STOCHASTIC P ARABOLIC MODEL FOR INFINITE-DIMENSIONAL FORWOARD RATE AND MEAN-VARIANCE OPTIMAL CONTROL

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Abstract: We consider the term structure modeling by using a appropriate stochastic parabolic system with boundary noises. After finding a sufficient condition for the no arbitrage opportunity, we solv e the mean-variance optimal control problem in the incomplete market.

Keywords: Stochastic parabolic systems, term structure, arbitrage-free, optimal control, self-financing, short rate

1. INTRODUCTION

Bonds are tradable assetes in a financial market. The market price $P(t,T)$ is a function of the time t and the maturity T and chages its value randomly as shown in Fig.1 (Shiryaev 1999). For
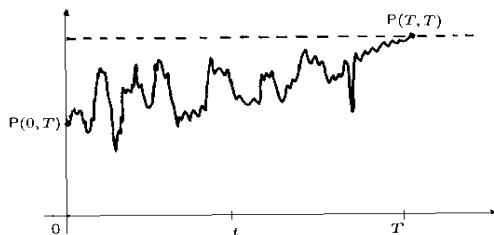

Fig. 1. The Market value $P(t,T)$ of a bond

the mathemtical modeling of this process, w e rewrite $P(t,T) = \exp\{-\int_t^T f(t,x)dx\}$ and the so-called forward rate dynamics for f is constructed in (Shiryaev 1999).

It is w ell kno wn that there are tw o different motivations for term structure of interest rates modeling. The first is concerned with the pricing of interest rate derivative securities. In the sense of arbitrage-free, the dynamics of the interest rates is dev eloped in (Kennedy 1994). Starting from the classical modeling of the short rate to the forward rate including the Musiela equation, there are many excellent books for example (Bjork 1998)

The second motivation is the actual "economet-ric" modeling of the real-world term structure dynamics which giv es the statistical description of the mov ements of interest rate in (Cont 1999).

In this paper, we consider the second viewpoint. First we present the motivating discussions of the term structure model. Then the parabolic stochas-tic equation with boundary noises is proposed for the term structure. After studying the existence of a unique solution to this stochastic partial differ-ential equation , we deriv e the sufficien condition to support the no arbitrage opportunity. The final section is devoted to consider the mean variance optimal control problem for the wealth process in the incomplete market situation.

The simple infinite-dimensional term structure model is given b y

$$df(t,x) = \frac{\partial f(t,x)}{\partial x}dt + \nu(t,x)dt + dw(t,x), \quad (1.1)$$

where $\nu(t,x)$ is identified by using the argument of absence of arbitrage and x denotes the time to maturity. Recently from empirical observations, Bouch et.al. and Cont (Cont 1999) proposed the parabolic type systems to support the smoothness property of $f(t,x)$ with respect to x. The natural extension of (1.1) to the parabolic system is

$$df(t,x) = \frac{k}{2}\frac{\partial^2 f(t,x)}{\partial x^2}dt$$
$$+ \frac{\partial f(t,x)}{\partial x}dt + \nu(t,x)dt + dw(t,x). \quad (1.2)$$

In Cont (Cont 1999), in order to formulate (1.2) first the two factors, i.e., the short rate and the spread are taken out of the model. The short rate and the spread are modeled by a bivariate diffusion process. In such a modeling procedure the short rate and the spread are independent of the new variable of the constructed model explicitly.

Here we formulate (1.2) without taking out these two terms. By constructing the boundary conditions, we get the short rate $r(t)$ and the long rate $\ell(t)$ as the boundary values $f(t,0)$ and $f(t,1)$.

