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MODELING AND CONTROL OF A TOWED SEISMIC STREAMER

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Abstract: This paper presents results on the modeling and control of a long towed cable. The cable is divided into cable elements connected by depth actuators. Each cable element is modeled as a three-dimensional string, where the cable is allowed to possess nonlinear tension. The passivity properties of the cable element are established. Exponetial decay of energy in the coupled transversal motion of the cable element is achieved by boundary control in the presence of vanishing perturbations. When subjected to nonvanishing perturbations, the energy growth in the cable is shown to be upper bounded. A cascaded cable system is then considered, and the passivity properties of the cascaded cable system are established. Using Lyapunov's direct method, asymptotic decay of energy is achieved in the cascaded system with passive controllers applied at the connection points.

Keywords: Modelling, Boundary Value Problem, Lyapunov Stability, Exponentially Stable, Cables, Marine Systems

1. INTRODUCTION

Offshore towing of seismic sensor arrays is a method used extensively in the search for hydrocarbon reservoirs under the seabed. Seismic surveying operations usually employ a cable towing configuration which consists of negatively buoyant lead-in cables attached to a towing vessel and, at the other end, to neutrally buoyant cables called streamers, which contain hydrophones to detect the reflected acoustic pulses from a towed acoustic source. A typical towed cable configuration is illustrated in Figure 1.

The dynamics of the cables/strings have been studied by many authors: (Choo and Casarella, 1973), (Päidoussis, 1973), (Ablow and Schlechter, 1983), (Triantafyllou and Chryssostodimis, 1988), (Dowling, 1988) and the references therein. Boundary control aspects of the distributed-parameter cable/string systems have been investigated by several authors. Among others, (Morgül, 1994) designed

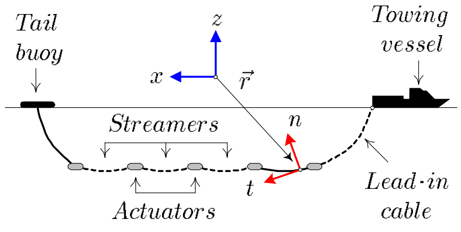

Fig. 1. A typical towing configuration for seismic surveys.

a boundary feedback controller for a system described by the wave equation where exponential stability of the closed loop is obtained for strictly proper transfer functions. (Baicu *et al.*, 1999) developed exponentially stabilizing controllers for the transverse vibration of a string-mass system modeled by one-dimensional wave equation. (Shahruz and Narasimha, 1997) and (Canbolat *et al.*, 1998) devised exponentially stabilizing controllers for a one-dimensional nonlinear string equation, allowing varying tension in the string. (Qu, 2000) devised a robust and adaptive controller to damp out the transverse oscillations of a stretched string, allowing nonlinear dynamics and their uncertainties in the model. In the works mentioned above, one typically uses a proper combination of the information available at the boundary, namely the boundary position, slope, slope-rate and velocity. In this way, simple and efficient controllers are achieved. Controllability and stabilizability of a system of coupled strings with control applied at the connection points is studied by Ho (Ho, 1993). By investigating the properties of certain exponential series, a sufficient condition for exact controllability is obtained in terms of the Riesz basis properties of the exponential series. A similar work on the energy decay problems in the design of a point stabilizer for coupled string vibrating systems has been carried out by Liu (Liu, 1988), using an approach of abstract semi-groups.

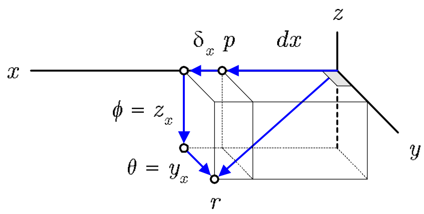

Fig. 2. Kinematic consideration of cable.

