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MINIMAL ORDER DISCRETE-TIME NONLINEAR SYSTEM INVERSION

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Abstract: The left-inverse system with minimal order of a discrete-time nonlinear system is studied in a linear algebraic framework. The general structure of the left inverse is described. Two algorithms are given for constructing left-inverse systems having minimal order.

Keywords: left-inverse, discrete-time system, nonlinear system, linear algebra.

1. INTRODUCTION

Inversion is one of the fundamental issues in systems theory. The construction of a minimal order left-inverse for a nonlinear system is of both theoretical and practical importance. Many signal processing problems can be thought of as dynamical system inversion problems. For example nonlinear inversion problems occur in digital communications where a coded signal can be represented as the output of a nonlinear dynamical system and finding an inverse system has practical significance. This paper focuses on the theory and the derivation of constructive procedures for determining a minimal order left-inverse for nonlinear systems.

The first systematic results relevant to nonlinear system inversion problems were given in (Hirschorn, 1979; Hirschorn, 1979), later followed by further results in (Di Benedetto,*et al.*, 1993; Conte *et al.*, 1999; Zheng & Cao, 1993; Cao & Zheng, 1992; Devasia *et al.*, 1998; Fliess, 1986; Singh, 1981; Singh, 1982). These papers all studied continuous-time nonlinear systems. For discrete-time nonlinear systems there are also some significant results. Kotta systematically considered right inversion problems for discrete-time nonlinear control in her book (Kotta, 1995), where many references are cited. In this paper we discuss the left-inversion problem for discrete-time systems. If the dimension of the input and output vectors are the same, then the invertibility conditions for left-inversion and right-inversion are equivalent. However, in general, right-inversion and left-inversion are different problems. Right-inversion relates to the input-output decoupling problem and is sometimes referred to as the decoupling controller problem. Left-inversion is mainly related to the system zeros, and a left-inverse gives the structure of the zero-dynamics of a system. Kotta studies the reduced-order right-inverse system of discrete-time systems (Kotta, 1995) only. The structure of the minimal order left-inverse for discrete-time systems has not been studied in the open literature to our knowledge. We use the linear algebraic approach introduced in (Grizzle, 1993; Aranda-Bricaire, *et al.*, 1996), which extends the results in (Di Benedetto,*et al.*, 1989). It is important to point out that the methods used for continuous-time nonlinear system inversion are not always readily applicable to the discrete-time case. The algorithms presented in this paper for the construction of a reduced order left-inverse for a discrete-time nonlinear system are not easily related to any counterparts in the continuous-time case.

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Section 2 presents certain required results on discrete-time nonlinear systems and differential linear algebra. Section 3 then uses these results to derive the structure of an inverse for a nonlinear system. Two procedures for constructing an inverse are then presented in Section 4 followed by a proof that the construction procedures lead to minimal order inverses and an example that demonstrates how to use the procedures.

2. PROBLEM FORMULATION AND MATHEMATICAL PRELIMINARIES

Consider the discrete-time nonlinear dynamical system

$$\begin{aligned} x[k+1] &= f(x[k],u[k]), \\ y[k] &= h(x[k],u[k]) \end{aligned} \tag{1}$$

where $x[k] \in \mathbf{R}^n,\ u[k] \in \mathbf{R}^m,\ y[k] \in \mathbf{R}^p$ are the state, input and output vectors, respectively. We assume that f and h are analytic vector functions over an open dense subset of $\mathbf{R}^{n+m}$.

Our aim is to construct a minimal order left inverse dynamical system

$$\begin{aligned} z[k+1] &= \psi(z[k],y[k],\cdots,y[k+r]) \\ w[k] &= \eta(z[k],y[k],\cdots,y[k+r]) \end{aligned} \tag{2}$$

where $z[k] \in \mathbf{R}^q$ and r is a properly chosen positive integer such that the output $w(k)$ of (2) is equal to the input of system (1) when the output vectors $\{y[k],y[k+1],\cdots,y[k+r]\}; k \geq 0$ of system (1) are taken as the input vectors of (2).

To avoid singularities, we assume that the map $f(\cdot,\cdot)$ is generically a submersion, i.e. f is a submersive map on an open and dense subset of $\mathbf{R}^{n+m}$. Under this assumption the discrete-time dynamics create a type of group action (refer to (Grizzle, 1993) for details). Thus given the initial state and input signal we can recursively calculate the output signal using (1).

