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NUMERICAL SOLUTION OF HYBRID OPTIMAL CONTROL PROBLEMS WITH APPLICATIONS IN ROBOTICS

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Abstract: Numerical solution techniques for a class of hybrid (discrete event / continuous variable) optimal control problems (HOCP) are described, and their potential use in robotic applications is demonstrated. HOCPs are inherently combinatorial due to their discrete event aspect which is one of the main challenges when numerically solving for optimal hybrid trajectories. One may associate a continuous nonlinear multi-phase problem with each possible discrete state sequence. Two solution techniques for obtaining suboptimal solutions are presented (both based on numerical direct collocation): one fixes interior point constraints on a grid, another uses branch-and-bound. Numerical results of a robotic multi-arm transport task and an underactuated robot are presented.

Keywords: Hybrid optimal control, mechatronics, underactuated robots.

1. INTRODUCTION

Solutions to nonlinear optimal control problems play a key role in modern mechatronics and robotics and particularly in the area of path, trajectory, and action planning. Some of the many applications include: walking pattern and trajectory planning (Hardt *et al.*, 2000), mobile robot path planning (Kondak and Hommel, 2001), optimal payload (weight) lifting and acrobatics (Martin and Bobrow, 1997; Albro and Bobrow, 2001), etc.

HOCPs with variable structure (switched) nonlinear differential equations describing a piecewise continuous subsystem coupled with discrete-event dynamical subsystems have recently received increased attention, see e.g. (Buss *et al.*, 2000; Buss, 2001; Tomlin, 1999). The key to numerically solving HOCPs appears to lie in the combination of efficient numerical solvers for optimal control problems – such as direct collocation – together with (heuristical) approaches to reduce the combinatorial complexity of the discrete event aspect (Buss *et al.*, 2000; Stryk and Glocker, 2000).

This paper presents numerical solution techniques for HOCPs with applications in mechatronics and robotics. An example problem of 3 robotic arms cooperatively transporting an object from an initial to a goal position is solved suboptimally by fixing interior point times and state constraints to fixed values on a grid. The trajectory planning problem of an underactuated robot with an unactuated joint equipped with only a holding brake is solved with branch-and-bound to obtain optimal hybrid trajectories including the optimal number of switches for the holding brake.

Both approaches rely on the efficient numerical tool DIRCOL implementing a direct collocation method to approximately solve nonlinear optimal control problems using advanced nonlinear programming methods (Stryk, 1999), see also (Stryk and Bulirsch, 1992; Hardt *et al.*, 2000) and related work by (Branicky *et al.*, 1999; Hedlund and Rantzer, 1999; Sussmann, 1999; Tomlin, 1999).

2. HYBRID OPTIMAL CONTROL—HOC

The discrete-continuous model of a HOCP consists of a set of ordinary differential or differential-algebraic equations of variable structure and variable constraint equations. The system structure varies among a (finite) discrete set of system descriptions each of which is associated with a specific discrete state of the considered hybrid system. The discrete state dynamics may be modeled, e.g., by a finite state automaton or a Petrinet. Modeling paradigms of hybrid dynamical systems are described in (Branicky *et al.*, 1998; Engell, 1997; Labinaz *et al.*, 1996; Nenninger *et al.*, 1999; Schlegl *et al.*, 2000).

Most of the hybrid models in the literature consider the hybrid state of the system as a combination of the continuous state $\mathbf{x}$ and a discrete state $\mathbf{q}$. Likewise, the control input is a combination of a continuous component $\mathbf{u}$ and a discrete-valued component $\mathbf{v}$. The hybrid system state and structure changes discontinuously when an autonomous or controlled discrete event at a particular time or state occurs.

The HOCP is to find optimal hybrid (i.e., continuous $\mathbf{u}$ and discrete $\mathbf{v}$) control trajectories such that an integral cost index, typically an integral of a function of the hybrid system state and control input, is minimized subject to the system dynamics, initial, terminal and further equality or inequality constraints.

