15th Triennial World Congress, Barcelona, Spain

POLES AND ZEROS OF MULTIVARIABLE LINEAR TIME-VARYING SYSTEMS

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Abstract: In the paper it is shown that the pole set of a linear system can be calculated by suitable polynomial factorizations. The theoretical issues related to poles and zeros of time-varying systems are discussed. Further, it is shown how the poles and zeros can be defined starting from a state-space realization of the input-output system.

Keywords: Time-varying systems, Linear systems, Poles, Polynomial transforms.

1. INTRODUCTION

The concept of poles and zeros of linear time-varying systems is not an obvious extension to the classical theory of time-invariant systems. For example, it is well known that the natural way of defining poles at each time instant from the "frozen" system matrix leads to a pole function, from which even the stability properties cannot generally be deduced (Rugh, 1993). Another way to define poles (or more specifically pole sets) was used by (Kamen, 1988) who used factorizations of operator polynomials to define the pole sets. Based on this analysis he obtained conditions for the stability of the system.

The method introduced by Kamen actually used the algebra of skew polynomials, which is a natural tool in the analysis of non-commutative rings. Kamen, however, did not formulate his results in a tight algebraic form, but merely used simple examples to show the idea, after which he generalized it to a more general setting by using pure calculus. Another way would be to use polynomial methods like that described in (Blomberg and Ylinen, 1983), and specifically in the time-varying case by (Ylinen, 1980).

Another approach to the problem would be to use state-space formalism and state transformations to study the stability of the system. The well-known theory of Lyapunov transformations (Lyapunov, 1966) is a powerful tool in this respect, because this transformation is known to preserve the stability properties of the original system in spite of the change of the state variable. The difficulty is to find a suitable Lyapunov transformation, which would make it possible to analyze the stability of the original system. For example, (Zenger and Orava, 1996) showed that any time-varying system matrix of a continuous linear state-space representation can be changed into a constant matrix, but the needed state transformation depends on the state-transition matrix, which is generally impossible to solve analytically. Hence it is not possible to know, whether the transformation is a Lyapunov transformation or not. The topic has been elaborated more extensively e.g. in (Harris and Miles, 1980).

In this paper the main focus is in continuous time linear systems following the ideas of (Kamen, 1988) and (O'Brien and Iglesias, 1998). It is shown that the factorization by Kamen to obtain the poles is only one special case from a more general approach, in which all Lyapunov transformations can be used to define the pole sets. The current work inspired by the papers of (O'Brien and Iglesias, 1998) and (Kamen, 1988) and is thus a natural extension to previous results; however it uses the state-space formalism instead of input-output representations.

2. POLES AND ZEROS OF TIME-INVARIANT SYSTEMS

Consider a time-invariant *SISO* input-output differential system

$$\sum_{i=0}^{n} a_i \frac{d^i y(t)}{dt^i} = \sum_{j=0}^{n} b_j \frac{d^j u(t)}{dt^j} \tag{1}$$

With input $u \in X$ and output $y \in X$ it can be described as

$$a(p)y = b(p)u \tag{2}$$

where $a(p)$ and $b(p)$ are polynomials over the complex field $\mathbf{C}$ in the differential operator p on a suitable signal space X. In multivariable (MIMO) case the corresponding *input-output description* is

$$A(p)y = B(p)u \tag{3}$$

where $A(p)$ and $B(p)$ are polynomial matrices with $\det A(p) \neq 0$ (Blomberg and Ylinen, 1983). Obviously, the system

$$\begin{aligned} S &= \{(u,y) | A(p)y = B(p)u\} \\ &= (\ker[A(p) \mid -B(p)])^{-1} \end{aligned} \tag{4}$$

is uniquely determined by the generator $[A(p) \mid -B(p)]$. On the other hand, there can be infinitely many generators $[A(p) \mid -B(p)]$ for the same system. Two generators determine the same system if and only if they are *row equivalent* as polynomial matrices i.e. they can be obtained from each other by premultiplication with a *unimodular matrix*,(Blomberg and Ylinen, 1983). Furthermore, the generators of some *canonical form* (e.g.*Canonical Upper Triangular Form*) for the row equivalence are *unique* descriptions of the system, (Blomberg and Ylinen, 1983)