Using the formal differential rule we have for each $j \in \underline{p} := \{1,2,\cdots,p\}$ and $0 \leq k \leq N+1$

$$dy_j[k] = \sum_{i=1}^{n} \frac{\partial y_j[k]}{\partial x_i} dx_i + \sum_{l=0}^{k} \sum_{i=1}^{m} \frac{\partial y_j[k]}{\partial u_i[l]} du_i[l] \tag{3}$$

Let $\mathcal{K}$ denote the field of meromorphic functions in variables $\{x,u[0],\cdots,u[N]\}$, where N is a sufficiently large positive integer. The field of meromorphic functions is defined as the quotient field of a ring of analytical functions (Conte *et al.*, 1999). The use of meromorphic function is essential for carrying out arithmetic operations, particularly division.

Define $dx := \{dx_1[0],\cdots,dx_n[0]\}$ and $du[0] := \{du_1[0],\cdots,du_m[0]\}$ and vector spaces

$$\begin{aligned} &\mathcal{X}^* := span_{\mathcal{K}}\{dx\}, \\ &\mathcal{U}^* := span_{\mathcal{K}}\{du[0],du[1],\cdots,du[N]\}, \\ &\mathcal{Y}^* := span_{\mathcal{K}}\{dy[0],dy[1],\cdots,dy[N]\}. \end{aligned} \tag{4}$$

We will sometimes refer to $\mathcal{X}^*$ and $\mathcal{U}^*$ as simply the state space and input space.

We need two notions in order to study the nonlinear system inversion problem as follows (Grizzle, 1993; Aranda-Bricaire, *et al.*, 1996). *Difference-field* $\mathcal{F}$, which is a field equipped with a shift-operation $\delta : \mathcal{F} \to \mathcal{F}$; and *Difference vector space* $\mathcal{V}^*$ over a difference-field $\mathcal{F}$, which is a vector space equipped with a shift operation $\delta : \mathcal{V}^* \to \mathcal{V}^*$.

Thus, we define the shift operator $\delta : \mathcal{K} \to \mathcal{K}, \eta \mapsto \delta\eta$ as follows. For any $\eta \in \mathcal{K}$

$$\begin{aligned} &\delta\eta(x,u[0],\cdots,u[k-1]) \\ &= \eta(f(x,u[0]),u[1],\cdots,u[k]) \end{aligned} \tag{5}$$

which plays the same role in a discrete-time system as the derivative of a function of the state and input variables with respect to time does in a continuous-time system.

It is easy to show that the meromorphic function field $\mathcal{K}$ with a shift-operation δ defined by (5) is a difference field.

For vector space $\mathcal{E}^* := \mathcal{X}^* \oplus \mathcal{U}^*$ and a vector $\omega = \alpha dx + \sum_{i=0}^{k} \beta_i du(k) \in \mathcal{E}^*$, where $\alpha, \beta_i;\ i = 0,1,\cdots,k$ are elements of $\mathcal{K}$, we define a shift operation $\delta : \mathcal{E}^* \to \mathcal{E}^*$ as

$$\begin{aligned} \delta\omega &= \delta\alpha d\delta x + \sum_{i=0}^{k} \delta\beta_k d\delta u(k) \\ &= \delta\alpha[\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial u(0)}du(0)] \\ &+ \sum_{i=0}^{k} \delta\beta_k du(k+1) \in \mathcal{E}^* \end{aligned} \tag{6}$$

where $\delta\alpha, \delta\beta_i, i = 0,1,\cdots,k$ are elements of $\mathcal{K}$ according to (5) . Thus, the vector space $\mathcal{E}^* := \mathcal{X}^* \oplus \mathcal{U}^*$ with shift operation δ defined by (6) is a difference vector space.