Definition 1. The HOCP is defined as the minimization of the hybrid cost index J

$$\min_{\mathbf{u},\,\mathbf{v}} J(\mathbf{u},\mathbf{v}) = \Theta + \int_{t_a}^{t_e} \psi(\mathbf{x},\mathbf{u},\mathbf{q},\mathbf{v},t)\ \mathrm{d}t\ , \qquad (1)$$

subject to

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\mathbf{u},\mathbf{q},\mathbf{v},t) \qquad \text{if } s_j(\mathbf{x},\mathbf{u},\mathbf{q},\mathbf{v},t) \neq 0 \quad (2)$$
$$j = 1,\ldots,n_s$$

$$\begin{bmatrix} \mathbf{x}(t_i^+) \\ \mathbf{q}(t_i^+) \end{bmatrix} = \phi_j(\mathbf{x},\mathbf{u},\mathbf{q},\mathbf{v},t_i^-) \ \text{ if } s_j(\mathbf{x},\mathbf{u},\mathbf{q},\mathbf{v},t_i^-) = 0 \ (3)$$
$$j \in \{1,\ldots,n_s\}$$

$$\mathbf{u}(t) \in \mathcal{U} \subset \mathbb{R}^{n_u},\ \mathbf{v}(t) \in \mathcal{V} \subset \mathbb{Z}^{n_v},$$
$$\mathbf{x}(t) \in \mathcal{X} \subset \mathbb{R}^{n_x},\ \mathbf{q}(t) \in \mathcal{Q} \subset \mathbb{Z}^{n_q}, \quad \forall t \in [t_a,t_e] \qquad (4)$$
$$0 \le \mathbf{h}(\mathbf{x},\mathbf{u},\mathbf{q},\mathbf{v},t), t \in [t_a,t_e] \text{ inequality constraints,} \quad (5)$$
$$\mathbf{x}(t_a) = \mathbf{x}_a,\ \mathbf{q}(t_a) = \mathbf{q}_a \text{ initial conditions,} \qquad (6)$$
$$\mathbf{x}(t_e) = \mathbf{x}_e,\ \mathbf{q}(t_e) = \mathbf{q}_e \ \text{ terminal conditions,} \qquad (7)$$

where the initial and final times t_a, t_e are free or fixed, s_j are the n_s switching functions and ϕ_j denotes the explicit phase transition conditions (jump maps) occurring at the zeros of one of the switching functions. The Mayer type part Θ of the performance index is a general function of the phase transition times (events) t_i, $i = 0,\ldots,N$, of the continuous $\mathbf{x}(t_i^-)$, $\mathbf{x}(t_i^+)$ and discrete states $\mathbf{q}(t_i^-)$, $\mathbf{q}(t_i^+)$ just before and just after the transition events written as

$$\Theta := \Theta[\ \mathbf{x}(t_0^-),\mathbf{x}(t_0^+),\ldots,\mathbf{x}(t_N^-),\mathbf{x}(t_N^+); \quad \mathbf{q}(t_0^-),\mathbf{q}(t_0^+),\ldots,\mathbf{q}(t_N^-),\mathbf{q}(t_N^+);t_0,\ldots,t_N]\ .$$

Here, $t_a = t_0$, $t_e = t_N$ is assumed while the number of phases N may be given or free. The integrand ψ is a real-valued function of the continuous/discrete state and control variables and of time.

The minimization of (1) is subject to the initial and terminal conditions (6), (7), admissible values for the continuous/discrete control variables (4), and inequality constraints (5). Obviously, valid hybrid optimal trajectories must obey the differential equations (2) and the discrete-based phase transition equations (3).

The optimization parameters to be determined are the continuous $\mathbf{u}(t)$ and discrete control input trajectories $\mathbf{v}(t)$ and all, some, or none of the phase transition times.

3. NUMERICAL SOLUTION STRATEGIES

The basis for the suboptimal solution strategies presented here is the highly efficient direct collocation method implemented in the software package DIRCOL (Stryk, 1999) which can approximately solve optimal control problems through their transcription into sparse, nonlinear programs. The strategy is to use DIRCOL in the inner optimization iteration and other strategies to solve for the combinatorial aspect of the discrete-event in an outer level optimization, see Figure 1. The key to cope with the possibly overwhelming combinatorial complexity of HOCPs is to reduce the number of candidates to be evaluated in the outer iteration.

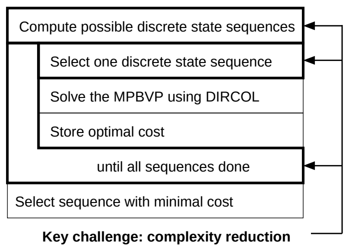

Fig. 1. Outer level optimization iteration.