In this paper, a long towed cable is modeled with a certain number of actuators placed along the cable. Due to the discontinuities at each point where an actuator is placed, the cable is divided into cable elements. The cable elements are assumed to be connected to each other through the actuators, which act transversely. The dynamics of the towing vessel and the actuators are not included in the model. Each cable element is modeled in a three-dimensional string-like form where the cable is allowed to possess nonlinear tension while being exposed to nonvanishing perturbations. The passivity property of the cable element in its nonlinear form is established to justify the use of linear passive controllers at its boundaries. Then, it is shown that the exponential decay of the total cable energy can be achieved for a cable system with two-degrees of freedom (the coupled transversal motion). When subjected to nonvanishing perturbations, an upper bound for the energy growth in the cable element is found. Thereafter a cascaded cable system is considered and the passivity property of the cascaded cable system is established. Using Lyapunov's direct method, it is shown that asymptotic decay of energy can be achieved in the cascaded system with passive controllers applied at the connection points.

Through out this paper we let $(\cdot)_x$, $(\cdot)_{xt}$, $(\cdot)_t$, $(\cdot)_{xx}$ and $(\cdot)_{tt}$ denote $\frac{\partial}{\partial x}(\cdot)$, $\frac{\partial^2}{\partial x \partial t}(\cdot)$, $\frac{\partial}{\partial t}(\cdot)$, $\frac{\partial^2}{\partial x^2}(\cdot)$ and $\frac{\partial^2}{\partial t^2}(\cdot)$, respectively.

2. EQUATIONS OF MOTION

2.1 Kinematics

An inertial reference frame i is defined with orthogonal unit vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ along the x, y and z axes, respectively. The spatial position of an arbitrary point on the center line of the *stretched* cable is given by the vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Consider an arbitrary point on the *undeformed* cable with coordinates $\mathbf{p} = [x, 0, 0]^T$ as shown in Figure (2). Due to external forces the cable is deformed so that the material point described by the material coordinates $\mathbf{p}$ will have the spatial coordinates $\mathbf{r} = [x + \delta, y, z]^T$. This gives

$$\|\mathbf{r}_x\| \approx 1 + \delta_x + \tfrac{1}{2}y_x^2 + \tfrac{1}{2}z_x^2 \tag{1}$$

where right-hand side of (1) is the binomial approximation of $\|\mathbf{r}_x\|$. A cable frame c of orthonormal vectors $\mathbf{t}$, $\mathbf{b}$ and $\mathbf{n}$ is defined where $\mathbf{t}$ is the unit vector tangent to the cable such that $\mathbf{r}_x = (1+\epsilon)\,\mathbf{t}$ where ϵ is the Lagrangian strain. Hence, from (1) the relation for the strain can be obtained as

$$\epsilon \approx \delta_x + \tfrac{1}{2}y_x^2 + \tfrac{1}{2}z_x^2 = \left(\boldsymbol{\xi}^T \mathbf{r}_x\right) \tag{2}$$

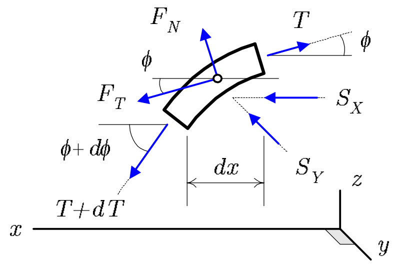

Fig. 3. The forces acting on an elemental length dx.

where

$$\boldsymbol{\xi} = [\delta_x/(1+\delta_x), \tfrac{1}{2}y_x, \tfrac{1}{2}z_x]^T$$

2.2 Dynamics

The dynamic of the cable is assumed to be determined by the tension in the cable, the inertial forces and the hydrodynamic normal and tangential drag forces as shown in Figure(3). Using Hooke's law, the tension T in the cable can be expressed in the form

$$T = \tau + e\epsilon = \tau + e\left(\boldsymbol{\xi}^T \mathbf{r}_x\right) \tag{3}$$

where τ is the constant tension in the undeformed cable, $e = EA$, E is the Young's modulus and A is the cross section of the cable.