The maps δ can be used to calculate $dy_i[k+1]$ for $k = 0,1,2,\cdots$, recursively as

$$\begin{aligned} dy_i[k+1] = d(\delta y_i[k]) &= \sum_{j=1}^{n} \delta(\frac{\partial y_i[k]}{\partial x_j})dx_j[1] \\ &+ \sum_{l=0}^{k} \sum_{j=1}^{m} \delta(\frac{\partial y_i[k]}{\partial u_j[l]})du_j[l+1] \end{aligned} \tag{7}$$

Given a set of functions $\{\theta_i \in \mathcal{K},\ i \in I\}$, where I is a countable index set, we order the vectors relevant to the set of functions in the following ordering set.

$$d\theta_1, d\theta_2, \cdots, d\theta_k, \cdots \tag{8}$$

In (8) we use the notation $d\theta_i < d\theta_j$ if $i < j$, and say that $d\theta_i$ is on the left of $d\theta_j$

Definition 2.1. Given an order of vector set $\mathcal{S} = \{d\theta_i,\ i \in I\}$ in the form (8), a vector $d\theta_i$ is called left-dependent in the ordering set if

$$d\theta_i \in span_{\mathcal{K}}\{d\theta_j \in \mathcal{S} \text{ with } d\theta_j < d\theta_i\}$$

otherwise, $d\theta_i$ is called left-independent (or independent).

Lemma 1. Assume that $\{\theta, \theta_1, \theta_2, \cdots, \theta_q\} \subset \mathcal{K}$ and $d\theta_i; i \in \underline{q}$, are independent vectors. If

$$d\theta = \sum_{i=1}^{q} \alpha_i d\theta_i;\ \alpha_i \in \mathcal{K},\ i = 1, 2, \cdots, q$$

then there exists a function, at least locally, such that

$$\theta = \phi(\theta_1, \theta_2, \cdots, \theta_q)$$

Remark 2.1. The function ϕ in Lemma 1 can be calculated numerically (see (Lang, 1999)).

In our linear algebraic framework observability of system (1) can be described as follows.

Definition 2.2. System (1) is observable if

$$dx \subset \mathcal{Y}^* + \mathcal{U}^* \tag{9}$$

Following Lemma 1 the initial state $x[0]$ can be determined from input-output data if the system is observable.

3. STRUCTURE OF LEFT-INVERSE SYSTEMS

The left invertibility of system (1) can be described as follows.

Definition 3.1. System (1) is left invertible if

$$du \in \mathcal{X}^* + \mathcal{Y}^* \tag{10}$$

By this definition of invertibility

$du (:= \{du_1[0], du_2[0], \cdots, du_m[0]\})$ can be spanned by dx, $dy[0], dy[1], \cdots, dy[r]$ for some integer $r \geq 1$. Then from Lemma 1 we have

$$u[0] = \varphi(x, y[0], y[1], \cdots, y[r]) \tag{11}$$

This further implies that the output data can be calculated from the state and input data if the system is left invertible. As the system is time-invariant, we have for $k \geq 1$

$$u[k] = \varphi(x[k], y[k], y[k+1] \cdots, y[k+r]) \tag{12}$$

Recursively applying (1) and (11), (12) we have

Proposition 1. System (1) is left-invertible if and only if

$$\mathcal{U}^* \subset \mathcal{X}^* + \mathcal{Y}^* \tag{13}$$

Definition 3.2. An inverse system is described by a dynamical system

$$\begin{aligned} z[k+1] &= \psi(z[k], y[k], \cdots, y[k+r]) \\ w[k] &= \eta(z[k], y[k], \cdots, y[k+r]) \end{aligned} \tag{14}$$

where $z[k] \in \mathbf{R}^q$ and r is a properly chosen positive integer. (14) is an inverse system for system (1) if there exists $z[0] = z_0$ such that the output $w(k)$ of (14) is equal to the input of system (1) when the output vectors $\{y[k], y[k+1], \cdots, y[k+r]\}; k \geq 0$ of system (1) are taken as the input vectors of (14).

Assume that system (1) is left invertible, then by (11), (12) we have

$$\begin{aligned} & x[k+1] = f(x[k], u[k]) \\ & = f(x[k], \varphi(x[k], y[k], \cdots, y[k+r])) \\ & = \phi(x[k], y[k], \cdots, y[k+r]) \end{aligned} \tag{15}$$

Therefore, (15) together with (12) implies the existence of a left inverse for system (1) if (1) is left-invertible. The order of an inverse system is not necessarily equal to n. In general, we can construct an inverse with order lower than n and the lowest order inverse is called the reduced inverse of system (1).