The system can also be *decomposed* to a *state space description*

$$\begin{bmatrix} pI - A & 0 \\ -C & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} B \\ D \end{bmatrix} u \tag{5}$$

i.e.

$$S = \left\{ (u,y) \,|\, \exists x \begin{bmatrix} px = Ax + Bu \\ y = Cx + Du \end{bmatrix} \right\} \tag{6}$$

Conversely, the input-output description $[A(p) \mid -B(p)]$ can be obtained from the state space description by bringing the equations (5) to a row equivalent upper triangular form

$$\begin{bmatrix} A_1(p) & A_2(p) \\ 0 & A(p) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} B_1(p) \\ B(p) \end{bmatrix} u \tag{7}$$

by *elementary row operations.* The state space description is *completely observable* if $A_1(p)$ is unimodular.

Consider the input-output description

$$A(p)y = B(p)u, \;\; \det A(p) \neq 0 \tag{8}$$

and suppose that the input u is of the form

$$u(t) = u_0 e^{\lambda t} \tag{9}$$

where u_0 is a constant vector and $\lambda \in \mathbf{C}$. Suppose that the output y is of the same form

$$y(t) = y_0 e^{\lambda t} \tag{10}$$

Then

$$A(\lambda) y_0 e^{\lambda t} = B(\lambda) u_0 e^{\lambda t} \tag{11}$$

Because $e^{\lambda t} \neq 0$ for all $t \in \mathbf{R}$, it can be cancelled so that

$$A(\lambda) y_0 = B(\lambda) u_0 \tag{12}$$

If $\det A(\lambda) \neq 0$ then y_0 can be solved

$$y_0 = A(\lambda)^{-1} B(\lambda) u_0 \tag{13}$$

Suppose now that $\lambda \in \mathbf{C}$ is such that

$$\text{rank} B(\lambda) < \text{rankB}(p) \tag{14}$$

Then there exist $u_0 \neq 0$ and $y_0 = 0$ satisfying (12). This kind of λ is a *zero of the system* and $u(t) = u_0 e^{\lambda t}$ is the corresponding *input mode.* Note that the premultiplication of a matrix by a unimodular matrix does not change the rank so that zeros are not depending on the description.

On the other hand, if

$$\text{rank} A(\lambda) < \text{rank} A(p) \Longleftrightarrow \det A(\lambda) = 0 \tag{15}$$

then there exist $u_0 = 0$ and $y_0 \neq 0$ satisfying (12), i.e. the equation (2) has nonzero solutions y_0, even though $u_0 = 0$. Then λ is a *pole of the system* and $y(t) = y_0 e^{\lambda t}$ is the corresponding *output mode.*

In the SISO case (2) the poles and zeros are simply the zeros (roots) of the *polynomial functions* $\lambda \mapsto a(\lambda), \quad \lambda \mapsto b(\lambda)$, respectively.

Suppose that the system (3) has $n = \deg(\det A(p))$ *distinct* poles $\lambda_1, \lambda_2, ..., \lambda_n$ with corresponding solutions $y_{01}, y_{02}, ..., y_{0n}$. Then an arbitrary solution y of equation $A(p)y = 0$ can be written as

$$y(t) = \sum_{i=1}^{n} c_i y_{0i} e^{\lambda_i t} \tag{16}$$

Thus the system is *asymptotically stable* if and only if the poles are located in the *open left half-plane* of the complex plane. The case of multiple poles is more complicated and is omitted here.

Consider next the state space description (5). According to the MIMO case above, the poles and zeros can be obtained from

$$\begin{bmatrix} \lambda I - A & 0 \\ -C & I \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} B \\ D \end{bmatrix} u_0 \tag{17}$$

The (internal) *poles of the state space description* are the solutions of

$$\det(\lambda I - A) = 0 \tag{18}$$

i.e. the *eigenvalues* of A. It is interesting to note that the state space description *does not have zeros* at all. From the equivalent equation

$$\begin{bmatrix} A_1(\lambda) & A_2(\lambda) \\ 0 & A(\lambda) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} B_1(\lambda) \\ B(\lambda) \end{bmatrix} u_0 \tag{19}$$

it is seen that the poles of the system are also poles of the state space description but the converse is not generally true because the poles obtained from

$$\det(A_1(\lambda)) = 0 \tag{20}$$

are not poles of the system. Instead, they are related to the *unobservable modes* of the state space description.