First we set the simple boundary condition to (1.2),

$$\frac{\partial f(t,0)}{\partial x} = 0. \quad (1.3)$$

The discrete version of (1.3) becomes

$$\frac{f(t,\Delta x) - f(t,0)}{\Delta x} = 0$$

i.e.,

$$r(t) = f(t,0) = f(t,\Delta x).$$

This implies that the short rate $r(t)$ is the same value as the nearest term structure value. It is well known that if the considered situation has no randomness, then $f(t,x)$ becomes a constant for all x. So we need to adjust (1.3) to fit the stochastic situation , i.e.,

$$\frac{k}{2}\frac{\partial f(t,0)}{\partial x} = \sigma_0(t)\frac{dw_0(t)}{dt} \quad (1.4)$$

where $w_0(t)$ is a standard Brownian motion process. The discrete version of (1.4) with respect to t and x is given by

$$\frac{k}{2}\frac{f(t_i,\Delta x) - f(t_i,0)}{\Delta x} = \sigma_0(t_{i-1})\frac{w_0(t_i) - w_0(t_{i-1})}{\Delta t}$$

i.e.,

$$r(t_i) = f(t_i,\Delta t) - \sigma_0(t_{i-1})\frac{w_0(t_i) - w_0(t_{i-1})}{\Delta t}\frac{2\Delta x}{k}.$$

Hence the spot rate $r(t_i)$ is fluctuated by w_0 and also depends on $f(t,x), x > 0$. It is also possible to generalize the boundary condition (1.4) to

$$\frac{k}{2}\frac{\partial f(t,0)}{\partial x} = \mu_1(f(t,0), f(t,1))$$

$$+\sigma_{00}(f(t,0), f(t,1))\frac{dw_0(t)}{dt}$$
$$+\sigma_{11}(f(t,0), f(t,1))\frac{dw_1(t)}{dt}$$

This formulation is exactly the generalization of the bivariate diffusion formulation given by Brennan & Schwarz (Cont 1999) to the stochastic boundary conditions.

2. MATHEMATICAL FORMULATION

In order to make our idea clear, we consider the following simple situation:

$$df(t,x) = \frac{k}{2}\frac{\partial^2 f(t,x)}{\partial x^2}dt + \frac{\partial f(t,x)}{\partial x}dt + \nu(t,x)dt$$
$$+ dw(t,x), (t,x) \in]0,t_f[\times G \quad (2.1)$$

$$f(0,x) = f_o(x), \quad x \in G \quad (2.2)$$

$$\frac{k}{2}\frac{\partial f(t,0)}{\partial x} = \sigma_0(t)\frac{dw_0(t)}{dt}, \quad t \in]0,t_f[\quad (2.3)$$

$$-\frac{k}{2}\frac{\partial f(t,1)}{\partial x} = \sigma_1(t)\frac{dw_1(t)}{dt}, \quad t \in]0,t_f[\quad (2.4)$$

We work in the following Hilbert spaces:

$$V = H^1(G) \subset H = L^2(G) \subset V' = \text{dual of } V.$$

Define $\forall \phi_1, \phi_2 \in V$

$$<A\phi_1, \phi_2> = \int_G \{\frac{k}{2}\frac{\partial \phi_1}{\partial x}\frac{\partial \phi_2}{\partial x} - \frac{\partial \phi_1}{\partial x}\phi_2\}dx.$$

The weak form of the proposed system is

$$(f(t),\phi) + \int_0^t <Af(s),\phi> ds + (\sigma w_b(t),\phi)_\Gamma$$

$$= (f_o,\phi) - \int_0^t (\nu(s),\phi)ds + (w(t),\phi)\forall \phi \in V \quad (2.5)$$

where $\Gamma = L^2(\partial G)$ and

$$(\sigma w_b(t),\phi)_\Gamma = \sigma_0 w_0(t)\phi(0) + \sigma_1 w_1(t)\phi(1)$$

and w, w_1 and w_0 are mutually independent Brownian motion processes; $\forall \phi_1, \phi_2 \in H$

$$E\{(w(t),\phi_1)(w(t),\phi_2)\} = t(\phi_1, Q\phi_2)$$
$$E\{|w_0(t)|^2\} = E\{|w_1(t)|^2\} = t$$

with

$$Tr\{Q\} < \infty. \quad (2.6)$$

Theorem 2.1. Under (2.6),

$$k > 0$$
$$f_o \in L^2(\Omega; H)$$

and
$$\nu \in L^2(\Omega \times]0, t_f[; V')$$
(2.5) has a unique solution in
$$L^2(\Omega; C([0, t_f]; H) \cap L^2(]0, t_f[; H)).$$

Proof. The parabolic type stochastic evolution equation with boundary noise has been studied by many authors. For example the method used can be found in the book by Rozovskii (Rozovskii 1983) and (Pardoux 1979).