2.2.1. Three-Dimensional Model

Consider a material cable element of spatial length dx. Writing equilibrium of the forces in the x, y and z directions and using (2) in (3) gives the *nonlinear coupled equations* of motion in the matrix form

$$\mathbf{M}\mathbf{r}_{tt} = e\mathbf{K}\mathbf{r}_{xx} + \mathbf{H} \tag{4}$$

where $\mathbf{r} = [x+\delta, y, z]^T$ and $\mathbf{M}$=$m\mathbf{I}_3$, m is the mass per unit length, $\mathbf{I}_3 \in \mathbb{R}^{3\times 3}$ is the identity matrix and

$$\mathbf{H}(\mathbf{r}_t, \mathbf{r}_x, \mathbf{S}) = \begin{bmatrix} F_T \\ F_B + F_T y_x \\ F_N - F_T z_x \end{bmatrix} \tag{5}$$

$$\mathbf{K} = \begin{bmatrix} 1 & y_x & z_x \\ y_x & y_x^2 + \frac{1}{e}T & y_x z_x \\ z_x & y_x z_x & z_x^2 + \frac{1}{e}T \end{bmatrix} \tag{6}$$

where $F_T(\mathbf{r}_t, \mathbf{r}_x, \mathbf{S})$, $F_B(\mathbf{r}_t, \mathbf{r}_x, \mathbf{S})$ and $F_N(\mathbf{r}_t, \mathbf{r}_x, \mathbf{S})$ are the hydrodynamic drag forces approximated by Morison's equation (Faltinsen, 1990) decomposed in the cable frame. $\mathbf{S} = [S_X, S_Y, 0]^T$ is the relative velocity between the surrounding fluid and towing vessel. $\mathbf{H}(\mathbf{r}_t, \mathbf{r}_x, \mathbf{S})$ is approximated to be an odd function of $\mathbf{r}_t$ such that $\mathbf{H}^T\mathbf{r}_t < 0$ when $\mathbf{S} = 0$. The initial conditions for (4) are

$$\mathbf{r}(x, 0) = \bar{\mathbf{r}}(x), \quad \mathbf{r}_t(x, 0) = \bar{\mathbf{r}}_t(x) \tag{7}$$

with boundary conditions

$$T(0)\,\mathbf{r}_x(0) = \mathbf{u}_0(t), \quad T(L)\,\mathbf{r}_x(L) = \mathbf{u}_L(t) \tag{8}$$

where $\mathbf{u}_0(t)$ and $\mathbf{u}_L(t)$ are the boundary control variables. The three-dimensional model will be used in the passivity analysis of the model dynamics in the following sections.

2.2.2. Two-Dimensional Model If only the transverse motions are of interest, the axial elongation may be ignored by simply setting $\delta = 0$ to obtain the two-dimentional model formulation

$$\mathbf{M}\mathbf{r}_{tt} = e\mathbf{K}\mathbf{r}_{xx} + \mathbf{H} \tag{9}$$

where $\mathbf{r} = (y,z)^T$ and $\mathbf{M}{=}m\mathbf{I}_2$ with

$$\mathbf{K} = \left[\mathbf{r}_x\mathbf{r}_x^T + \tfrac{1}{e}T\mathbf{I}\right] \tag{10}$$

$$\mathbf{H} = \left[(-F_B - F_T y_x),\ (-F_N + F_T z_x)\right]^T \tag{11}$$

where $\mathbf{I}_2 \in \mathbb{R}^{2\times 2}$ is the identity matrix. Similarly, the initial conditions for (9) are

$$\mathbf{r}(x,0) = \bar{\mathbf{r}}(x), \quad \mathbf{r}_t(x,0) = \bar{\mathbf{r}}_t(x) \tag{12}$$

with boundary conditions

$$T(0)\mathbf{r}_x(0) = \mathbf{u}_0(t), \quad T(L)\mathbf{r}_x(L) = \mathbf{u}_L(t) \tag{13}$$

where $\mathbf{u}_0(t)$ and $\mathbf{u}_L(t)$ are the boundary control variables. The two-dimensional model will be used in the stability analysis of the model dynamics in the following sections.

3. STABILITY ANALYSIS OF A CABLE ELEMENT

3.1 Passivity of the Coupled Cable Dynamics

In this section, the passivity properties of the cable dynamics given in (4) will be established. The inputs are the end point forces and the outputs are the end point velocities.