After providing some insights into the tool DIRCOL, two alternatives HOCP solution strategies will be shown: i) suboptimal solution with interior event time and state constraints fixed on a grid combined with graph search, and ii) branch-and-bound algorithm for mixed-binary-optimal control problems.

3.1 *Sparse Direct Collocation* DIRCOL

The numerical method of sparse direct collocation implemented in DIRCOL can efficiently solve multiphase optimal control problems with a fixed discrete state trajectory. The state $\mathbf{x}$ is approximated by cubic Hermite polynomials $\tilde{\mathbf{x}}(t) = \sum_j \alpha_j \hat{\mathbf{x}}_j(t)$ and the control vector $\mathbf{u}$ by piecewise linear functions $\tilde{\mathbf{u}}(t) = \sum_k \alpha_k \hat{\mathbf{x}}_k(t)$ on a discretization grid $t_i^c = t_1^{(i)} < t_2^{(i)} < \ldots < t_{n_t^{(i)}}^{(i)} = t_i^c$ in each phase. The state differential equations (2) are pointwise fulfilled at the grid points and at the grid midpoints resulting in a set of nonlinear NLP equality constraints $\mathbf{a}(\mathbf{y}) = 0$ (collocation at Lobatto points). The control or state inequality constraints are to be satisfied at the grid points resulting in a set of nonlinear NLP inequality constraints $\mathbf{b}(\mathbf{y}) \ge 0$. The vector $\mathbf{y}$ contains the n_y parameters $\mathbf{y} = (\alpha_1,\alpha_2,\ldots,\beta_1,\beta_2,\ldots,p,t_1^c,\ldots,t_{n_c}^c,t_f)^T$ where $p_i \in [0,1]$, $i = 1,\ldots,n_p$ denotes the set of relaxed binary variables. With ϕ as the parameterized cost index (9), the nonlinearly constrained optimization problem may be written as the NLP

$$\min_{\mathbf{y}} \phi(\mathbf{y}) \quad \text{subject to} \quad \mathbf{a}(\mathbf{y}) = 0,\ \mathbf{b}(\mathbf{y}) \ge 0\,.$$

This NLP can be solved very efficiently with the advanced SQP-based sparse nonlinear program solver SNOPT (Gill *et al.*, 1997); details about DIRCOL may be found in (Stryk, 1999).

3.2 *Suboptimal Solution Technique*

Suboptimal solutions may be obtained by fixing interior point times and states to fixed values on a (fine) grid. Between all these grid points standard optimal control problems with fixed boundary conditions are solved. Finally, the suboptimal solution to the HOCP is obtained by a graph search with each grid point forming nodes and the optimal cost weighing the vertices of this graph. This solution strategy is applied to solve the cooperative multi-arm transport problem in Section 4.1, see also (Buss *et al.*, 2000; Buss, 2001; Denk, 1999). Disadvantages of this approach are the possibly high number of multi-point boundary value problems to be solved and the inherent suboptimality of the obtained solution. On the other hand, an appealing advantage is that by problem understanding one often has good insight as to how the grids need to be specified, and that useful solutions usually can be obtained easily.

3.3 *Branch-and-Bound*

The solution method for mixed-binary optimal control problems (MBOCP) using a combination of sparse direct collocation and branch-and-bound was first presented in (Buss *et al.*, 2000; Stryk and Glocker, 2000). Given certain assumptions, the HOCP may be transformed into a MBOCP with a simple transformation of its discrete variables. For this we assume:

(A1) The number $n_c \geq 0$ of event times t_i^c and, thus, the number n_c+1 of phases are finite and known (this assumption may be circumvented with yet another “outer” iteration to vary n_c).

(A2) The discrete state variable $\mathbf{q}$ and the discrete control variable $\mathbf{v}$ are constant in each phase and may only change at an event t_i^c.

Each discrete variable $q_k(t)$ (or $v_l(t)$), $0 \leq t \leq t_f$, is described by an integer variable $\mathbf{z}_k \in \mathbb{Z}^{n_c+1}$ with $q_k(t) = z_{k,i}$ in the i-th phase. A scalar, integer variable z_1 with given lower and upper bounds $z_1 \in [z_{1,\min}, z_{1,\max}] \subset \mathbb{Z}$ can be transformed into a binary variable $\omega \in \{0,1\}^{n_{z_1}}$ of dimension n_{z_1} by

$$z_1 = z_{1,\min} + \omega_1 + 2^1\omega_2 + \ldots + 2^{n_{z_1}-1}\omega_{n_{z_1}}, \quad (8)$$

with $n_{z_1} = 1 + \mathrm{INT}\{\log(z_{1,\max} - z_{1,\min})/\log 2\}$. In this manner, a binary control vector ω may be used to represent both the unknown discrete state $\mathbf{q}$ in each phase and the discrete control variable $\mathbf{v}$ which controls the order and types of phase transitions.