Proposition 2.1. Under

$$f_o \in L^2(\Omega; V), \quad \nu \in L^2(\Omega \times]0, t_f[; H) \quad (2.7)$$

and

$$Tr\{\frac{\partial}{\partial x}(\frac{\partial Q}{\partial x})^*\} < \infty, \qquad (2.8)$$

we have

$$f \in L^2(\Omega; C([0, t_f]; V) \cap L^2(]0, t_f[; H^2) \quad (2.9)$$

and the spot rate $r(t) = f(t, 0)$ and the long rate $\ell(t) = f(t, 1)$ respectively satisfy

$$r, \ell \in L^2(\Omega; C([0, t_f]; R^1)). \qquad (2.10)$$

Proof: By using the technique proposed by Bardos (Bardos 1971), it is easy to derive the above regularity property.

3. MARKET MODEL

In this section G is replaced as $G =]0, T_d[$. Our market consists of a bank account $B(\cdot)$ and bonds for the maturity $T, t \leq T \leq t + T_d$ where t is a present time, i.e.,

$$P(t, T) = \exp\{-\int_0^{T-t} f(t, x)dx\} \qquad (3.1)$$

and the bank account B is set as

$$\frac{dB(t)}{dt} = r(t)B(t), B(0) = B_o, \qquad (3.2)$$

where $r(t)$ is a spot rate and is given by

$$r(t) = f(t, 0).$$

Proposition 3.1. The bond price $P(t, T)$ is a solution of

$$dP(t, T) = \{r(t) - \frac{k}{2}f_x(t, T-t)$$
$$+ g(T-t)\}P(t, T)dt$$
$$+ \sigma_0 P(t, T)dw_0(t) - P(t, T)d\tilde{w}(t, T) \quad (3.3)$$

where

$$g(T-t) = \frac{1}{2}\int_0^{T-t}\int_0^{T-t} q(x, y)dxdy - \int_0^{T-t} \nu(t, x)dx$$

$$f_x(t, T-t) = \frac{\partial f(t, x)}{\partial x}\Big|_{x=T-t}$$

$$\tilde{w}(t, T) = \int_0^t \Sigma_{i=1}^{T-s}\int_0^{T-s}\sqrt{\lambda_i}e_i(\tau)d\tau d\beta_i(s)$$

and $q(x, y)$ is a kernel of Q, i.e.

$$Q = \int_G q(x, y)(\cdot)dy \qquad (3.4)$$

Proof: By using Ito' formulat to (3.1), (3.3) can be obtained.

4. NO ARBITRAGE OPPORTUNITY

In the field of mathematical finance, it is important that the proposed model is arbitrage free . In order to prevent the free-lunch opportunity, we must ahow that the proposed system can be transformed to the local martingale by using the Girsanov theorem.

Mathematically speaking, the discounted bond price $\bar{P}(t, T) = \frac{P(t, T)}{B(t)}$-process should be a local martingale. This means that the original parabolic system should be transformed to the hyperbolic one such that

$$(f(t), \phi) = (f_o, \phi) + \int_0^t (\frac{\partial f(t)}{\partial x}, \phi)ds$$

$$+ \int_0^t (\Sigma_{i=1}^\infty \sqrt{\lambda_i}e_i(x)\int_0^x e_i(y)dy, \phi)ds + (\tilde{w}(t), \phi)$$

where

$$(\tilde{w}(t, x), \phi) = \int_0^t \frac{k}{2}(\frac{\partial f(t, x)}{\partial x}, \frac{\partial \phi}{\partial x})ds$$

$$+ \int_0^t [\nu(s, x) - \Sigma_{i=1}^\infty \sqrt{\lambda_i}e_i(x)\int_0^x e_i(y)dy]ds$$
$$- \sigma(w_b(t), \phi)_\Gamma + (w(t), \phi).$$

If we can show that the process $\tilde{w}(t, x)$ is a Brownian motion process under the suitable measure, $\bar{P}$ becomes a local martingale.

