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STABILIZATION OF OSCILLATIONS IN THE INVERTED PENDULUM

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Abstract: This paper addresses the problem of obtaining robust and stable oscillations in an electromechanical system. These oscillations are associated to a limit cycle that is born through a supercritical Hopf bifurcation. The method proposed in the paper works well for fully actuated systems, and even for certain underactuated ones. In order to illustrate the method, we have chosen an underactuated system that is well known in the literature and in control systems laboratories: the inverted pendulum. Actual stable and robust oscillations have been obtained experimentally in a rotating Furuta pendulum.

Keywords: Inverted pendulum, Non-linear Oscillations, Hopf bifurcation.

1. INTRODUCTION

This paper focuses on the possibility of leading an electromechanical system to a state of oscillation by means of an appropriate control law. To accomplish this, the proposed methodology matches the original system with a generalized hamiltonian system (Van der Schaft, 1989) that is able to exhibit robust oscillations. The new system shows very interesting behaviors because it may undergo a supercritical Hopf bifurcation when a parameter takes suitable values. Therefore, the system can display robust oscillations associated with a limit cycle. The method introduced consists in finding a control law that matches the open loop system with that desired behavior.

In a previous paper (Aracil *et al.*, 1998) a Hopf bifurcation was detected in an inverted pendulum. However, the limit cycle born through this bifurcation had no physical meaning as it was associated to positive damping. In the family of systems considered in the present paper the limit cycle is easily implementable in an experimental framework.

The method used to get the control law belongs to the family of energy shaping methods (Ortega *et al.*, 2001). Currently, these methods try to find control laws that drive the controlled system to an isolated equilibrium point. However, here our goal is to reach a closed curve (a limit cycle) that produces a stable oscillating behavior. It is worthy to mention that this limit cycle is born through a supercritical Hopf bifurcation. Therefore, for certain values of the bifurcation parameter, the system has an attractor point, which is the case normally considered in conventional energy shaping control systems. Nevertheless, for other values of that parameter the limit set of the dynamical system changes to a limit cycle and, therefore, the system oscillates in a stable and robust way.

Other attempts have been reported to obtain oscillations in nonlinear systems, see (Fradkov and Pogromsky, 1998) and references therein. However, to the best of our knowledge, the generalized hamiltonian systems formalism has not been considered in the literature to deal with limit cycles.

The method proposed in the current paper works well for fully actuated second-order systems, and even for certain underactuated ones. In order to illustrate the method we have been chosen an underactuated system that is well known in the literature and in control systems laboratories: the inverted pendulum.

The paper is organized as follows. In Section 2, a generalized hamiltonian system that exhibits a pertinent supercritical Hopf bifurcation is introduced. This system will be used as the desired closed loop behavior. In Section 3, a control law that matches the behavior of the pendulum on a cart with that desired system behavior is obtained. The case of the rotating Furuta pendulum is also analyzed both by simulation and on an experimental framework. In Section 4 conclusions are given. Finally, the analysis of the limit cycle is performed in the Appendix.

2. OSCILLATIONS IN A GENERALIZED HAMILTONIAN SYSTEM

In this section a generalized hamiltonian system (Van der Schaft, 1989), which presents a supercritical Hopf bifurcation, is introduced. The generalized hamiltonian system formalism is particularly well suited to solve the problems of designing controllers by energy shaping. With this formalism the desired closed loop behavior can be stated as follows. Define the hamiltonian function

$$H_d = \frac{1}{4}(\omega_c^2 x^2 + \dot{x}^2)^2 - \frac{\mu}{2}(\omega_c^2 x^2 + \dot{x}^2).$$

Making $x_1 = x$ and $x_2 = \dot{x}$, using the generalized hamiltonian system formalism and including damping it is obtained

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & \dfrac{1}{\omega_c^2 x_1^2 + x_2^2 - \mu} \\ -\dfrac{1}{\omega_c^2 x_1^2 + x_2^2 - \mu} & -k_a \end{bmatrix} \begin{bmatrix} D_{x_1} H_d \\ D_{x_2} H_d \end{bmatrix} \qquad (1)$$

being $k_a > 0$ the damping coefficient.