The MBOCP is to minimize the real-valued, hybrid performance index

$$\begin{aligned} J[\mathbf{u},\omega] = &\sum_{i=1}^{n_c+1} \varphi_{(i)}\left(\mathbf{x}(t_i^c-0), \mathbf{x}(t_i^c+0), \omega, t_i^c\right) \\ &+ \sum_{i=1}^{n_c+1} \int_{t_{i-1}^c}^{t_i^c} L_{(i)}(\mathbf{x}(t), \mathbf{u}(t), \omega, t)\,\mathrm{d}t \qquad (9) \end{aligned}$$

subject to (2)-(7) with the discrete variables $\mathbf{q}$ substituted by the binary control vector $\omega \in \{0,1\}^{n_\omega}$. The solutions of the MBOCP are the optimal (open loop) trajectories of $\mathbf{x}^*(t)$, $\mathbf{u}^*(t)$, $0 \leq t \leq t_f$, the optimal phase transition times t_i^{c*}, the possibly free final time t_f^* , and the optimal binary control vector ω^*.

To avoid solving all $\{0,1\}^{n_w}$ MBOCPs a branch-and-bound strategy in combination with a binary search tree is employed:

(i) Find a global upper bound. Make an initial guess for ω and solve the resulting control problem with ω fixed;

(ii) At the root node, relax all binary variables ($0 \leq \omega_i \leq 1$, $i \in \{1,2,\ldots,n_\omega\}$) and solve to obtain a lower bound to the solution;

(iii) Select the branching variable ω_i and solve both subproblems with that component set to 0 and 1 thereby creating two offspring to the current node;

(iv) Select the next node where to continue the branching process by either: Breadth First Search (node with minimal performance out of those with the least amount of fixed components), Depth First Search (node with minimal performance out of those with the maximum amount of fixed components), Minimum Bound Strategy (node with minimal performance);

(v) If the lower bound in a node is greater than the current best upper bound of the whole search tree, then all subsequent branches from this node are cut off.

4. APPLICATIONS

4.1 *Multi-Arm Transportation Task*

Figure 2 shows a cooperative multi-arm transport task. The square object is initially on the right and is to be transported to the elevated goal position on the left. This is to be accomplished by picking up the object with transport arm 1, handing it over to arm 2, then to arm 3, and finally placing it in the goal position. Each transport arm j has two rotational joints $\theta_{j,i}$ driven by control input torques $u_{j,i}$, $j = 1,2,3$, $i = 1,2$. The effector of each transport arm can be opened/closed to grasp/release the object by a discrete control input v_j. The transportation task should be performed such that the cost index of quadratic power consumption is minimized

$$\min_{u_{j,i}(t), v_j(t)} J = \int_0^T \sum_{j=1}^{3} \sum_{i=1}^{2} (u_{j,i}\,\dot{\theta}_{j,i})^2 \,\mathrm{d}t \,.$$

To solve this HOCP we need to determine the optimal hybrid control trajectories $u_{j,i}^*(t)$, $v_j^*(t)$, the positions, velocities and times of object handover. The physical parameters of the multi-arm system are assumed as: mass $m_1 = m_2 = 5$, length $l_1 = l_2 = 1$ of link 1, 2, respectively, object mass $m_o = 10$, ground distance from arm mount point $x_g = 1.5$. The distance between two arms is $d = 1.5$, the grid points for handover of arm 1 are at $y_{1,ho} = -0.75$, $x_{1,ho} = 1.5/x_{1,ho} = 1$ (ground/air), and likewise for the other arms.

For each arm $i = 1,2,3$ the hybrid model has 4 discrete states $q_i = 1,2,3,4$ as follows: $q_i = 1$: arm has no

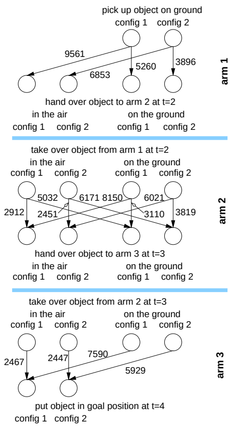

Fig. 3. Feasible handover TPBVPs for each arm.