Theorem 1. The system dynamic in (4)–(6) with initial and boundary conditions (7)–(8) and $\mathbf{S} = \mathbf{0}$, is state strictly passive from $\mathbf{u} \mapsto \mathbf{v}$ where

$$\mathbf{u} = \begin{bmatrix} \mathbf{u}_L(L,t) \\ -\mathbf{u}_0(0,t) \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} \mathbf{r}_t(L,t) \\ \mathbf{r}_t(0,t) \end{bmatrix} \tag{14}$$

and

$$\mathbf{u}_0(x,t) = \begin{bmatrix} e\boldsymbol{\xi}^T(0)\mathbf{r}_x(0) \\ T(0)y_x(0) \\ T(0)z_x(0) \end{bmatrix}$$

$$\mathbf{u}_L(x,t) = \begin{bmatrix} e\boldsymbol{\xi}^T(L)\mathbf{r}_x(L) \\ T(L)y_x(L) \\ T(L)z_x(L) \end{bmatrix}$$

.

Proof. Consider the following natural energy function $V = V_k + V_p$ for $(x,t) \in [0,L]\times[0,\infty)$ where

$$V_k = \tfrac{1}{2}m\int_0^L \mathbf{r}_t^T\mathbf{r}_t dx \tag{15}$$

$$V_p = \tfrac{1}{2}\tau\int_0^L \left[\mathbf{r}_x^T\mathbf{r}_x - (1+\delta_x)^2\right]dx$$
$$+\tfrac{1}{2}e\int_0^L \left(\boldsymbol{\xi}^T\mathbf{r}_x\right)^2 dx \tag{16}$$

where V_k is the kinetic energy and V_p is the potential energy of the system. Clearly, $V(\mathbf{r}_x,\mathbf{r}_t) > 0$ for all $\mathbf{r}_x, \mathbf{r}_t \neq \mathbf{0}$. The time derivative of V along the solution of the system is

$$\begin{aligned}\dot{V} = {} & e\left[\delta_t\delta_x\right]_0^L + \tfrac{1}{2}e\left[\delta_t y_x^2\right]_0^L + \tfrac{1}{2}e\left[\delta_t z_x^2\right]_0^L \\ & +\tau\left[y_t y_x\right]_0^L + e\left[y_t y_x \delta_x\right]_0^L \\ & +\tfrac{1}{2}e\left[y_t y_x y_x^2\right]_0^L + \tfrac{1}{2}e\left[y_t y_x z_x^2\right]_0^L \\ & +\tau\left[z_t z_x\right]_0^L + e\left[z_t z_x \delta_x\right]_0^L \\ & +\tfrac{1}{2}e\left[z_t z_x y_x^2\right]_0^L + \tfrac{1}{2}e\left[z_t z_x z_x^2\right]_0^L \\ & +\int_0^L \mathbf{H}^T\mathbf{r}_t dx \end{aligned} \tag{17}$$

where the integration by parts has been used. This can be written

$$\dot{V} = \mathbf{u}^T\mathbf{v} + \int_0^L \mathbf{H}^T\mathbf{r}_t dx \tag{18}$$

where $\mathbf{u}$, $\mathbf{v}$ are defined in (14). Since $\int_0^L \mathbf{H}^T\mathbf{r}_t dx < 0$ it can be concluded that the system dynamic in (4)–(6) is passive, strictly speaking, $\mathbf{u} \mapsto \mathbf{v}$ is state strictly passive according to ((Khalil, 1996), def. 10.4, pp.439). ■

3.2 Transversal Boundary Control

In this section transversal boundary control of the nonlinear coupled cable dynamics given in (9)–(13) is studied. A cross term V_c is introduced (Shahruz and Narasimha, 1997) in the energy function V to show exponential stability :

$$V = V_0 + V_c, \quad V_0 = V_k + V_p \tag{19}$$

for $(x,t) \in [0,L]\times[0,\infty)$ where

$$V_k = \tfrac{1}{2}m\int_0^L \mathbf{r}_t^T\mathbf{r}_t dx \tag{20}$$

$$V_p = \tfrac{1}{2}\tau\int_0^L \mathbf{r}_x^T\mathbf{r}_x dx + \tfrac{1}{2}e\int_0^L \left(\tfrac{1}{2}\mathbf{r}_x^T\mathbf{r}_x\right)^2 dx \tag{21}$$

$$V_c = \gamma m\int_0^L x\mathbf{r}_x^T\mathbf{r}_t dx \tag{22}$$

Theorem 2. Consider the system (9) with $\mathbf{S} = \mathbf{0}$, that is, the system