As $D_{x_1} H_d = \omega_c^2 x_1 (\omega_c^2 x_1^2 + x_2^2 - \mu)$ and $D_{x_2} H_d = x_2(\omega_c^2 x_1^2 + x_2^2 - \mu)$, the system can be expressed as

$$\ddot{x} = -\omega_c^2 x - k_a(\omega_c^2 x_1 + \dot{x} - \mu)\dot{x}. \qquad (2)$$

This is a very interesting system which undergoes a

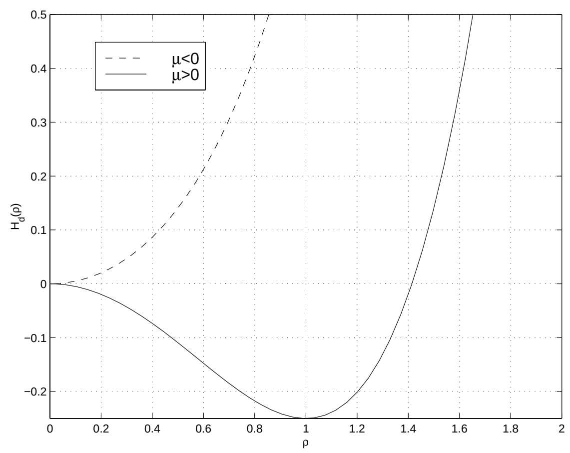

Fig. 1. Curve $H_d(\rho)$, for $\rho = \sqrt{\omega_c^2 x_1^2 + x_2^2}$

Hopf bifurcation when $\mu = 0$. The geometrical shape of H_d as a function of $\rho = \sqrt{\omega_c^2 x_1^2 + x_2^2}$ for different values of parameter μ is shown in Fig. 1. The shape of $H_d(x_1, x_2)$ can be obtained by means of a ellipsoidal

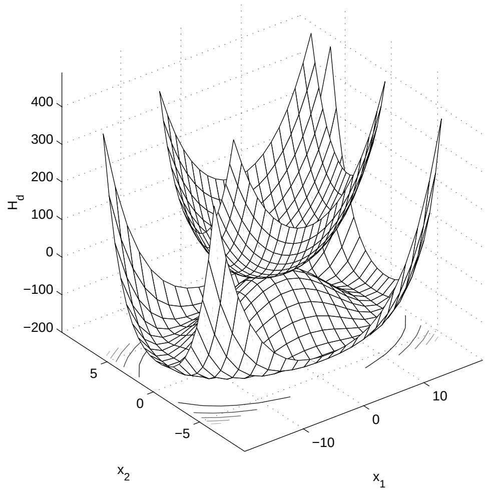

Fig. 2. Hamiltonian function for closed-loop system. The upper surface with a single minimum point corresponds to $\mu < 0$; and the lower surface with a minimum set formed by a closed curve corresponds to $\mu > 0$.

rotation of these curves (Fig. 2). From this figure it is clear that for $\mu < 0$ H_d has a single minimum at the origin, but for $\mu > 0$ the minimum turns into a maximum, and the minimum now is reached in a closed elipsoidal curve that surrounds the origin. These shapes give an intuitive geometrical insight into the expected system behaviors. They provide the desired energy shape that will be the objective of our control problem.

The linearization of system (1) at the origin is given by

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_c^2 & k_a\mu \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \qquad (3)$$

whose characteristic polynomial is $\lambda^2 - k_a\mu\lambda + \omega_c^2$ and the corresponding eigenvalues are

$$\lambda_{1,2} = \frac{k_a\mu \pm \sqrt{(k_a\mu)^2 - 4\omega_c^2}}{2}.$$

Then, for $\mu < 0$, we have $\mathrm{Re}[\lambda] < 0$ and the origin is stable; for $\mu = 0, \lambda = \pm j\omega_c$ and the origin is a center; and for $\mu > 0$, $\mathrm{Re}[\lambda] > 0$ and the origin is unstable. Therefore the system has a single equilibrium at the origin for $\mu < 0$, and for $\mu > 0$ this equilibrium becomes unstable and the trajectories tend to a limit cycle. Therefore, a supercritical Hopf bifurcation (Hale and Koçak, 1991; Kuznetsov, 1995) is produced for $\mu = 0$, where the stability of the origin changes from stable to unstable.