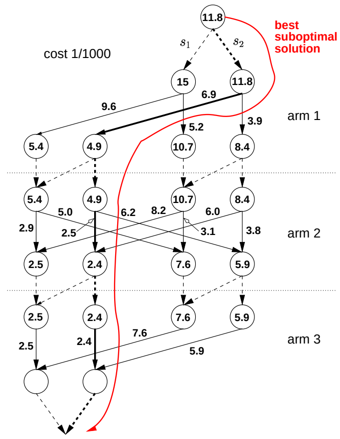

Fig. 4. Graph connecting all feasible discrete sequence candidates.

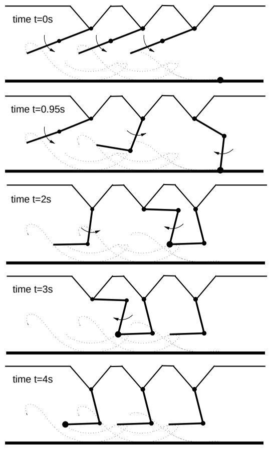

Fig. 5. Snapshot sequence of suboptimal transport solution.

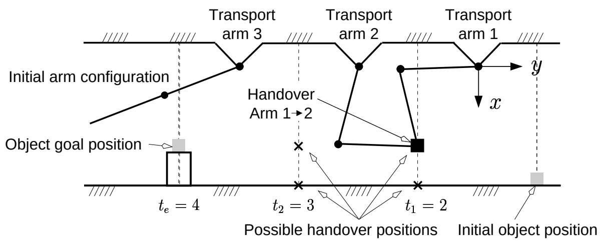

Fig. 2. Cooperative multi-arm transport task.

contact with environment, effector open; $q_i = 2$: arm holds object in configuration 1 (elbow right), object has contact to ground; $q_i = 3$: arm holds object in configuration 2 (elbow left), object has contact to ground; $q_i = 4$: arm holds object in the air, no contact with environment. The variable structure q_i dependent motion differential equation for arm i then are:

$$\dot{\mathbf{x}}_i = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i, q_i) = \begin{cases} \mathbf{f}_1(\mathbf{x}_i, \mathbf{u}_i) & \text{if } q_i = 1 \\ \mathbf{f}_{21}(\mathbf{x}_i, \mathbf{u}_i) & \text{if } q_i = 2 \\ \mathbf{f}_{22}(\mathbf{x}_i, \mathbf{u}_i) & \text{if } q_i = 3 \\ \mathbf{f}_3(\mathbf{x}_i, \mathbf{u}_i) & \text{if } q_i = 4 \end{cases} \quad (10)$$

Note that if $q_i = 2,3$ the arm is also subject to a kinematic equality constraint as ground contact needs to be maintained. Environment forces must also be taken into account during such phases.

Applying the suboptimal solution strategy outlined in Section 3.2, the coupling of the optimal control problems is first eliminiated for each of the transport arms by fixing the possible times and states of handover to constant values on a grid, see Figure 2. The object handover time from arm 1 to 2 is fixed to $t_1 = 2$ and only two possible handover positions (on the ground and in the air) are considered. Some of the handover possibilities can be excluded because of internal arm collision problems, e.g. handover in the air between arms 1, 2 with configuration 2, 1, respectively.

All remaining feasible handover TPBVPs and the cost of the optimal solutions obtained by DIRCOL are shown in Figure 3. The 3 subgraphs are then combined into the complete graph in Figure 4, in which the best suboptimal solution is obtained by minimum path search; also marked in Figure 4.

The best suboptimal solution to the transport task is to pick up the object by arm 1 and hand it over to arms 2/3 in the air at the fixed positions and times as shown in Figure 2. Figure 5 shows some snapshots of the suboptimal coordinated transportation task [1] .