3. TIME-VARYING LINEAR SYSTEMS

The time-varying differential systems can be described by the same kind of models than the time-invariant ones but the models are mathematically more complicated. In the SISO case the models are of the form

$$a(p)y = b(p)u \tag{21}$$

where $u, y \in X$ are input and output signals of the system, p is the differential operator on X , and $a(p), b(p)$ are polynomials in p with coefficients from a suitable space K of complex-valued infinitely differentiable functions. The existence and uniqueness of the solutions as well as the *realizability* of the models are difficult mathematical questions depending on the signal and coefficient spaces but they are not considered in this paper.

Now the multiplication of arbitrary polynomials defined by composition of operators is no more commutative but is constructed using the property

$$pa = ap + \frac{da}{dt} \tag{22}$$

This is an example of multiplication of *skew polynomials.* An other example is the discrete time case where the *unit prediction operator* q satisfies

$$qa = (q_K a)q \tag{23}$$

where $q_K$is the unit prediction operator in the space of coefficients (Ylinen 1980, Kamen 1988).

The algebraic structure of skew polynomials is a noncommutative ring with coefficient space K as a subring. Most of the concepts and properties of ordinary polynomials can be applied to skew polynomials. However, for stronger structures the coefficient ring K should be a field which is difficult to satisfy in the time-varying case without *extension* of coefficients and signals to corresponding *fractions* with nonzero coefficients as denominators.

In particular, this holds for the *division algorithms.* For instance, the *left division algorithm*

$$\begin{gathered} a(p) = b(p)q(p) + r(p) \\ \deg(r(p)) < \deg(b(p)) \end{gathered} \tag{24}$$

is satisfied for *all* $a(p), b(p) \neq 0$ only if the coefficient ring K is a field. This is important because the division algorithm is needed for manipulation of skew polynomial matrices used in descriptions of multivariable systems.

4. POLES AND ZEROS OF TIME-VARYING SYSTEMS

Analogously to the time-invariant case, the modes and poles are defined by means of solutions of first order differential equations

$$(p - \lambda)y = 0 \Leftrightarrow y(t) = y_0 e^{\int_0^t \lambda(t)dt} \tag{25}$$

where $y_0$is a *constant.*

Putting the mode to the equation $a(p)y = 0$ leads to

$$a(p)y_0 e^{\int_0^{(.)} \lambda(t)dt} = a^S(\lambda) e^{\int_0^t \lambda(t)dt} \tag{26}$$

where $\lambda \mapsto a(p)^S(\lambda) \widehat{=} a^S(\lambda)$ is a *skew polynomial function* $K \to K$ associated with $a(p)$. The skew polynomial functions have the following properties

$$(a_0)^S(\lambda) = a_0 \quad (27)$$
$$(p)^S(\lambda) = \lambda \quad (28)$$
$$(a(p) + b(p))^S(\lambda) = a^S(\lambda) + b^S(\lambda) \quad (29)$$
$$(a(p)b(p))^S(\lambda) = (a(p)b^S(\lambda))^S(\lambda) \quad (30)$$

Now the poles (pole functions) can be solved from

$$a^S(\lambda) = 0 \quad (31)$$

This is a *nonlinear differential equation* of order $(n-1)$ and the existence of its solutions depends on the chosen coefficient ring and initial values.

Using the right division algorithm of skew polynomials it can be proven:

Proposition 1. $a^S(\lambda) = 0$ if and only if $(p - \lambda)$ is a right factor of $a(p)$.

Proof 1. According to the division algorithm $a(p)$ can be presented in the form

$$a(p) = q(p)(p - \lambda) + r(p) \deg(r(p)) < \deg(p - \lambda) = 1 \quad (32)$$

i.e. $r(p) = r_0$. Furthermore

$$\begin{aligned} a^S(\lambda) &= (q(p)(p-\lambda))^S(\lambda) + r^S(\lambda) \\ &= (q(p)(\lambda - \lambda))^S(\lambda) + r_0 = r_0 \end{aligned}$$

This means that all *modes are related to linear right factors.*

The zeros and the input modes of the SISO system (21) can be defined analogously, i.e. the zeros are the roots of

$$b^S(\lambda) = 0 \quad (33)$$

The extension to multivariable systems needs some more sophisticated tools for manipulation of skew polynomial matrices.