The conditions for a Hopf bifurcation are fulfilled since

$$\frac{d\mathrm{Re}[\lambda(\mu)]}{d\mu}\Big|_{\mu=0} = \frac{k_a}{2} \neq 0.$$

The initial period (of the zero-amplitude oscillation) is

$$T_0 = \frac{2\pi}{\omega_c}.$$

This gives an approximate value of the period of the expected oscillations, approximation which is reasonably valid for small values of μ.

It is interesting to note that for $k_a = 0$ system (2) reduces to $\ddot{x} = -\omega_c^2 x$ that has a state portrait formed by cycles that fill the whole state space. However these oscillating behaviors are not structurally stable. The effect of the damping is just to select from all these cycles a single one given by a limit cycle that produces stable and robust oscillations.

The fact that there is actually a limit cycle for $\mu > 0$ is easily checked by the Poincaré-Bendixon criterion. Consider in the (x_1, x_2) plane the family of curves $\rho^2 = \omega_c^2 x_1^2 + x_2^2$, where $\rho \geq 0$ is a radius-like coordinate. Then, it is straightforward to show that

$$\rho\dot{\rho} = -k_a(\rho^2 - \mu)x_2^2$$

which means that for $\rho < \sqrt{\mu}$, $\dot{\rho} > 0$ and then ρ grows. On the other hand for $\rho > \sqrt{\mu}$, $\dot{\rho} < 0$ and then ρ decreases. The curve corresponding to $\rho = 0$ is invariant ($\dot{\rho} = 0$) and, therefore, is a limit cycle. This limit cycle corresponds to the closed curve defined by the minimum of H_d in Fig. 2. For a more detailed discussion of this limit cycle see the Appendix.

3. APPLICATION TO THE PENDULUM ON A CART

The system introduced in the previous section, which presents robust oscillations, suggests a method to get controlled systems that fit into it. The aim is to design a controller that matches the original open loop system with the behavior given by Eq. (1). The method will be presented by its application to the pendulum on a cart, which is one of the most studied cases of underactuated control systems.

The main physical parameters of the system are M and m, which stand for the masses of the cart and the pendulum respectively, and l that is the distance from the pivot of the pendulum to its center of mass. For the sake of completeness the equations are included. The Lagrangian of the pendulum on a cart is given by

$$L = \frac{1}{2}(\alpha\dot{\theta}^2 + 2\beta\cos\theta\dot{\theta}\dot{s} + \gamma\dot{s}^2) - \omega_0^2\cos\theta, \quad (4)$$

where θ is the angle of the pendulum with respect to the upright position and s is the linear displacement of the cart. The parameters are $\alpha = ml^2$, $\beta = ml$, $\gamma = M + m$, and $\omega_0^2 = mgl$. From Eq. (4) the Euler-Lagrange equations become

$$\alpha\ddot{\theta} + \beta\cos\theta\ddot{s} - \omega_0^2\sin\theta = 0 \quad (5)$$

$$\beta\cos\theta\ddot{\theta} + \gamma\ddot{s} + \beta\sin\theta\dot{\theta}^2 = u, \quad (6)$$

where u is the force applied to the cart.

In the following, a control law $u(\theta, \dot{\theta}, s, \dot{s})$ will be obtained so that system (5) and (6) matches the desired closed loop system given by Eq. (1). In this way, the pendulum will oscillate around the upright position when $\mu > 0$.

The first step is to partially linearize (Khalil, 1996) the open loop system given by Eqs. (5) and (6). Equation (5) can be written as

$$\ddot{\theta} = -\frac{1}{\alpha}(\beta\cos\theta\ddot{s} - \omega_0^2\sin\theta), \quad (7)$$

which together with (6) leads to

$$\gamma\ddot{s} + \frac{\beta\cos\theta}{\alpha}(\omega_0^2\sin\theta - \beta\cos\theta\ddot{s}) - \beta\sin\theta\dot{\theta}^2 = u, \quad (8)$$

and so to

$$\ddot{s}\left(\gamma - \frac{\beta^2\cos^2\theta}{\alpha}\right) + \frac{\beta\omega_0^2}{\alpha}\sin\theta\cos\theta - \beta\sin\theta\dot{\theta}^2 = u. \quad (9)$$