4.2 *Underactuated Robot R2D1*

Now, we consider the trajectory planning example application of a 2-link SCARA robotics arm with two rotational degrees-of-freedom, one in each joint with the torque u_1 in the first joint and a holding brake controlled by $v_1(t) \in \{0,1\}$ in the second joint, see see Figure 6 and (Mareczek *et al.*, 1999) for details. The brake can only be activated when the second joint has reached a zero relative velocity. A discrete control action can switch back and forth between the passive and locked modes for the second joint while a continuous control force is applied to the first joint actuator. We are interested in finding not only the optimal continuous state and control trajectories, but also the optimal discrete strategy composed of the optimal *number* and *times* of the switches necessary to move the R2D1 from a given initial state to a goal state.

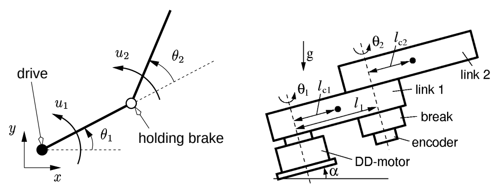

Fig. 6. Kinematic structure of R2D1.

The following $\mathcal{H}_2$ performance index is considered

$$J[u_1, v_1] = \int_0^{t_f} (\mathbf{x}(t) - \mathbf{x}_f)^T \mathbf{W} (\mathbf{x}(t) - \mathbf{x}_f)$$

[1] An animated movie of the suboptimal solution is available at `http://www.rs.tu-berlin.de`

$$+\,\alpha(u_1(t)-u_{1,f})^2\,\mathrm{d}t \qquad (11)$$

where $\mathbf{W}\in\mathbb{R}^{4\times4}$, $\mathbf{W}\geq 0$, and $\alpha>0$. Here, we use $\mathbf{W}=\mathbf{I}$ and $\alpha=1$. Furthermore, $\mathbf{x}_f\in\mathbb{R}^4$ denotes a desired final state, and $u_{1,f}$ is the control value for which the system is at equilibrium at $\mathbf{x}_f$. The final time is constrained, e. g., by $t_f\leq 10\,$s. The HOCP is to minimize J subject to the robot dynamics

$$\begin{aligned}
\ddot{\theta} &= \begin{pmatrix} u_1 \\ 0 \end{pmatrix} - v_1(t)\,\mathbf{F}_1(\theta(t),\dot{\theta}(t)) \\
&\quad -(1-v_1(t))\,\mathbf{F}_2(\theta(t),\dot{\theta}(t)) \qquad (12)\\
\mathbf{F}_i(\theta,\dot{\theta}) &= \mathbf{M}_i^{-1}(\theta)(\mathbf{C}_i(\theta,\dot{\theta})+\mathbf{g}_i(\theta)+\mathbf{r}_i\theta),\\
i &= 1,2,
\end{aligned}$$

$$\begin{aligned}
&\mathbf{x}(t)=(\theta(t),\dot{\theta}(t))\\
&\mathbf{x}(0)=\mathbf{x}_0=(1.2,\ 0,\ 0.8,\ 0)^T,\\
&\mathbf{x}(t_f)=\mathbf{x}_f=(\pi/2,\ 0,\ -\pi/2,\ 0)^T,\\
&v_1(t_f)=2 \quad \text{(brake on)},
\end{aligned}$$

where $\mathbf{M}_i$ are the mass-inertia matrices for each dynamical configuration, $\mathbf{C}_i$ are the vectors of Coriolis and centrifugal forces, $\mathbf{g}_i$ are the vectors of gravitational forces, and $\mathbf{r}_i$ are the friction forces. The physical parameters in standard units are: $l_1=0.300$, $l_{c1}=0.206$, $l_{c2}=0.092$, $I_1=0.430$, $I_2=0.127$, $m_1=10.2$, $m_2=5.75$.

The optimal control problem for R2D1 is formulated as a MBOCP, and the numerical approach discussed in Section 3.3 is applied. The time $t_f\leq 10$ is initially divided into a fixed number $m=8$ of phases, though the intermediate times corresponding to the phase transitions may vary freely. Included in the problem formulation are a set of constant, unknown binary parameters $p_i\in\{0,1\}$, $i=\{1,\ldots,n_p\}$ which are related to the unknown binary variables ω_i. They determine the total number of switches and indicate at which of the pre-defined phase transitions a switch occurs. The first component p_1 indicates in which discrete state the system starts, $\{p_1=0$, brake off; $p_1=1$, brake on$\}$. The remaining components of p are a binary representation for the total number of switches taking place during the time interval. For example, if five switches occur beginning with the brake off, then $p=[p_1\ p_2\ p_3\ p_4]=[0\ 1\ 0\ 1]$ and the switches are assigned to the predefined phase transitions using the scheme:

$p_k=1:\quad 2^{(n_p-k)}$ switches with one every 2^{k-1} phase transitions beginning with no. $2^{(k-2)th}+1$.