$$\mathbf{M}\mathbf{r}_{tt} = e\mathbf{K}\mathbf{r}_{xx} + \mathbf{H} \tag{23}$$

with the initial and the boundary conditions (12)–(13) where the inputs have been set to

$$\mathbf{u}_0(t) = \kappa_1\mathbf{r}_t(0,t) \tag{24}$$

$$\mathbf{u}_L(t) = -\kappa_2\mathbf{r}_x(L,t) - \kappa_3\mathbf{r}_t(L,t) \tag{25}$$

Suppose that there exist nonnegative constants L, κ_1, κ_2 and κ_3 so that

$$\kappa_2 > \kappa_3, \quad \kappa_2 - \kappa_3 > \frac{2}{3L} \tag{26}$$

and $L \geq 1$. Then, the function $V(x,t)$ in (19) decays exponentially to zero along the trajectories of (23) with a decay rate $\gamma/(1+\lambda\gamma)$, where γ and λ satisfy

$$0 < \gamma < \frac{2\kappa_2 + 4\kappa_3}{3L\kappa_3\left(\kappa_2 + \kappa_3 + \frac{2}{3\kappa_3}m\right)} < 1 \tag{27}$$

$$\gamma < 1/\lambda, \quad \lambda = L\max\{1, m/\tau, m/e\} \tag{28}$$

Next, let $\mathbf{H}$ be a nonvanishing perturbation of (23) with $\mathbf{S} \neq \mathbf{0}$, and satisfy the inequality

$$\|\mathbf{H}\|_{L_2(0,L)} \leq \alpha(t)\|\mathbf{r}_t\|_{L_2(0,L)} + \beta(t) \tag{29}$$

where $\alpha(t)$ is nonnegative and continuous for $t \in [0,\infty)$, and $\beta(t)$ is a nonnegative, continuous and bounded for $t \in [0,\infty)$. Assume that $\alpha(t)$ satisfies the inequality

$$\int_{t_0}^{t} \alpha(\tau)\, d\tau \le \sigma_1 (t - t_0) + \sigma_2 \tag{30}$$

for some nonnegative constants σ_1 and σ_2, where

$$\sigma_1 < \frac{c_1}{\sqrt{2}\eta\mu}\frac{\gamma}{c_2} \tag{31}$$

and

$$c_1 = 1 - \gamma\lambda, \quad c_2 = 1 + \gamma\lambda \tag{32}$$

$$\mu^2 = \max\{1/m, 1/\tau, 1/e\} \tag{33}$$

$$\eta^2 = \max\{2c_2/m, 2c_2\gamma L/\tau, 2/e\} \tag{34}$$

Under these conditions, if the following conditions are satisfied

$$V(t_0) < (R/\varrho)^2 \tag{35}$$

$$\sup_{t \ge t_0} \beta(t) < \frac{2c_1\zeta}{\mu\eta}(R/\varrho) \tag{36}$$

for some nonnegative constants $R > 0$, $\varrho \ge 1$ and

$$\zeta = \left(\gamma/2c_2 - \sigma_1\mu\eta/\sqrt{2}c_1\right)$$

then the energy growth in $V(x,t)$ given by (19) will be upper bounded by

$$V(x,t) \le R^2 \tag{37}$$

Proof. Let $0 < \gamma < 1$. Taking the time derivatives of (20)–(22) along the solutions of (23) and applying integration by parts to the results yields

$$\begin{aligned}
\dot{V} = {} & \tau\left[\mathbf{r}_t^T\mathbf{r}_x\right]_0^L + \tfrac{1}{2}e\left[\left(\mathbf{r}_x^T\mathbf{r}_x\right)\mathbf{r}_x^T\mathbf{r}_t\right]_0^L \\
& + \tfrac{1}{2}\gamma\tau\left[x\mathbf{r}_x^T\mathbf{r}_x\right]_0^L - \tfrac{1}{2}\gamma\tau\int_0^L \mathbf{r}_x^T\mathbf{r}_x dx \\
& + \tfrac{3}{8}\gamma e\left[x\left(\mathbf{r}_x^T\mathbf{r}_x\right)^2\right]_0^L - \tfrac{3}{8}\gamma e\int_0^L \left(\mathbf{r}_x^T\mathbf{r}_x\right)^2 dx \\
& + \tfrac{1}{2}\gamma m\left[x\mathbf{r}_t^T\mathbf{r}_t\right]_0^L - \tfrac{1}{2}\gamma m\int_0^L \mathbf{r}_t^T\mathbf{r}_t dx \\
& + \int_0^L \mathbf{H}^T\mathbf{r}_t dx
\end{aligned} \tag{38}$$