Proposition 4.1. In addition to (2.8) and (2.9), we assume that

$$Q = \Sigma_{i=1}^m \sqrt{\lambda_i}e_i \otimes e_i, \quad e_i \in H^2. \qquad (4.5)$$

Hence

$$E\{\int_0^{t_f} |Q^{-1/2}\frac{\partial^2 f(t)}{\partial x^2}|_H^2 dt\} < \infty \qquad (4.6)$$

where

$$Q^{-1/2} = \Sigma_{i=1}^m \frac{1}{\sqrt{\lambda_i}} e_i \otimes e_i \qquad (4.7)$$

Proof: It is easy to show that

$$E\{\int_0^{t_f} |Q^{-1/2}\frac{\partial^2 f(t)}{\partial x^2}|_H^2 dt\}$$

$$< m \max_i \{\frac{1}{\lambda_i}\} E\{\int_0^{t_f} |\frac{\partial^2 f(t)}{\partial x^2}|_H^2 dt\}$$

$$< \text{Const. } E\{\int_0^{t_f} |\frac{\partial^2 f(t)}{\partial x^2}|_H^2 dt\}$$

$$\text{(from (2.9))}$$

$$< \text{Const.}$$

It should be noted that the operator Q is an m-dimensional operator but the state process $f(t,x)$ is still an infinite-dimensional one, because the initial condition f_o is still an infinite- dimensional state.

Theorem 4.1. Under (4.5) we can define a Martingale measure $\tilde{\mathcal{P}}$

$$\frac{d\tilde{\mathcal{P}}}{d\mathcal{P}} = \exp\left\{\int_0^t (\frac{k}{2}\frac{\partial^2 f(s,x)}{\partial x^2} - \nu(s,x) \right.$$

$$+ \tilde{q}(x,m), d\tilde{w}(s))$$

$$\left. -\frac{1}{2}\int_0^t |\frac{k}{2}\frac{\partial^2 f(s,x)}{\partial x^2} - \nu(s,x) + \tilde{q}(x,m)|_H^2 ds\right\}$$

and $\tilde{w}(t,x)$ is a Brownian motion process with respect to $\tilde{\mathcal{P}}$ where

$$\tilde{q}(x,m) = \Sigma_{i=1}^m \sqrt{\lambda_i} e_i(x) \int_0^x e_i(y) dy.$$

Proof: This theorem is exactly the Girsanov theorem, because (4.6) is the Novikov condition.

5. MEAN-VARIANCE OPTIMAL CONTROL

We consider a portfolio comprised of β shares of the money market fund, and $\gamma(\cdot, T)$ shares of bond maturing at dates T:

$$c(t) = \beta(t)B(t) + \int_t^{\hat{T}} \gamma(t,T)P(t,T)dT, \quad (5.1)$$

where we assume that

$$\hat{T} < T_d. \qquad (5.2)$$

For the self-financing portfolio, the derivatives of $\beta(t)$ and $\gamma(t,\cdot)$ with respect to t become zero. So in our case, the instantaneous change in portfolio value is

$$dc(t) = \beta(t)dB(t) + \int_t^{\hat{T}} \gamma(t,T)dP(t,T)dT. (5.3)$$

It is easy to show that

$$dc(t) = r(t)c(t)dt - \int_t^{\hat{T}} (\frac{k}{2}f_x(t,T-t)$$

$$-g(T-t))\gamma(t,T)P(t,T)dTdt$$

$$+\sigma_0 \int_t^{\hat{T}} \gamma(t,T)P(t,T)dT dw_0(t)$$

$$-\int_t^{\hat{T}} \gamma(t,T)P(t,T)d\tilde{w}(t,T)dT. \quad (5.4)$$

Setting

$$u(t,T) = \begin{cases} \gamma(t,T)P(t,T) & \text{for } t \le T \\ 0 & \text{for otherwise} \end{cases} \quad (5.5)$$