Lemma 2. Dynamical system (14) is an inverse of system (1) if and only if there are variables denoted by $\{z_1[k], z_2[k], \cdots, z_q[k]\} \subset \mathcal{K}$ such that $du_1[k], \cdots, du_m[k]$ and $dz_1[k+1], dz_2[k+1], \cdots, dz_q[k+1]$ are left-dependent in the following two ordering sets in factor space $\frac{\mathcal{E}^*}{\mathcal{Y}^*}$.

$$\begin{aligned} & dz_1[k], dz_2[k], \cdots, dz_q[k], du_1[k], \\ & du_2[k] \cdots, du_m[k] \pmod{\mathcal{Y}^*} \end{aligned} \tag{16}$$

and

$$\begin{aligned} & dz_1[k], dz_2[k], \cdots, dz_q[k], dz_1[k+1], \\ & dz_2[k+1], \cdots, dz_q[k+1] \pmod{\mathcal{Y}^*} \end{aligned} \tag{17}$$

where $k \geq 0$

Since the systems considered in this paper are time-invariant, the dependence and independence of a vector in a ordering set is independent of the choice of the time instant k hence below we let $k = 0$.

4. CONSTRUCTION OF LEFT INVERSE SYSTEMS AND MINIMALITY

We are now in a position to describe algorithms for constructing dynamical systems with are the left-inverse of system (1). By Lemma 2 the construction of an inverse can be carried out using the following two steps:

Step 1: Find a subset $\{z_i; i \in \underline{\gamma}\}$ of $\mathcal{K}$, which can be written as $\{z_1[0], z_2[0], \cdots, z_\gamma[0]\}$, such that $du_1[0], \cdots, du_m[0]$ are left-dependent in the following ordering set:

$$\begin{aligned} &dz_1[0], dz_2[0], \cdots, dz_\gamma[0], du_1[0], \\ &du_2[0] \cdots, du_m[0] \ (\text{mod } \mathcal{Y}^*) \end{aligned} \tag{18}$$

Step 2: Check the left-dependence of each vector from left to right in the following ordering set:

$$\begin{aligned} &dz_1[0], dz_2[0], \cdots, dz_\gamma[0], \cdots, dz_1[k], \\ &dz_2[k] \cdots, dz_\gamma[k], \cdots \ (\text{mod } \mathcal{Y}^*) \end{aligned} \tag{19}$$

By abuse of notation, we denote all left-independent vectors $\{dz_i[s], i \in \underline{\gamma}, s \in \underline{k}\}$ in ordering set (19) by $dz_{\gamma+1}[0], dz_{\gamma+2}[0], \cdots, dz_q[0]$.

We show how to construct variables $\{z_1, z_2, \cdots, z_\gamma\}$ when system (1) is left-invertible. We also show that the left-independent vectors in (19) are finite, i.e. $q < \infty$. Once the q vectors $dz[0] = \{dz_1[0], dz_2[0], \cdots, dz_q[0]\}$ are obtained, the q vectors $dz[1] = \{dz_1[1], dz_2[1], \cdots, dz_q[1]\}$ will be left-dependent in ordering set (17). Therefore, by Lemma 2, $dz_i[0]; i \in \underline{q}$ can be taken as a basis for the (differential) state space of an inverse system.

We now present two methods for determining $\{z_i; i \in \underline{\gamma}\}$.

4.1 *Method One*

Consider the following ordering set:

$$\begin{aligned} &du_1[0], du_2[0], \cdots, du_m[0], \cdots, du_1[n-1], \\ &du_2[n-1] \cdots, du_m[n-1] \ (\text{mod } \mathcal{Y}^*) \end{aligned} \tag{20}$$

where n(=dim x) is the dimension of the state space of system(1).

Let $\mathcal{R} := \{du_i[j], dy_k[l];\ 1 \leq i \leq m,\ 0 \leq j \leq n-1,\ 1 \leq k \leq p, 0 \leq l \leq N\}$, then under the assumption that system (1) is observable we have

$$\mathcal{X}^* \subset span_{\mathcal{K}}\{\mathcal{R}\}$$

and from the definition of $\mathcal{R}$

$$\mathcal{Y}^* \subset span_{\mathcal{K}}\{\mathcal{R}\}$$

If system (1) is left invertible, then

$$\mathcal{X}^* + \mathcal{Y}^* \subset span_{\mathcal{K}}\{\mathcal{R}\} \subset \mathcal{X}^* + \mathcal{Y}^* \tag{21}$$