Making $\ddot{s} = v$ in this last equation it yields

$$v = \frac{-\frac{\beta\omega_0^2}{\alpha}\sin\theta\cos\theta + \beta\sin\theta\dot{\theta}^2 + u}{\left(\gamma - \frac{\beta^2\cos^2\theta}{\alpha}\right)} \quad (10)$$

which defines a partial linearizing controller. This controller converts Eqs. (5) and (6) into

$$\alpha\ddot{\theta} - \omega_0^2\sin\theta = -\beta\cos\theta v \quad (11)$$

$$\ddot{s} = v, \quad (12)$$

which are the partially linearized form of the equations of the pendulum on a cart. The form of these equations is very nice as variables θ and s have been decoupled. The problem of controlling θ has been made independent of the one of controlling s.

The desired closed loop behavior is given by Eq. (2), where x should be replaced by θ. The open loop is given by Eq. (11), that can be written as

$$\ddot{\theta} = \frac{\omega_0^2}{\alpha}\sin\theta - \frac{\beta}{\alpha}\cos\theta v. \quad (13)$$

Matching the open loop with the desired closed loop behaviors (Eqs. (13) and (2) respectively) yields

$$-\omega_c^2\theta - k_a(\omega_c^2\theta^2 + \dot{\theta}^2 + \mu)\dot{\theta} = \frac{\omega_0^2}{\alpha}\sin\theta - \frac{\beta}{\alpha}\cos\theta v,$$

which leads to

$$v = \frac{\alpha}{\beta\cos\theta}\Big(\omega_c^2\theta + \frac{\omega_0^2}{\alpha}\sin\theta + k_a(\omega_c^2\theta^2 + \dot{\theta}^2 + \mu)\dot{\theta}\Big). \quad (14)$$

This law is not valid at $\theta = \pm\pi/2$, but it is for all $\theta \neq \pm\pi/2$.

Therefore, applying control law (14) to system (11) oscillations ruled by Eq. (2) are obtained. It should be noticed that the pendulum oscillates around the upright position.

3.1 *Oscillations in the controlled variable*

So far, only Eq. (11), which is related with variable θ, has been taken into account. When the obtained control law (14) is applied, the movement of the cart is given by

$$\ddot{s} = \frac{\alpha}{\beta\cos\theta}\Big(\omega_c^2\theta + \frac{\omega_0^2}{\alpha}\sin\theta + k_a(\omega_c^2\theta^2 + \dot{\theta}^2 + \mu)\dot{\theta}\Big). \tag{15}$$

By simulation, it can be seen that variable s presents oscillations together with a drift.

In order to eliminate such drift, the following modification of control law (14) is proposed

$$v = \frac{\alpha}{\beta\cos\theta}\Big(\omega_c^2\theta + \frac{\omega_0^2}{\alpha}\sin\theta + k_a(\omega_c^2\theta^2 + \dot{\theta}^2 + \mu)\dot{\theta}\Big) + k_s\dot{s}, \tag{16}$$

with $k_s > 0$.

Figures 3 and 4 show the results of the simulations with $\mu = -15$, $k_a = 0.2$ and $k_s = 0.01$. The systems parameters were chosen as: $M = 0.44$ Kg, $m = 0.14$ Kg and $l = 0.215$ m. This behavior is satisfactory since both variables θ and s oscillate without drift. Nevertheless, variable s oscillates around a position that has not be pre-specified.

Another interesting result is obtained for negative values of μ. In this case control law (16) stabilizes the pendulum at the upright position.

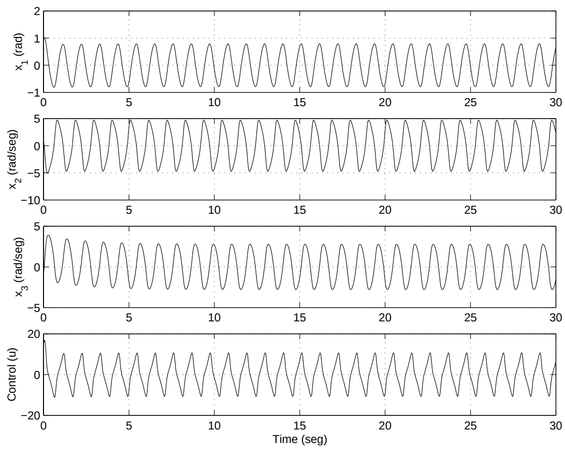

Fig. 3. Pendulum on a cart. Simulations with initial conditions $[x_1, x_2, x_3] = [\theta, \dot{\theta}, \dot{s}] = [1, 1, -1]$.