To be able to simplify (38) even more, consider the following:

Remark 1. Note that $V(x,t)$ is lower and upper bounded by

$$c_1 V_0 \le V \le c_2 V_0 \tag{39}$$

where $c_1 = 1 - \gamma\lambda$, $c_2 = 1 + \gamma\lambda$. Let $\gamma < 1/\lambda$ and consider the following inequalities

$$\begin{aligned}
V_c &= \gamma m \int_0^L x\mathbf{r}_t^T\mathbf{r}_x dx \\
&\le \gamma m \int_0^L |x|\,\|\mathbf{r}_t\|\,\|\mathbf{r}_x\|\, dx \\
&\le \gamma L \int_0^L \tfrac{1}{2}m\left[\|\mathbf{r}_t\|^2 + \|\mathbf{r}_x\|^2 + \left(\tfrac{1}{2}\mathbf{r}_x^T\mathbf{r}_x\right)^2\right] dx \\
&\le \gamma\lambda V_0
\end{aligned} \tag{40}$$

where $\lambda = L\max\{1, m/\tau, m/e\}$. Thus, (39) follows from (19) and (40).

For simplicity of notation, let $[\mathbf{r}(x,t)]_0^L = \mathbf{r}(L) - \mathbf{r}(0)$ and $T(L) = T_L, T(0) = T_0$ represent the values at the boundaries. Evaluation of (38) at the boundaries leads to

$$\begin{aligned}
\dot{V} \le {} & -\gamma V_0 + T_L\mathbf{r}_x^T(L)\,\mathbf{r}_t(L) - T_0\mathbf{r}_x^T(0)\,\mathbf{r}_t(0) + \\
& \tfrac{3}{4}\gamma LT(L)\,\|\mathbf{r}_x(L)\|^2 + \tfrac{1}{2}\gamma mL\,\|\mathbf{r}_t(L)\|^2
\end{aligned} \tag{41}$$

where T is defined in (3). Inserting (24)–(25) into (41) yields

$$\begin{aligned}
\dot{V} \le {} & -\gamma V_0 - \kappa_1\|\mathbf{r}_t(0)\|^2 \\
& + \tfrac{3}{4}\gamma LT_L\left\|-\frac{\kappa_2}{T_L}\mathbf{r}_x(L) - \frac{\kappa_3}{T_L}\mathbf{r}_t(L)\right\|^2 \\
& - \kappa_2\mathbf{r}_x^T(L)\,\mathbf{r}_t(L) - \kappa_3\|\mathbf{r}_t(L)\|^2 \\
& + \tfrac{1}{2}\gamma mL\,\|\mathbf{r}_t(L)\|^2 + \int_0^L \mathbf{H}^T\mathbf{r}_t dx
\end{aligned}$$

which can be further simplified as

$$\begin{aligned}
\dot{V} \le {} & -\gamma V_0 \\
& - \left[\tfrac{1}{2}\kappa_2 - \tfrac{3}{4}\gamma L\left(\kappa_2^2 + \kappa_2\kappa_3\right)\right]\|\mathbf{r}_x(L)\|^2 \\
& - \left[-\tfrac{3}{4}\gamma L\kappa_2\kappa_3 - \tfrac{3}{4}\gamma L\kappa_3^2 + \right. \\
& \left.\tfrac{1}{2}\kappa_2 + \kappa_3 - \tfrac{1}{2}\gamma mL\right]\|\mathbf{r}_t(L)\|^2 \\
& + \int_0^L \mathbf{H}^T\mathbf{r}_t dx
\end{aligned} \tag{42}$$