Consider first the system described by

$$A(p)y = B(p)u \quad (34)$$

where $A(p), B(p)$ are skew polynomial matrices with full column rank and $A(p)$ square. The output modes

$$y(t) = y_0 e^{\int_0^t \lambda(t)dt} \quad (35)$$

where y_0 's are *constant vectors*, satisfy the equation

$$A^S(\lambda)y_0 = 0 \quad (36)$$

where $A^S(\lambda)$ is defined in the obvious way. Now the problem is to find a λ and a $y_0 \neq 0$ such that for *all t*

$$A^S(\lambda)(t)y_0 = 0 \quad (37)$$

Note that only in few special cases (e.g. if $A(p)$ is diagonal) λ can be obtained from $\det A^S(\lambda)(t) = 0$.

Consider next the model

$$P(p)A(p)y = P(p)B(p)u \quad (38)$$

equivalent to (34), where $P(p)$ is a unimodular skew polynomial matrix. Then

$$\begin{aligned} (P(p)A(p))^S(\lambda)y_0 &= (P(p)A^S(\lambda))^S(\lambda)y_0 \\ &= (P(p)A^S(\lambda)y_0)^S(\lambda) \\ &= 0 \end{aligned} \quad (39)$$

Thus the modes and poles are *independent of descriptions.*

In terms of the state-space realizations consider

$$\begin{bmatrix} pI - A(t) & 0 \\ -C(t) & I \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} B \\ D \end{bmatrix} u \quad (40)$$

where A, B, C, D are matrices over K of time-varying coefficients. Now the (internal) modes satisfy for *all t*

$$\begin{aligned} &\begin{bmatrix} pI - A & 0 \\ -C & I \end{bmatrix}^S (\lambda)(t) \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \\ &= \begin{bmatrix} \lambda(t)I - A & 0 \\ -C & I \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = 0 \end{aligned} \quad (41)$$

i.e. the poles of the state space description are such *pointwise eigenvalues* for which there exist *constant eigenvectors* x_0 with $C(t)x_0$ constant. Furthermore, only those eigenvalues satisfying $C(t)x_0 = \text{constant} \neq 0$ are also poles of the system.

Suppose that the model (40) can be brought to upper triangular form

$$\begin{bmatrix} A_1(p) & A_2(p) \\ 0 & A(p) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} B_1(p) \\ B(p) \end{bmatrix} u \quad (42)$$

by elementary row operations (note that the left division algorithm is needed). Further, assume that λ is a pole of the system generated by (34) i.e. it holds that $A^S(\lambda)y_0 = 0$ with $y_0 \neq 0$. Then λ is also a pole of the state space description only if the equation

$$A_1^S(\lambda)x_0 + A_2^S(\lambda)y_0 = 0 \quad (43)$$

has a non-zero constant solution x_0. That means that in the time-varying case there are state-space descriptions the poles of which are different from the poles of the system even in the observable case, where $A_1(p)$ can be taken equal to I. The problem is to find such state-space descriptions, which have the same poles as the system itself. This question is discussed in the next section.

5. STATE REPRESENTATIONS AND POLYNOMIAL FACTORIZATIONS

Consider a *SISO* input-output differential system

$$\sum_{i=0}^{n} a_i(t)\frac{d^i y(t)}{dt^i} = \sum_{j=0}^{n} b_j(t)\frac{d^j u(t)}{dt^j} \tag{44}$$

where it is assumed that for all time instants $a_n(t) \equiv 1$ and the functions $a_i(\cdot)$ and $b_j(\cdot)$ are differentiable at least $n-1$ times. The system has the realization (Wiberg, 1971)

$$\begin{array}{l} \dot{x}(t) = A(t)x(t) + B(t)u(t), \quad x(t_0) = x_0 \\ y(t) = C(t)x(t) + D(t)u(t) \end{array} \tag{45}$$

with

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0(t) & -a_1(t) & \cdots & \cdots & -a_{n-1}(t) \end{bmatrix}$$

$$B(t) = \begin{bmatrix} \gamma_1(t) \\ \gamma_2(t) \\ \vdots \\ \gamma_n(t) \end{bmatrix}$$

$$C(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} \quad D(t) = \gamma_0(t)$$

and

$$\begin{cases} \gamma_0(t) = b_n(t), \\ \gamma_i(t) = b_{n-i}(t) \\ -\displaystyle\sum_{k=0}^{i-1}\sum_{j=0}^{i-k} \frac{(n+j-i)!}{j!(n-i)!} a_{n-i+k+j}(t)\frac{d^j \gamma_k(t)}{dt^j} \end{cases}$$