3.2 *Extension to the Furuta pendulum*

The same procedure can be applied to the case of the rotating Furuta pendulum (Åström and Furuta, 1996). For this last pendulum, matching the open and closed loop equations we have

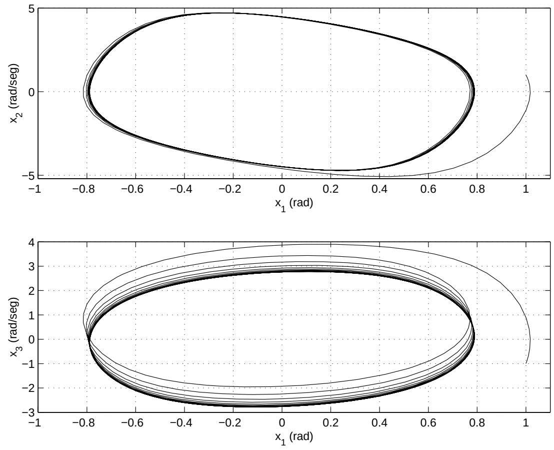

Fig. 4. Phase portraits of the pendulum on a cart. Simulations with initial conditions $[x_1, x_2, x_3] = [\theta, \dot{\theta}, \dot{s}] = [1, 1, -1]$.

$$-\omega_c^2\theta - k_a(\omega_c^2\theta^2 + \dot{\theta}^2 + \mu)\dot{\theta} = \frac{\omega_0^2}{\alpha}\sin\theta + \frac{1}{\alpha}\sin\theta\cos\theta\dot{\varphi}^2 - \frac{\beta}{\alpha}\cos\theta v.$$

Therefore, the control law is

$$v = \frac{\alpha}{\beta\cos\theta}\Big(\omega_c^2\theta + \frac{\omega_0^2}{\alpha}\sin\theta + k_a(\omega_c^2\theta^2 + \dot{\theta}^2 + \mu)\dot{\theta}\Big) + \frac{1}{\beta}\sin\theta\dot{\varphi}^2, \tag{17}$$

that is, the same law as in Eq. (14) but with the additional term $\frac{1}{\beta}\sin\theta\dot{\varphi}^2$, which can be interpreted as a cancellation of the perturbation introduced in the rotating pendulum by the rotating effects.

In order to eliminate the drift the following law is proposed

$$v = \frac{\alpha}{\beta\cos\theta}\Big(\omega_c^2\theta + \frac{\omega_0^2}{\alpha}\sin\theta + k_a\,(\omega_c^2\theta^2 + \dot{\theta}^2 + \mu)\dot{\theta}\Big) + \frac{1}{\beta}\sin\theta\dot{\varphi}^2 + k_\varphi\dot{\varphi}, \tag{18}$$

which also gives satisfactory results in simulations and experiments. In the following, the results of two real experiments are presented. Figures 5 and 6 show the results of experiment 1, in which the initial value of parameter μ is -15 and at $t = 3$ sec. it is changed to 15. It can be seen that for this last value of μ the system begins to swing and reaches a stable oscillation. The period is $T_0 \approx 0.5$ sec. Figure 7 shows a sequence of actual pictures obtained by experimentation corresponding to one period of the oscillations.

The objective of experiment 2 is to show the robustness of the oscillations. In this experiment an external disturbance on the system is introduced at $t \approx 5$ sec. The results are shown in Figs. 8 and 9. As in the case of the pendulum on a cart, for $\mu < 0$ control law (16) is able to stabilize the pendulum at the upright position.