and

$$d\hat{w}(t,T) = \sigma_0 dw_0(t) - d\tilde{w}(t,T), \qquad (5.6)$$

we have

$$dc(t) = r(t)c(t)dt - (\frac{k}{2}f_x(t,\cdot - t)$$

$$-g(\cdot - t), u(t))_{\hat{T}}dt - (u(t), d\hat{w}(t,\cdot))_{\hat{T}} \quad (5.7)$$

where

$$(\phi,\psi)_{\hat{T}} = \int_0^{\hat{T}} \phi(T)\psi(T)dT.$$

From (5.2) we can not treat the bond $P(t,T)$ for all maturity $T \le T_d$ and the market becomes incomplete (Shiryaev 1999). In such a situation, we need to consider the mean-variance control problem instead of the usual option pricing. Hence instead of finding a portfolio (β,γ), we want to construct a control $u(t,T)$ to achieve $c(t_f) = \xi$ for *a-priori* given ξ. It is almost impossible to achieve $c(t_f) = \xi$ a.s. for the stochastic process $c(t)$. So our control problem is to find the control u to minimize

$$J(u) = \frac{1}{2}E\{|c(t_f) - \xi|^2\}. \qquad (5.8)$$

For (5.8) an admissible control is set as

$$u(\cdot) \in L^2_{\mathcal{F}}(]0, t_f[; \hat{H}), u(t) \in U_{ad} \text{ a.e., a.s.} \quad (5.9)$$

where U_{ad} is a convex closed non empty subset of $\hat{H} = L^2(0, \hat{T})$.

It should be noted that our control problem is not a usual linear one, because the system (5.7) contains the random coefficients $r(t)$ and f_x.

We introduce the adjoint system:

$$-dp(t) = r(t)p(t)dt - (h(t), d\hat{w}(t, \cdot))_{\hat{T}} \quad (5.10)$$

$$p(t_f) = c(t_f) - \xi \quad (5.11)$$

where

$$h \in L^2(\Omega \times]0, t_f[; \hat{H}) \quad (5.12)$$

For the details about the above stochastic adjoint equation, we refer to Bennsousann (Bensoussan 1983) , Kohlmann (Kohlmann and Tang 2001) .

Define Z to be the solution of

$$\begin{cases} dZ(t) = r(t)Z(t)dt - (\dfrac{k}{2}f_x(t, \cdot - t) \\ \qquad -g(\cdot - t), v(t))_{\hat{T}}dt - (v(t), d\hat{w}(t, T))_{\hat{T}} \\ Z(0) = 0, \quad v \in U_{ad} \end{cases}$$

Noting that $p(t)$ is $\mathcal{F}_t$-measurable, we get

$$p(t)Z(t) - p(0)Z(0) = -\int_0^t (\frac{k}{2}f_x(s, \cdot - s)$$

$$-g(\cdot - s), v(s))_{\hat{T}}p(s)ds - \int_0^t p(s)(v(s), d\hat{w}(s))_{\hat{T}}$$

$$-\int_0^t Z(s)(h(s), d\hat{w}(s))_{\hat{T}} + \int_0^t (v(s), \tilde{\tilde{Q}}h(s))_{\hat{T}}ds, (5.13)$$

where $\tilde{\tilde{Q}}$ is an incremental covariance operator of $\hat{w}$. Taking the mathematical expectation to (5.13) and using the initial condition $Z(0) = 0$ and the terminal condition $p(t_f) = c(t_f) - \xi$, we get

