (Boyd et al, 1994).
- Matrix bounded uncertain structure
$$\delta A^T \delta A \leq \gamma_a Q_a, \quad \delta B^T \delta B \leq \gamma_b Q_b \qquad (5)$$
where γ_a, γ_b are nonnegative constants, Q_a, Q_b are nonnegative definite matrices.
- Uncertainties satisfying "matching conditions"
$$\delta A = UWA_1, \quad \delta B = UWA_2 \qquad (6)$$
where U, A_1 and A_2 are known matrices and W is an unknown matrix satisfying $W^T W \leq$

I, I being an identity matrix of corresponding dimension.

The problem studied in this paper can be formulated as follows. For a continuous linear time invariant system described by (1) a robust static output feedback controller is to be designed with the control algorithm in the form

$$u = FCx \tag{7}$$

so that the closed loop system

$$\dot{x} = (A + BFC)x + (\delta A + \delta BFC)x \tag{8}$$

is stable for all admissible uncertainties described by (2)- (6). A cost function associated with the system (1) is

$$J = \int_0^\infty (x^T Qx + u^T Ru)dt \tag{9}$$

where $Q = Q^T \geq 0$, and $R = R^T > 0$ are matrices of compatible dimensions.

Definition. Consider the uncertain system (1). If there exist a control law u^* and a positive scalar J^* such that for all admissible uncertainties, the closed loop system is stable, and the cost function (9) satisfies $J \leq J^*$, then J^* is said to be the guaranteed cost and u^* is said to be the guaranteed cost control law for uncertain system (1). □

The nominal model of the system (1) is given

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{10}$$

Let us recall some commonly used notions for continuous time systems. Matrix $D \in R^{n \times n}$ is called stable when all its eigenvalues lie in the left half complex plane, $Re\{\lambda_i(D)\} < 0$ for $i = 1, 2, ..., n$. The system (10) with a stable A is called a stable system. System (10) is called to be output feedback stabilizable if there exists a real output feedback gain matrix F such that $A + BFC$ is a stable matrix. The pair (A, C) is called detectable if there exists a real matrix X such that $A + XC$ is stable. The following lemma is well known (Lankaster 1969).

Lemma 1. Suppose P to be a solution to the following Lyapunov matrix equation

$$A^T P + PA + Q = 0 \tag{11}$$

Then A is stable iff $P > 0$ and $Q > 0$. □

If there exists such a P, the matrix A is said to be quadratically stable. A linear time invariant system is stable if and only if it is quadratically stable. It is possible, however, for example for polytopic linear systems to be stable without being quadratically stable (Boyd et al,1994). In the next development we will consider exclusively quadratically stable systems.

3. DESIGN OF ROBUST CONTROLLERS

In this paragraph we will present a design procedure of a robust static output feedback controller for the continuous time system (1). Note that it is well known (Syrmos et al, 1997) that a fixed order dynamic output feedback of order less or equal to n is a special case of the static output feedback problem. The well known results are summarized in the following lemma.

Lemma 2. Consider the linear uncertain continuous time system (1). Then, the following statements are equivalent.

- The system (1) is robust static output feedback stabilizable.
- There exist a positive definite matrix $P = P^T > 0$ and a matrix F satisfying the following matrix inequality

$$(A + BFC)^T P + P(A + BFC) + Q_o < 0 \tag{12}$$

where

$$Q_o = (\delta A + \delta BFC)^T P + P(\delta A + \delta BFC)$$

□

In general, a polytope description of uncertainties results in a less conservative controller design than other characterizations of uncertainty (Boyd et al, 1994). However, with the increasing of uncertain parameters the number of vertices increases exponentially, and the design time increases exponentially, too. Let the system be represented by the state realization (1) with uncertainties (4)

$$\dot{x} = (A + \sum_{i=1}^{p} \epsilon_i A_i)x + (B + \sum_{i=1}^{p} \epsilon_i B_i)u \tag{13}$$

$$y = (C + \sum_{i=1}^{p} \epsilon_i C_i)x$$

The system represented by (13) is a polytope of linear systems. The genetic algorithm approach requires the system (13) to be described by a list of its vertices, i.e., in the form

$$\{(A_{v1}, B_{v1}, C_{v1}), ..., (A_{vN}, B_{vN}, C_{vN})\} \tag{14}$$

where $N = 2^p$.