The branch-and-bound search strategy was used together with a minimum-bound node selection strategy. Figure 7 displays the complete binary search path for the problem. An initial solution with p fixed at $[0\ 1\ 0\ 0]$ (4 switches) is first calculated to obtain a upper bound of $J^*=41.156$. Lower bounds were first calculated for the root node's children, and the second binary variable is arbitrarily first selected as the branching variable. The final optimal solution has a discrete solution of $p^*=[0\ 1\ 1\ 1]$ corresponding to 7 switches starting with the brake off and an objective value of $J^*=39.062$. Normally by such a branch-and-bound search, if an integer solution is obtained when solving a relaxed problem which provides the best lower bound, the search procedure ends. In this case, our optimal solution was obtained already at node 2, after the third optimization run. The search though was continued here to verify the solution and ensure that it did not correspond to a local minimum.

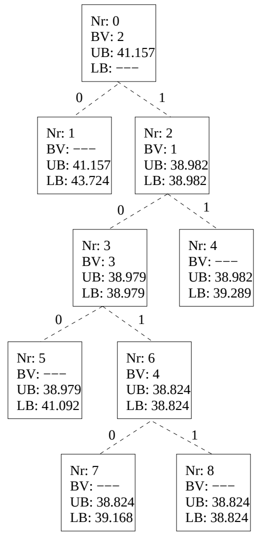

Fig. 7. Branch-and-bound search using *minimum bound* strategy. *Nr* – node number from search order, *BV* – branching variable, *UB* – global upper bound, *LB* – lower bound for branch.

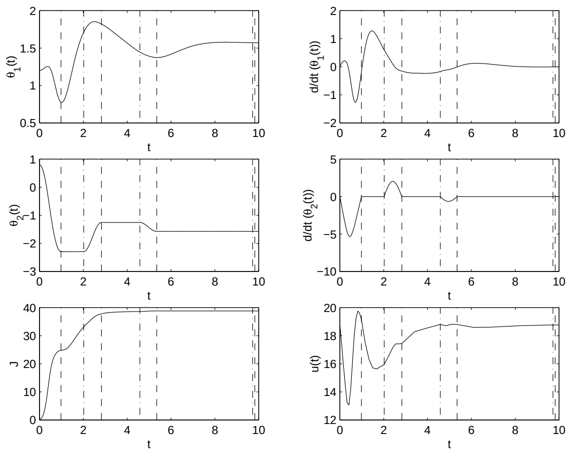

Fig. 8. Final optimal hybrid switching solution with 7 switches.

The final solution displayed in Figure 8 has an optimality error of $\tilde{w}=0.567$ [2]. The incremental difference in the objective decreases rapidly with an increasing number of switches: $N_s=0,2,4,5,6,7$, cost $J^*(N_s)=104.09, 43.724, 41.157, 41.092, 39.168, 38.824$, respectively. The average computational time by DIRCOL for each optimal control problem (the solution at a given node) was 19.6 seconds on a Pentium III 500 MHz computer, the average grid size used was $\sum_{i=1}^{n_c+1} n_t^{(i)}=56.3$, and the average NLP dimension was $n_y=278$, $n_a=230$.

[2] An animated movie of the final solution is available at `http://www.sim.informatik.tu-darmstadt.de`

5. CONCLUSIONS

A class of hybrid (discrete-continuous) optimal control problems has been defined and solution strategies have been proposed. The first approach decouples HOCPs by fixing interior point time and state constraints to a grid of possible values. Useful grid assumptions are likely to be available from problem insight. Solutions to the decoupled TPBVPs may be obtained with available numerical software, and their optimal costs are assigned to a graph whose nodes represent the grid points and vertices the optimal cost. In this graph the best suboptimal solution is found by minimum path search. Alternatively, a bround-and-bound strategy is proposed based on the decomposition of HOCPs into MBOCPs. Binary variables are successively relaxed to obtain upper and lower bounds on the solutions. The search in the resulting solution tree is performed with branch-and-bound. Two robotic applications of a cooperative multi-arm transport task and a trajectory planning problem of the underactuated robot R2D1 have been presented to validate the two suboptimal solution strategies for HOCPs.

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