Define a subset of $\mathcal{U}^*$ such that

$$\begin{aligned} \mathcal{L} := \{&du_i[k];\ du_i[k] \text{ is left independent} \\ &\text{in(20) for},\ i \in \underline{m},\ 0 \leq k \leq n-1.\} \end{aligned} \tag{22}$$

Thus,

$$\mathcal{X}^* + \mathcal{Y}^* = span_{\mathcal{K}}\{\mathcal{L}\} + \mathcal{Y}^* = span_{\mathcal{K}}\{\mathcal{R}\}$$

If we let $\gamma := |\mathcal{L}|$ denote the cardinality of the set $\mathcal{L}$, then (21) implies that γ is independent of the chosen N provided that N is large enough ((Zheng & Cao, 1990)). Thus, we define $\{dz_i;\ i \in \underline{\gamma}\} := \mathcal{L}$.

Letting dz denote $\{dz_1, dz_2, \cdots, dz_\gamma\}$, it follows that for $i \in \underline{m}$

$$\begin{aligned} &u_i[0] \in \mathcal{X}^* + \mathcal{Y}^* \\ &= span_{\mathcal{K}}\{dz, dy[0], dy[1], \cdots, dy[N]\} \end{aligned} \tag{23}$$

This implies that there exists, at least locally, a map such that for some $r \geq 1$

$$u[0] = \eta(z[0], y[0], \cdots, y[r]) \tag{24}$$

where $z_i[0] = z_i; i \in \underline{\gamma}$.

Since $dz[0] = \mathcal{L}$, (21) implies that $dz[1] \in span_{\mathcal{K}}\{\mathcal{L}\} + \mathcal{Y}^*$. Therefore, by Lemma 2 there exists a dynamical system, at least locally, such that

$$z[1] = \psi(z[0], y[0], y[1], \cdots, y[r+1]) \tag{25}$$

As system (1) is time-invariant, (24) and (25) forms an inverse system for system (1).

Thus, a left-inverse system has been constructed.

4.2 *Method Two*

In this method we do not assume that system (1) is observable. Consider the following two ordering sets:

$$du_1[0], \cdots, du_m[0], \cdots, du_1[n-1], \cdots, \\ du_m[n-1], dx_1, dx_2, \cdots, dx_n \ (\text{mod } \mathcal{Y}^*) \quad (26)$$

and

$$dx_1, dx_2, \cdots, dx_n \ (\text{mod } \mathcal{Y}^*) \quad (27)$$

Define $\gamma_1 := |\mathcal{L}_1|$, where

$\mathcal{L}_1 := \{dx_i; dx_i$ is left independent in (26), $i \in \underline{n}\}$

Remark 4.1. It is easy to see that all vectors in $\mathcal{L}_1$ are unobservable under definition (9).

Define $\gamma_2 := |\mathcal{L}_2|$, where

$\mathcal{L}_2 := \{dx_i; dx_i \ i \in \underline{n}\}\}$ with $dx_i \ i \in \underline{n}$ is left independent in (27) and observe that since $\mathcal{L}_1 \subset \mathcal{L}_2$ it follows that $\gamma_1 \leq \gamma_2$. Under the assumption that system (1) is left invertible, we have the following

Theorem 4.1. $\gamma_2 - \gamma_1 = \gamma$, where the γ is defined in section 4.1 above.

Proof: By left invertibility of system (1) and (13)

$$\mathcal{X}^* + \mathcal{U}^* + \mathcal{Y}^* = \mathcal{X}^* + \mathcal{Y}^* = \mathcal{X}^* + \mathcal{U}^*$$

Delete dependent vectors of dx in (26), (27) and change, if necessary, the order of coordinates of state variables, we obtain

$$\begin{aligned} &\mathcal{X}^* + \mathcal{U}^* + \mathcal{Y}^* \\ &= span_{\mathcal{K}}\{dy[0], dy[1], \cdots, \cdots, dy[N], du[0], \\ &\quad \cdots, du[n-1], dx_1, dx_2, \cdots, dx_{\gamma_1}\} \\ &= span_{\mathcal{K}}\{dy[0], dy[1], \cdots, \cdots, dy[N], \\ &\quad dx_1, dx_2, \cdots, dx_{\gamma_1}, \cdots, dx_{\gamma_2}\} \end{aligned} \quad (28)$$