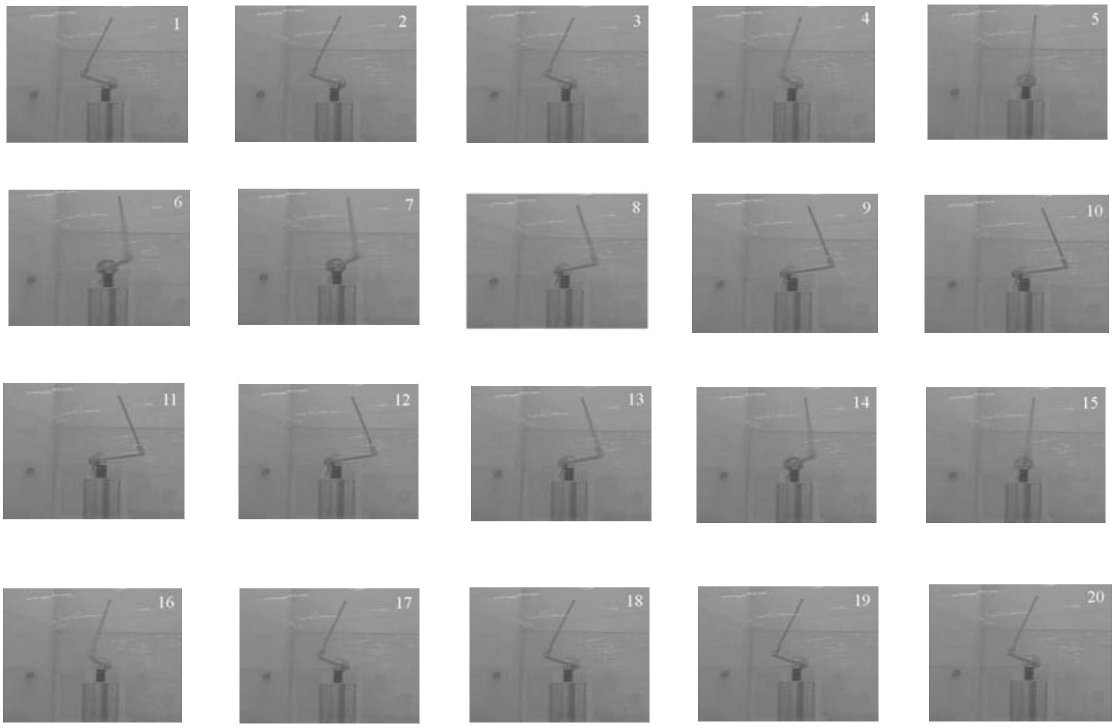

Fig. 7. Sequence of pictures showing one period of the oscillations for a experiment 1 when $\mu = 15$. The period is $T_0 \approx 0.5$ sec.

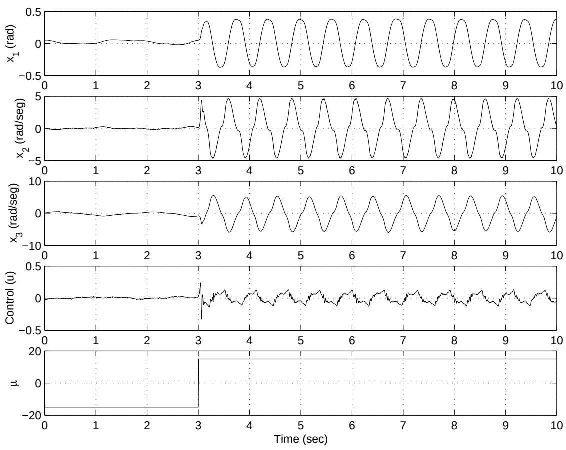

Fig. 5. Time response for experiment 1.

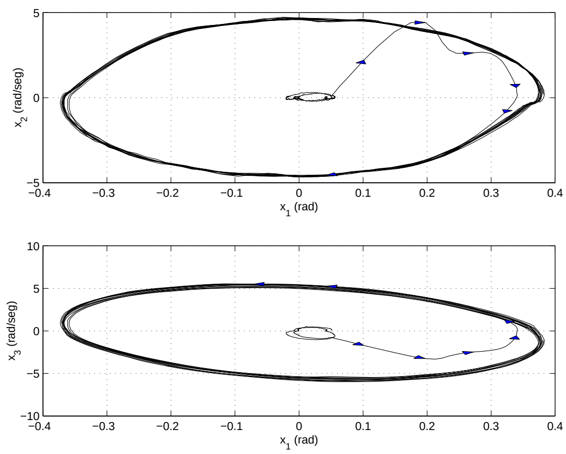

Fig. 6. Phase portraits for experiment 1.