The system represented by (14) is quadratically stable if and only if there is a Lyapunov matrix $P > 0$ such that (Boyd et al, 1994)

$$A_{vi}^T P + PA_{vi} < 0, \tag{15}$$

$i = 1, 2, ..., N$.

Consequently, the system (14) is static output feedback quadratically stabilizable if and only if there is a Lyapunov matrix $P > 0$ and a feedback matrix F such that

$$(A_{vi} + B_{vi}FC_{vi})^T P + P(A_{vi} + B_{vi}FC_{vi}) < 0, \quad i = 1, 2, ..., N \tag{16}$$

If (16) holds for $P > 0$ and some F, then the vertices of the polytope (14) are said to be simultaneously quadratically stabilized by F. It is well known that if P is a common Lyapunov matrix for the vertices of the polytope (14), it serves as a common Lyapunov function for the uncertain system (13) for all admissible uncertainties $\epsilon_i \in < \underline{\epsilon_i}, \overline{\epsilon_i} >, i = 1, 2, ..., p$. Each vertex in (14)is computed for a different permutation of the p variables ϵ_i, alternatively taken at their maximum and minimum values.

Theorem 1. Consider the system (14). Then the following statements are equivalent.

- The system (14) is static output feedback simultaneously stabilizable with a guaranteed cost

$$\int_0^\infty (x^T Qx + u^T Ru)dt \leq x_0^T Px_0 = J^* \tag{17}$$

  and $P > 0$.
- There exist matrices $P > 0, R > 0, Q > 0$ and a matrix F such that the following inequality holds

$$(A_{vi} + B_{vi}FC_{vi})^T P + P(A_{vi} + B_{vi}FC_{vi}) + Q + C_{vi}^T F^T RFC_{vi} \leq 0 \tag{18}$$

  for $i = 1, 2, ..., N$.

Proof. Consider the output feedback control algorithm to have the form

$$u = Fy = FC_{vi}x$$

then for the closed loop system results

$$\dot{x} = (A_{vi} + B_{vi}FC_{vi})x, \quad i = 1, 2, ..., N$$

For $V = x^T Px$, the time derivative of V along the system (14) is

$$\frac{dV}{dt} = x^T [(A_{vi} + B_{vi}FC_{vi})^T P + P(A_{vi} + B_{vi}FC_{vi})]x$$

If the inequality (18) holds then there exist matrices $P > 0, R > 0, Q > 0$ and F such that

$$\frac{dV}{dt} \leq -x^T (Q + C_{vi}^T F^T RFC_{vi})x < 0$$

for $i = 1, 2, ..., N$. Therefore the closed loop system is asymptotically stable. Furthermore, by integrating both sides of the inequality from 0 to T and using the initial condition x_0 we obtain

$$V(0) - V(T) \geq \int_0^T x^T (Q + C_{vi}^T F^T RFC_{vi})xdt$$

As the closed loop system is asymptotically stable for $T \to \infty$

$$x(T)^T Px(T) \to 0$$

Hence, we get

$$\int_0^\infty x^T (Q + C_{vi}^T F^T RFC_{vi})xdt \leq x_0^T Px_0 \tag{19}$$

and the control algorithm $u = Fy$ is a guaranteed cost control law and

$$J^* = x_0^T Px_0$$

is a guaranteed cost value for the uncertain closed loop system. □

Based on the *Theorem 1* it is possible to reduce the robust controller design procedure to search for a matrix P and a feedback matrix F such that the condition (18) holds. As mentioned before, for the solution of this problem the genetic algorithm (GA) has been used.

Genetic Algorithms are numeric, stochastic-based and robust search or optimization procedures derived from the principles of natural search and natural genetics. They are sufficiently described in (Goldberg, 1989; Michalewicz, 1992 and others). GA are able to approach the best solution or the best representative from the space of all possible solutions. Normally, their only limitation is the required computation time or computation effort.