(26) follows from the second term on the right-hand side of (42). From the third term inequality (27) is obtained. Hence, using (39) it can be shown that $\dot{V} \le -\gamma/(1+\gamma\lambda)\,V$ which implies that the energy stored in the system in (23) decays exponentially to zero. Next, consider the system in (23) with the perturbation term satisfying (29). $\dot{V}(x,t)$ can be rewritten as

$$\dot{V} \le -\frac{\gamma}{c_2}V + \int_0^L \mathbf{H}^T(\mathbf{r}_t + \gamma x\mathbf{r}_x)\, dx \tag{43}$$

Now, consider the last term in (43). Applying Hölder's inequality to this term gives

$$\begin{aligned}
\int_0^L \mathbf{H}^T(\mathbf{r}_t + \gamma x\mathbf{r}_x)\, dx \le & \left\{\int_0^L |\mathbf{r}_t + \gamma x\mathbf{r}_x|^2\, dx\right\}^{1/2} \\
& \times\left\{\int_0^L |\mathbf{H}|^2\, dx\right\}^{1/2}
\end{aligned} \tag{44}$$

At this moment, consider the following

Remark 2. Similar to the derivations in Remark 1, one can write

$$\begin{aligned}
|\mathbf{r}_t + \gamma x\mathbf{r}_x|^2 \le {} & \|\mathbf{r}_t\|^2 + (\gamma L)^2\|\mathbf{r}_x\|^2 \\
& + \gamma L\left(\|\mathbf{r}_t\|^2 + \|\mathbf{r}_x\|^2\right)
\end{aligned} \tag{45}$$

For $\eta^2 = \max\{2c_2/m, 2c_2\gamma L/\tau, 2/e\}$, the first multiplicand in (44) satisfies the inequality

$$\int_0^L |\mathbf{r}_t + \gamma x\mathbf{r}_x|^2\, dx \le \eta^2 V_0 \tag{46}$$

Remark 3. The perturbation term $\mathbf{H}$ in (29) is upper bounded by

$$\|\mathbf{H}\|_{L_2(0,L)} \le \sqrt{2/c_1}\mu\alpha(t)\,V^{1/2} + \beta(t) \tag{47}$$

where $\mu^2 = \max\{1/m, 1/\tau, 1/e\}$. Here again, it is desirable to reconstruct the energy function V_0. Thus, from (29) the following inequalities can be written

$$\begin{aligned}
\|\mathbf{H}\|_{L_2(0,L)} &\le \alpha(t)\left\{\int_0^L \|\mathbf{r}_t\|^2\, dx\right\}^{1/2} + \beta(t) \\
&\le \sqrt{2}\alpha(t)\left\{\mu^2 V_0\right\}^{1/2} + \beta(t) \\
&\le \sqrt{2/c_1}\mu\alpha(t)\,V^{1/2} + \beta(t)
\end{aligned} \tag{48}$$

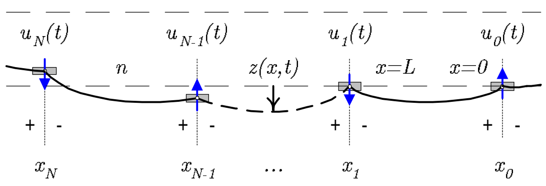

Fig. 4. Cascaded cable configuration.

Hence, from (43), (44), (46) and (48) the following inequality can be obtained

$$\dot{V} \leq -\left[\frac{\gamma}{c_2} - \frac{\sqrt{2}\mu\eta}{c_1}\alpha(t)\right]V + \frac{\eta}{c_1}\beta(t)V^{1/2} \quad (49)$$

where the fact $c_1 < \sqrt{c_1}$ is used to increase the upper bound of $\dot{V}$ in the last term. To obtain a linear differential inequality, set $W = V^{1/2}$ so that $\dot{W} = \dot{V}/(2V^{1/2})$. When $V \neq 0$, that gives

$$\dot{W} \leq -\frac{1}{2}\left[\frac{\gamma}{c_2} - \frac{\sqrt{2}\mu\eta}{c_1}\alpha(t)\right]W + \frac{\eta}{2c_1}\beta(t) \quad (50)$$