$(i = 1, 2, \cdots, n)$. The transformation

$$x(t) = P(t)s(t) \tag{46}$$

where $P(\cdot)$ is an invertible square matrix changes the system representation (45) into the form

$$\begin{array}{l} \dot{s}(t) = E(t)s(t) + F(t)u(t) \\ y(t) = G(t)s(t) + H(t)u(t) \end{array} \tag{47}$$

$(s(t_0) = P^{-1}(t_0)x_0)$ where

$$\begin{array}{l} E(t) = P^{-1}(t)[A(t)P(t) - \dot{P}(t)] \\ F(t) = P^{-1}(t)B(t) \\ G(t) = C(t)P(t) \\ H(t) = D(t) \end{array} \tag{48}$$

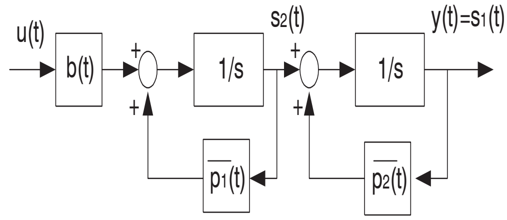

Fig. 1. Realization by the pole set

It is easy to show, (Zenger and Orava, 1996), that the matrix $E(\cdot)$ of the target system can be chosen arbitrarily by choosing

$$P(t) = \Phi_A(t, t_0)P(t_0)\Phi_E^{-1}(t, t_0) \tag{49}$$

where $\Phi_A(\cdot,\cdot)$, $\Phi_E(\cdot,\cdot)$ are the state transition matrices related to $A(\cdot)$ and $E(\cdot)$, respectively.

To proceed, consider first the two-dimensional system

$$\ddot{y}(t) + a_1(t)\dot{y}(t) + a_0(t)y(t) = b(t)u(t) \tag{50}$$

which has a realization of the form (45)

$$\begin{array}{l} A(t) = \begin{bmatrix} 0 & 1 \\ -a_0(t) & -a_1(t) \end{bmatrix}, \quad B(t) = \begin{bmatrix} 0 \\ b(t) \end{bmatrix} \\ C(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D(t) = 0 \end{array} \tag{51}$$

Let us try to find a realization corresponding to the system structure in Fig. 1. Then the system matrix of the target representation will have the form

$$E(t) = \begin{bmatrix} \bar{p}_2(t) & 1 \\ 0 & \bar{p}_1(t) \end{bmatrix} \tag{52}$$

and the transformation matrix P can be calculated from

$$\dot{P}(t) = A(t)P(t) - P(t)E(t) \tag{53}$$

If the elements of the matrix are denoted as p_{ij}, the equations become

$$\begin{array}{l} \dot{p}_{11} = p_{21} - p_{11}\bar{p}_2 \\ \dot{p}_{12} = p_{22} - p_{11} - p_{12}\bar{p}_1 \\ \dot{p}_{21} = -a_0 p_{11} - a_1 p_{21} - p_{21}\bar{p}_2 \\ \dot{p}_{22} = -a_0 p_{12} - a_1 p_{22} - p_{21} - p_{22}\bar{p}_1 \end{array} \tag{54}$$

(The notation of time t has been deleted in the previous equation to save space.) The equations have a solution

$$P(t) = \begin{bmatrix} 1 & 0 \\ p_2(t) & 1 \end{bmatrix} \tag{55}$$

leading to

$$\begin{array}{l} E(t) = \begin{bmatrix} \bar{p}_2(t) & 1 \\ 0 & \bar{p}_1(t) \end{bmatrix}, \quad F(t) = \begin{bmatrix} 0 \\ b(t) \end{bmatrix} \\ G(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad H(t) = 0 \end{array} \tag{56}$$

where

$$\begin{aligned}\bar{p}_2(t) &= p_2(t)\\ \bar{p}_1(t) &= -p_2(t) - a_1(t)\end{aligned} \quad (57)$$

and

$$-p_2^2(t) - a_0(t) - a_1(t)p_2(t) - \dot{p}_2(t) = 0 \quad (58)$$

using an arbitrary initial condition.