$$E\{p(t_f)Z(t_f)\} = E\{(c(t_f) - \xi)Z(t_f)\}$$

$$= E\{\int_0^t [(v(s), -(\frac{k}{2}f_x(s, \cdot - s) - g(\cdot - s))p(s)$$

$$+ \tilde{\tilde{Q}}(s)h(s))_{\hat{T}}ds\} \quad (5.14)$$

Hence from the stochastic maximum principle, the optimal process h satisfies

$$\tilde{\tilde{Q}}h(s) = \{\frac{k}{2}f_x(s, \cdot - s) - g(\cdot - s)\}p(s)$$

Now we set the following assumption:

$$\tilde{\tilde{Q}}^{-1}\{\frac{k}{2}f_x - g\}p \in L^2(\Omega \times]0, t_f[; \hat{H}). \quad (5.15)$$

Hence the optimal system becomes

$$dc(t) = r(t)c(t)dt - (u^o(t), \hat{w}(t))_{\hat{T}}$$

$$-(\frac{k}{2}f_x(t, \cdot - t) - g(\cdot - t)), u^o(t))_{\hat{T}}dt, \quad (5.16)$$

$$-dp(t) = r(t)p(t)dt - (\tilde{\tilde{Q}}^{-1}(\frac{k}{2}f_x(t, \cdot - t)$$

$$-g(\cdot - t))p(t), d\hat{w}(t))_{\hat{T}}. \quad (5.17)$$

Here we set

$$p(t) = P(t)(c(t) - q(t)). \quad (5.18)$$

Hence from (5.16) we get

$$dp(t) = \dot{P}(t)(c(t) - q(t))dt + P(t)(dc(t) - \dot{q}(t)dt)$$

$$= (\dot{P}(t)c(t) - \dot{P}(t)q(t) - P(t)\dot{q}(t))dt$$

$$-(\frac{k}{2}f_x(t, \cdot - t) - g(\cdot - t), u^o(t))_{\hat{T}}P(t)dt$$

$$+P(t)r(t)c(t)dt - P(t)(u^o(t), d\hat{w}(t))_{\hat{T}} \quad (5.19)$$

and substituting (5.18) into (5.17), we obtain

$$dp(t) = -r(t)P(t)(c(t) - q(t))dt$$

$$+(\tilde{\tilde{Q}}^{-1}(\frac{k}{2}f_x(t, \cdot - t) - g(\cdot - t))$$

$$\times P(t)(c(t) - q(t)), d\hat{w}(t))_{\hat{T}}. \quad (5.20)$$

Comparing (5.19) with (5.20) and assuming

$$P(t) > 0 \quad (5.21)$$

we obtain

$$u^o(t) = -\tilde{\tilde{Q}}^{-1}(\frac{k}{2}f_x(t, \cdot - t) - g(\cdot - t))$$

$$\times (c(t) - q(t)) \quad (5.22)$$

and

$$\dot{P}(t)c(t) - \dot{P}(t)q(t) - P(t)\dot{q}(t) + P(t)r(t)c(t)$$

$$+|\tilde{\tilde{Q}}^{-1/2}\frac{k}{2}f_x(t, \cdot - t) - g(\cdot - t))|^2$$

$$\times (c(t)P(t) - q(t)P(t))$$

$$= -r(t)P(t)c(t) + r(t)P(t)q(t). \quad (5.23)$$

Hence we have the following two equations

$$\dot{P}(t) + 2r(t)P(t) + |\tilde{\tilde{Q}}^{-1/2}(\frac{k}{2}f_x(t, \cdot - t)$$

$$-g(\cdot - t))|^2P(t) = 0, P(t_f) = 1 \quad (5.24)$$

and

$$\dot{P}(t)q(t) + P(t)\cdot q(t) + |\frac{k}{2}\tilde{\tilde{Q}}^{-1/2}(\frac{k}{2}f_x(t, \cdot - t)$$

$$-g(\cdot - t))|^2q(t)P(t) = -r(t)P(t)q(t) \quad (5.25)$$

Substituting (5.24) into (5.25), we have

$$\begin{cases} \dot{q}(t) = r(t)q(t) \\ q(t_f) = \xi \end{cases} \quad (5.26)$$

It is obvious that $P(t) > 0$ is satisfied from (5.24). At this point we should notice that the solution