In the application of GA for our controller design the task is to find such a symmetric and positive definite matrix P and a feedback matrix F, that for chosen matrixes Q and R the condition (18) is fulfilled for each i of the polytopic description of the system. A general form of a chromosome, which is a linear string of potential solution parameters is in our case

$$string = \{p_{11}, p_{12}, ..., p_{nn}, f_{11}, ..., f_{mm}\}$$

where p_{ij} are the entries of the matrix P and f_{ij} are entries of the matrix F. That means that the matrix P and F are searched concurrently. The cost function J_G, to be minimized is calculated for each string using the following algorithm:

if min(eig(P))> 0
if max(eig(M_{vi}*)* ≤ 0 *for all i*
J_G *=-sum(eig(*M_{vi}*))*

else $J_G = 10^4 * (\pi + \sigma)$
else $J_G = 10^6 * \mu$

where: $eig(P)$ is the vector of matrix P eigenvalues, $eig(M_{vi})$ is the vector of the matrix M_{vi} eigenvalues, which is equal to the left hand side of the inequality (18), π is the number of positive eigenvalues of M_{vi}, σ is the sum of all positive eigenvalues of M_{vi} and μ is the number of all non-positive eigenvalues of P. Using this cost function the GA has the following effect: During the solution the non-negative eigenvalues of the left hand side of (18) are moved to the left half-plane and the negative eigenvalues are moved to zero from left. The algorithm allows to find such a feedback matrix F, which ensures the negative semidefinitness of the left hand side of (18) and minimizes the absolute value of its eigenvalues. It is known, that GA's converge to global optima (Holland,1975; Goldberg, 1989).
Remark: The used cost function, which ensures fulfilment of the closed-loop robust stability condition can be extended, using computer simulation of the minimization of some other performance index (e.g. absolute control error, input energy, settling time, overshoot, their combinations etc.). This can ensure also fulfilment of additional performance requirements.

4. EXAMPLE

The example has been borrowed from (Benton and Smith, 1999) to demonstrate the use of the proposed algorithm. Note that in (Benton and Smith, 1999) the LMI based algorithm has been used. It is known that the presented system is static output feedback stabilizable. Let (A, B, C) in (1) be defined as

$$A = \begin{bmatrix} -0.036 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.010 & 0.0024 & -4.0208 \\ 0.1002 & q_1(t) & -0.707 & q_2(t) \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.4422 & 0.1761 \\ q_3(t) & -7.59222 \\ -5.520 & 4.490 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$$

with parameters bounds $-0.6319 \le q_1(t) \le 1.3681, 1.22 \le q_2(t) \le 1.420$, and $2.7446 \le q_3(t) \le 4.3446$. Find a stabilizing output feedback matrix F. The nominal model of (A, B) is given by the above matrices when we substitute for the entries $A(3,2) = 0.3681, A(3,4) = 1.32$ and $B(2,1) = 3.5446$. The structured model uncertainty (4) $(A1, A2, B1)$ are matrices with the following entries $A1(3,2) = 1, A2(3,4) = 0.1$ and $B1(2,1) = 0.8$ with $\epsilon_i \in< -1, 1 >, i = 1, 2$ and $\gamma_1 \in< -1, 1 >$. Other entries of the above uncertain matrices are equal to zero. The nominal model is unstable with eigenvalues:

$$eig\{-2.0516, 0.2529 \pm 0.3247i, -0.2078\}$$

Let the structure of F be defined as

$$F^T = [F(1,1) \quad F(2,1)]$$

For $R = I, \epsilon_m = 1$ the matrix Q has been in small steps changed from $Q = .1 * I$ to $Q = 1.5 * I$. The corresponding closed loop maximal eigenvalues of the four polytopic system ($\epsilon_1 = \gamma_1$) moves as follows:
for $Q = .1 * I \quad CLosedEIG = \{-.055, -.2 \pm 1.55i, -12\}$,
for $Q = 1.5 * I \quad CLosedEIG = \{-.086, -.3 \pm 1.1i, -81\}$.
For the first case the gain matrix F is