4. CONCLUSIONS

In this paper, we have presented a technique for obtaining stable and robust oscillations around the upright position in an inverted pendulum. To accomplish this a control law has been introduced that drives the system to a stable limit cycle. This control law belongs to the family of the energy shaping methods. The limit cycle is associated to the occurrence of a Hopf bifurcation in a generalized hamiltonian system. The results have been checked both by simulation and by experimentation on an actual rotating pendulum.

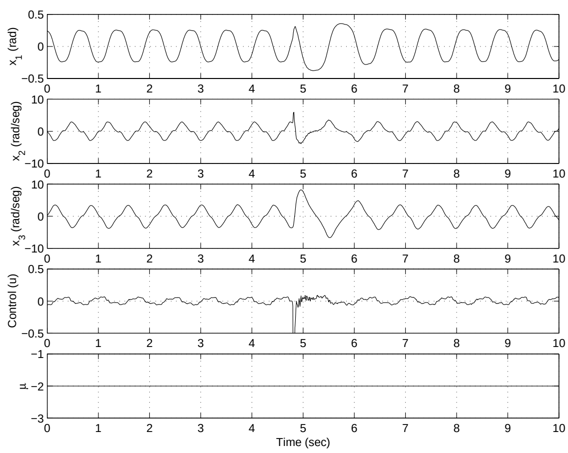

Fig. 8. Time response for experiment 2. A disturbance is added at $t \approx 5$ sec.

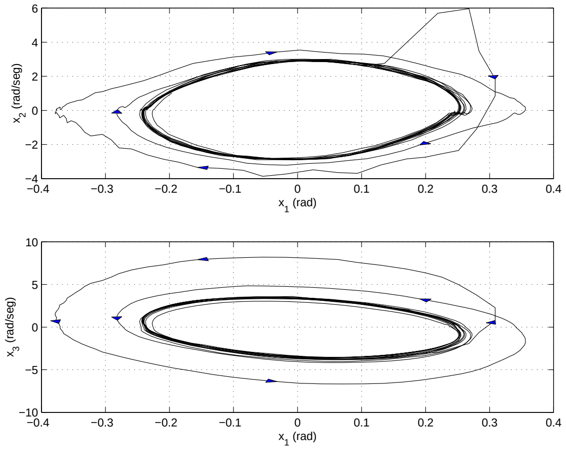

Fig. 9. Phase portraits for experiment 2. A disturbance is added at $t \approx 5$ sec.

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APPENDIX

Equation (2) can be written as

$$\dot{x}_1 = x_2 \tag{19}$$

$$\dot{x}_2 = -\omega_c^2 x_1 - k_a(\omega_c^2 x_1^2 + x_2^2 - \mu)x_2. \tag{20}$$

This system has as limit sets the point $(x_1, x_2) = (0,0)$ and, when $\mu > 0$, the closed curve $\Gamma = \{x_1, x_2 | \omega_c^2 x_1^2 + x_2^2 = \mu\}$. In the following, it is shown that curve Γ is a stable limit cycle for $\mu > 0$. Applying the quasi-polar transformation

$$x_1 = \frac{\rho}{\omega_c}\cos\eta \quad x_2 = \rho\sin\eta$$

to Eqs. (19) and (20) yields

$$\dot{\rho}\cos\eta - \rho\sin\eta\,\dot{\eta} = \omega_c\rho\sin\eta$$

$$\dot{\rho}\sin\eta + \rho\cos\eta\,\dot{\eta} = -\omega_c\rho\cos\eta - k_a(\rho^2 - \mu)\rho\sin\eta$$

Adding the first equation multiplied by $\cos\eta$, to the second one multiplied by $\sin\eta$ yields

$$\dot{\rho} = -k_a(\rho^2 - \mu)\rho\sin^2\eta \tag{21}$$

Similarly, adding the first equation multiplied by $\sin\eta$, to the second one multiplied by $-\cos\eta$ yields

$$\dot{\eta} = -\omega_c - k_a(\rho^2 - \mu)\rho\sin\eta\cos\eta \tag{22}$$

From Eq. (21) it is clear that $\rho \to \sqrt{\mu}$. Furthermore Eq. (22) shows that this happens in a monotone form.

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