$q(t)$ of (5.26) must be a $\mathcal{F}_t$-measurable solution. So we shall introduce the following process:

$$q(\tilde{t}, t) = E\{\exp(-\int_{\tilde{t}}^{t_f} r(s)ds)|\mathcal{F}_t\}\xi, \ \text{for} \tilde{t} \geq t, (5.27)$$

i.e.,

$$\frac{dq(\tilde{t}, t)}{d\tilde{t}} = E\{r(\tilde{t}) \exp(\int_{\tilde{t}}^{t_f} r(s)ds)\xi|\mathcal{F}_t\} \quad (5.28)$$

and

$$\lim_{\tilde{t} \to t} q(\tilde{t}, t) = q(t, t).$$

Hence we get

$$\lim_{\tilde{t} \to t} \frac{dq(\tilde{t}, t)}{d\tilde{t}} = r(t)q(t, t).$$

So the derivative of the left hand side of (5.26) should be as stated above and

$$q(t) = q(t, t). \quad (5.29)$$

In order to realize the optimal control (5.22) we only need to obtain the explicit form of (5.29). The Riccati equation (5.24) is not used anymore as stated in Kohlmann et. al. (Kohlmann and Zhou 1999)

Proposition 5.1. Define the Dirichlet map D such that

$$D\phi(s) = \phi(s, 0), \ \text{for} \ \phi(s) \in H^1(]0, \hat{T}[). (5.30)$$

The $\mathcal{F}_t$-measurable solution of (5.26) is given by

$$q(t) = \exp\{\int_t^{t_f} -D(\Phi(s, t)f(t) + \int_t^s \Phi(s, \tau)\nu(\tau)ds\}$$

$$\times \exp\{\frac{1}{2}\int_t^{t_f}[D\int_\tau^{t_f} \Phi(s, \tau)ds\tilde{Q}(D\int_\tau^{t_f} \Phi(s, \tau)ds)^*$$

$$+\sigma_0^2|D\int_\tau^{t_f} (D\Phi^*(s, \tau))^*ds|^2]ds\}\xi, \quad (5.31)$$

where $\Phi(s, t)$ is a semigroup generated by A.

Proof: By using the semi-group and the Diriclet map, the solution f of (2.5) can be represented by

$$f(s) = \Phi(s, t)f(t) + \int_t^s \Phi(s, \tau)\nu(\tau)d\tau$$

$$+\int_t^s \Phi(s, \tau)d\tilde{w}(\tau) + \sigma_0 \int_s^t (D\Phi^*(s, \tau))^*dw_0(\tau).$$

Hence it follows from (5.27) that

$$q(t) = E\{\exp(-\int_t^{t_f} Df(s)ds)|\mathcal{F}_t\}\xi$$

$$=\exp\{-\int_t^{t_f} D(\Phi(s, t)f(t) + \int_t^s \Phi(s, \tau)\nu(\tau)d\tau)ds\}$$

$$\times E\{\exp(-D\int_t^{t_f} (\int_t^s \Phi(s, \tau)d\tilde{w}(\tau)$$

$$+\sigma_0 \int_t^s (D\Phi^*(s, \tau))^*dw_0(\tau))ds\}\xi.$$

Hence we derive (5.31).

Remark 5.1. From above proposition, to realize the optimal control $u(t)$,we need the information for $c(t)$ and $f(t)$ processes.

6. CONCLUSIONS

From the empirical consideration of (Cont 1999), the term structure is modeled by the stochastic parabolic systems with boundary noises. The arbitrage-free opportunity can be found under the condition that the number of the random source is finite (4.13). In the mean-variance optimal control problem, we set the assumption (5.15). This assumption is also recovered under the same arbitrary-free condition.

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