The system structure in Fig. 1 is seen to be analogous to that used in (Kamen, 1988). It corresponds to the polynomial factorization

$$\begin{aligned}&[p^2 + a_1(t)p + a_0(t)]y(t)\\ &= [p - \bar{p}_1(t)]\{[p - \bar{p}_2(t)]y(t)\}\end{aligned} \quad (59)$$

where $\bar{p}_2(t)$ is the *right pole* of the system. It is interesting to note that if $p_2(t)$ is a bounded function, $P(t)$ is a *Lyapunov transformation*, which is known to preserve the stability properties of the original and transformed systems, (Lyapunov, 1966), (Harris and Miles, 1980).

The concept of 'right pole' is analogous to the theory presented in the previous section, because it has a direct correspondence to the mode of the system. However, the pole p_1, does not have this property, and the definitions differ in this respect. It is interesting to note that in (Kamen, 1988) two different right poles are used to study the stability of the system. This means the determination of $p_2(t)$ with two different initial conditions.

Note that it is also possible to use a transformation that leads to a diagonal matrix $E(t)$. The the main diagonal elements corresponds to the modes of the system as described earlier. The transformation can then be written by a generalized Vandermonde matrix (Kamen, 1988). A similar approach has been used in (O'Brien and Iglesias, 1998), where it is further required that the state transformation matrix is a Lyapunov transformation. However, that is not an easy condition to meet in this case.

The above idea can be extended to the n-dimensional case as follows. If the transformation matrix P is chosen to have the form of a lower triangular matrix as follows

$$P(t) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0\\ p_n(t) & 1 & 0 & \cdots & 0\\ x_{31}(t) & p_{n-1}(t) & 1 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ x_{n1}(t) & x_{n2}(t) & \cdots & p_2(t) & 1\end{bmatrix} \quad (60)$$

the system matrix E in the target representation becomes

$$E(t) = \begin{bmatrix} p_n & 1 & \cdots & 0\\ \bullet & -p_n + p_{n-1} & 1 & 0\\ \bullet & \bullet & \ddots & \vdots\\ \bullet & \bullet & \bullet & -p_2 - p_1\end{bmatrix} \quad (61)$$

with $p_1 = a_{n-1}$ and the positions marked with a bullet represent suitable differential equations, which are functions of the terms $x_{ij}(t)$ and $p_i(t)$ (compare to the equation (58)). Setting these equal to zero the 'poles' in the main diagonal can be calculated. The terms correspond to the factorization of the pole polynomial. Formally, the zeros can be calculated in a similar manner by setting the coefficients $b_i(t)$ in a fictitious system matrix.

6. CONCLUSION

The concept of poles and zeros of time-varying linear differential systems has been discussed in the paper. By using the modes of the system representation the poles and zeros have been defined in a natural way, which is shown to lead to a polynomial factorization of the pole and zero polynomials. The relations to previous work has been discussed, and a new way to define or calculate the poles and zeros starting from a canonical state-space representation of the system has been presented.

7. REFERENCES

Blomberg, H. and R. Ylinen (1983). *Algebraic Theory for Multivariable Linear Systems.* Academic Press. London.

Harris, C. J. and J. F. Miles (1980). *Stability of Linear Systems: Some Aspects of Kinematic Similarity.* Academic Press. London.

Kamen, E. W. (1988). The poles and zeros of a linear time-varying system. *Linear Algebra and its Applications* **98**, 263–289.

Lyapunov, A. M. (1966). *Stability of Motion.* Academic Press. New York.

O'Brien, R. T. and P. A. Iglesias (1998). Pole/zero cancellations in time-varying systems. *International Journal of Control* **71**, 103–130.

Rugh, W. J. (1993). *Linear System Theory.* Prentice-Hall. New Jersey.

Wiberg, D. M. (1971). *State Space and Linear Systems.* McGraw-Hill. New York.

Ylinen, R. (1980). An algebraic theory for analysis and synthesis of time-varying linear differential systems. *Acta Polytechnica Scandinavica, Mathematics and Computer Science Series.*

Zenger, K. and J. Orava (1996). Time-varying transformations on linear system representations. *Proceedings of the 13th IFAC World Congress.*