which further implies that

$$\begin{aligned} &\dim span_{\mathcal{K}}\{dy[0], dy[1], \cdots, \cdots, dy[N], du[0], \\ &\quad \cdots, du[n-1], dx_1, dx_2, \cdots, dx_{\gamma_1}\} \\ &-\dim span_{\mathcal{K}}\{dy[0], dy[1], \cdots, \cdots, dy[N]\} \\ &= \dim span_{\mathcal{K}}\{dy[0], dy[1], \cdots, \cdots, dy[N], \\ &\quad dx_1, dx_2, \cdots, dx_{\gamma_1}, dx_{\gamma_1+1}, \cdots, dx_{\gamma_2}\} \\ &-\dim span_{\mathcal{K}}\{dy[0], dy[1], \cdots, \cdots, dy[N]\} \end{aligned} \quad (29)$$

As $\{dx_i; i \in \underline{\gamma_1}\}$ and $\{dx_j; j \in \underline{\gamma_2}\}$ are left-independent in ordering set (26) and (27), repectively, we obtain that

$$\gamma + \gamma_1 = \gamma_2$$

□

We can now present the second method for constructing a left-inverse of system (1). By ordering sets (26), (27) we define $z_1 = x_{\gamma_1+1}, z_2 = x_{\gamma_1+2}, \cdots, z_\gamma = x_{\gamma_2}$. By definition $dx_1, dx_2, \cdots, dx_{\gamma_1}$ are left independent in both (26) and (27).

It is known from invertibility and the definition of $\mathcal{L}_2$ that

$$\begin{aligned} &span_{\mathcal{K}}\{du[0], \cdots, du[n-1]\} \\ &\subset span_{\mathcal{K}}\{dz, dy[0], \cdots, \cdots, dy[N]\} \end{aligned}$$

It must therefore hold that

$$du[0] \subset span_{\mathcal{K}}\{dz, dy[0], \cdots, \cdots, dy[N]\} \quad (30)$$

On the other hand, $dx_{\gamma_1+1} \cdots, dx_{\gamma_2}$ are left dependent in ordering set (26) even if $dx_1, dx_2, \cdots, dx_{\gamma_1}$ are deleted from (26) because they are observable, i.e. $\{dx_j; \gamma_1 < j \leq \gamma_2\} \subset \mathcal{Y}^* + \mathcal{U}^*$ by the definition of $\mathcal{L}_1$. Thus,

$$\begin{aligned} &z[1] \subset span_{\mathcal{K}}\{du[0], \cdots, du[n-1]\} \\ &+span_{\mathcal{K}}\{dz, dy[0], dy[1], \cdots, \cdots, dy[N]\} \\ &\subset span_{\mathcal{K}}\{dz, dy[0], dy[1], \cdots, \cdots, dy[N]\} \end{aligned} \quad (31)$$

Therefore, by (30) and (31) another structure for a left inverse system is obtained by Lemma 2.

Theorem 4.2.

(1) The minimal order of inverse systems of system (1) is $\gamma = |\mathcal{L}|$.

(2) The inverse systems constructed in the previous section are minimal order inverse systems of system (1)

Proof: Assume there exists an inverse system described by (14). For $N \geq k \geq 0$

$$span_{\mathcal{K}}\{dw[k]\} \subset span_{\mathcal{K}}\{dz, dy[0], \cdots, \cdots, dy[N]\}$$

Let $w[k] = u[k]$ and dim $z = q$, then for $i \in \underline{m}, N \geq k \geq 0$

$$du_i[k] \in span_{\mathcal{K}}\{dz, dy[0], dy[1], \cdots, \cdots, dy[N]\}$$

Thus, $du_i[k]$ can be represented as a function of some variables of $\{dz, dy[0], dy[1], \cdots, \cdots, dy[N]\}$ for each i, k.

By definition

$$\begin{aligned} &\dim span_{\mathcal{K}}\{du_i[k]\} = \gamma \\ &(\text{mod } span_{\mathcal{K}}\{dy[0], dy[1], \cdots, \cdots, dy[N]\}) \end{aligned}$$

then $\gamma \leq q$. (1) is proved. (2) is a consequence of (1). □

Example 4.1. Consider the following system.