$$F^T = [-1.0389 \quad 0.7613]$$

Moreover, for the guaranteed cost we obtain $\lambda_M(P)\|x_0\|^2 = 7.405\|x_0\|^2$.
Above results have been compared with the results of the V-K iteration method (El Ghaoui and Balakrishnan,1994). For this example the V-K iteration method does not give feasible solution. Note that the number of GA generations are 20000. As the closed loop system is quadratically stable in all its vertices, robust stability of the uncertain system (13) with the above designed static output feedback gain matrix F and uncertainties (4) has been proved.

5. CONCLUSIONS

The main aim of this paper has been to propose new, alternative to BMI and LMI method genetic algorithm for solving the robust controllers design via static output feedback for linear continuous time systems and guaranteed cost.

6. REFERENCES

Benton I.E. and Smith D. (1999): A non iterative LMI based algorithm for robust static output feedback stabilization. *IJC*, Vol.72, N14, 1322-1330.
Boyd S., El Ghaoui L., Feron E. and Balakrishnam V. (1994): Linear Matrix Inequalities in System and Control Theory. *SIAM*, Vol15, Philadelphia.
El Ghaoui L. and Balakrishnan V. (1994) : Synthesis of fixed structure controllers via numerical optimization. *Proc. of the* 33^{rd} *Conference on Decision and Control*, Lake Buena Vista (FL), 2678-2683.

Goh K.C., Safonov M.G. and Papavassilopoulos G.P. (1995): Global Optimization for the Biaffine Matrix Inequality Problem. *Journal of Global Optimization* , 7, 365-380.

Goldberg D.E. (1989): Genetic Algorithms in search, Optimization and Machine Learning, Addisson-Wesley.

Hei Ke Tam amd Lam J.(1999): Robust Output Stabilization Using Genetic Algorithm. In:14^{th} *World Congres of IFAC*, paper C-2a-10-4, 343-346.

Holland J.(1975): Adaptation in Natural and Artificial Systems, *University of Michigan Press*, Ann Arbor, MI, Reissued by the MIT Press.

Imai H. (1997): Robust output regulation for linear systems with structured uncertainty. *IJC*, Vol 66, N4, 499-515.

Iwasaki T., Skelton R.E. and Geromel J.C. (1994): Linear quadratic suboptimal control with static output feedback. *Systems and Control Letters*, 23, 421-430.

Kose I.E. and Jabbari F. (1999): Robust control of linear systems with real parametric uncertainty. *Automatica*, 35, 679-687.

Kozák Š. (1995) : Simple and robust PID controller.*Appl. Math. Comp.*, 70, 2-8.

Kučera V. and De Souza C.E. (1995): A necessary and sufficient conditions for output feedback stabilizability. *Automatica*, Vol31, N9, 1357-1359.

Lankaster P. (1969): Theory of matrices. *Academic press*, New-York, London.

Li Yu and Jian Chu (1999): An LMI approach to guaranteed cost control of linear uncertain time-delay systems. *Automatica*, 35, 1155-1159.

Michalewiz Z. (1992); Genetic Algorithms+Data Structures= Evolution Programs, Springer.

Syrmos V.L., Abdalah C.T., Dorato P. and Grigoriadas K. (1997): Static output feedback. A survey. *Automatica*, 33, 203-210.

Toker O. and Özbay H. (1995): On the NP-hardness of solving Bilinear Matrix Inequalities and Simultaneous Stabilization with Static Output Feedback. *Proc. ACC*, 2525-2526.

Veselý V. and Sekaj I. (2000) :Robust Controller Design Using Genetic Algorithms. In:*Preprints of the IFAC Conference Control Systems Design*, Bratislava, 500-504

Xu S.J. and Darouch M. (1998) : On the robustness of linear systems with nonlinear uncertain parameters. *Automatica*, Vol34, N8, 1005-1008.

Yong-Yan Gao and You-Xian Sun (1998): Static output feedback simultaneous stabilization: LMI approach. *IJC*, Vol70, N5, 803-814.