$$\begin{aligned} x_1[k+1] &= x_2[k] + x_4[k]u_1[k] \\ x_2[k+1] &= x_1[k] + x_3[k] \\ x_3[k+1] &= x_3[k] + x_2[k]u_2[k] \\ x_4[k+1] &= \frac{1}{2}x_1[k]x_3[k] \end{aligned} \quad (32)$$

$$\begin{aligned} y_1[k] &= x_1[k] \\ y_1[k] &= x_2[k] \end{aligned}$$

Calculate the outputs of the system for $k \geq 0$,

$$
\begin{aligned}
&y_1[0] = x_1,\\
&y_2[0] = x_2,\\
&y_1[1] = x_2 + x_4 u_1[0],\\
&y_2[1] = x_1 + x_3,\\
&y_1[2] = x_1 + x_3 + \frac{1}{2} x_1 x_3 u_1[1],\\
&y_2[2] = x_2 + x_3 + x_4 u_1[0] + x_2 u_2[0],\\
&y_1[3] = x_2 + x_3 + x_4 u_1[0] + x_2 u_2[0]\\
&+\frac{1}{2}(x_2 x_3 + x_2^2 u_2[0] + x_3 x_4 u_1[0]\\
&+x_2 x_4 u_1[0] u_2[0]) u_1[2],\\
&y_2[3] = x_1 + 2x_3 + x_2 u_2[0] + \frac{1}{2} x_1 x_3 u_1[1] +\\
&(x_1 + x_3) u_2[1],\\
&\quad \vdots
\end{aligned}
\tag{33}
$$

It can be shown that

$$\{du_1[0], du_2[0]\} \subset \mathcal{X}^* + \mathcal{Y}^*$$

then the system is left invertible.

It can be check that $du_1[0]$ is the only left-independent vector in the following ordering set

$$
\begin{aligned}
&dy_1[0], dy_2[0], dy_1[1], dy_2[1], dy_1[2], dy_2[2], \cdots,\\
&du_1[0], du_2[0], du_1[1], du_2[1], \cdots
\end{aligned}
$$

In fact, $u_2[0] = \frac{y_2[2] - y_1[1] - y_2[1] + y_1[0]}{y_2[0]}$.

As $u_1[1] = 2\frac{y_1[2] - y_2[1]}{y_1[0] y_2[1] - y^2[0]}$, and

$$
\begin{aligned}
du_2[1] \in\ & span_{\mathcal{K}}\{dy_1[0], dy_2[0], dy_1[1], dy_2[1],\\
& dy_1[2], dy_2[2], dy_2[3], du_1[0], du_2[0], du_1[1]\}
\end{aligned}
$$

By recursive calculation, we see that for $k \geq 1,\ i = 1, 2$

$$
\begin{aligned}
du_i[k] \in\ & span_{\mathcal{K}}\{du_1[0], du_2[0], \cdots,\\
& du_1[k-1], du_2[k-1]\} + \mathcal{Y}^*
\end{aligned}
$$

$$
\begin{aligned}
du_2[k] \in\ & span_{\mathcal{K}}\{du_1[0], du_2[0], \cdots, du_2[k-1],\\
& du_2[k-1], du_1[k]\} + \mathcal{Y}^*
\end{aligned}
$$

Thus, $\gamma = 1$. Now we construct a minimal order inverse as follows. Let $z = u_1[0]$, then $z[1] = u_1[1]$. The left inverse for $k \geq 0$ becomes

$$
\begin{aligned}
&z[k+1] = 2\frac{y_1[k+2] - y_2[k+1]}{y_1[k] y_2[k+1] - y^2[k]}\\
&w_1[k] = z[k]\\
&w_2[k] (= u_2[k])\\
&= \frac{y_2[k+2] - y_1[k+1] - y_2[k+1] - y_1[k]}{y_2[k]}
\end{aligned}
$$

We let $\mathcal{L}_2 = \{dx_4\}$ and $z = x_4$. Thus, $z[1] = x_4[1] = \frac{1}{2} x_1 x_3 = \frac{1}{2} y_1[0](y_2[1] - y_1[0])$.

As $u_1[0] = \frac{y_1[1] - y_2[0]}{x_4}$, the left-inverse by the second method is

$$
\begin{aligned}
z[k+1] &= \frac{1}{2} y_1[k](y_2[k+1] - y_1[k])\\
w_1[k]\ (= u_1[k]) &= \frac{y_1[k+1] - y_2[k]}{z[k]}\\
w_2[k]\ (= u_2[k]) &=\\
&\frac{y_2[k+2] - y_1[k+1] - y_2[k+1] + y_1[k]}{y_2[k]}
\end{aligned}
$$

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