When $V = 0$, the right-hand side derivative of $W(t)$ is $D^+W(t) \leq \eta\beta(t)/2c_1$. Hence, $D^+W(t)$ satisfies (50) for all values of $V(t)$. By the comparison lemma ((Khalil, 1996), lemma 2.5, pp.85) $W(t)$ satisfies the inequality

$$\begin{aligned} W(t) \leq{}& \varphi(t,t_0)W(t_0) \\ &+\frac{\eta}{2c_1}\int_{t_0}^{t}\varphi(t,\tau)\beta(\tau)d\tau \end{aligned} \quad (51)$$

where the transition function $\varphi(t,\tau)$ is given by

$$\varphi = \exp\left[-\frac{\gamma}{2c_2}(t-t_0) + \frac{\mu\eta}{\sqrt{2}c_1}\int_{t_0}^{t}\alpha(\tau)d\tau\right]$$

Now, suppose that α satisfies (30) for some nonnegative constants σ_1 and σ_2 as given in (31). Defining the constants $\zeta = \left(\gamma/2c_2 - \sigma_1\mu\eta/\sqrt{2}c_1\right)$ and $\varrho = \exp\left(\mu\eta\sigma_2/\sqrt{2}c_1\right) \geq 1$ and using (30) and (31) in (51) yields

$$\begin{aligned} W(t) \leq{}& \varrho W(t_0)\exp\left[-\zeta(t-t_0)\right] \\ &+\frac{\eta\varrho}{2c_1}\int_{t_0}^{t}\exp\left[-\zeta(t-\tau)\right]\beta(\tau)d\tau \end{aligned} \quad (52)$$

If $W(t_0) < R/\varrho$ and $\sup_{t\geq t_0}\beta(t) < (2c_1\zeta/\eta)(R/\varrho)$ then $W(t) \leq R$ for all $t \geq t_0$. (35) follows from $W(t) = V^{1/2}$ and hence, (37) is satisfied.

Remark 4. Note that if the perturbation term $\mathbf{H}$ satisfies (29) with $\beta(t) = 0$, that is, it has a vanishing nature then the function $V(x,t)$ in (19) decays exponentially to zero along the trajectories of (23).

■

4. CASCADED CABLE DYNAMICS

This section deals with the modelling and stability analysis of a cascaded cable system, based on the results of the previous sections. Figure (4) illustrates a cascaded cable system. A certain number of actuators are placed along the cable to perform the boundary control action. The transversal motion of each cable element is described by (9) with the corresponding initial and boundary conditions that are defined below. Let $x_0 < \ldots < x_i < \ldots < x_N$ be the position of each actuator where x_0 is the closest to the vessel. Let $(\cdot)^+$ denote the upstream end of the cable element i for $x \in \left[x_{i-1}^+, x_i^-\right]$. The model for a cascaded cable system composed of N elements can now be written

$$\Lambda_i = \{\mathbf{M}\mathbf{r}_{tt} - e\mathbf{K}\mathbf{r}_{xx} - \mathbf{H}\}_i \quad (53)$$

for $i = 1, \ldots, N$. The initial conditions are

$$\mathbf{r}(x,0) = \bar{\mathbf{r}}(x), \quad \mathbf{r}_t(x,0) = \bar{\mathbf{r}}_t(x) \quad (54)$$

and the boundary conditions can be written

$$\mathbf{r}\left(x_i^+\right) = \mathbf{r}\left(x_i^-\right) \quad (55)$$

$$T\left(x_0^+\right)\mathbf{r}_x\left(x_0^+\right) = \mathbf{u}_0(t) \quad (56)$$

$$T\left(x_i^-\right)\mathbf{r}_x\left(x_i^-\right) - T\left(x_i^+\right)\mathbf{r}_x\left(x_i^+\right) = \mathbf{u}_i(t) \quad (57)$$

$$T\left(x_N^-\right)\mathbf{r}_x\left(x_N^-\right) = \mathbf{u}_N(t) \quad (58)$$

for $t > 0$, $x \in \left[x_{i-1}^+, x_i^-\right]$ and $i = 1, \ldots, N-1$.

4.1 Passivity of the Cascaded Cable Dynamics

Corollary 3. In the view of Theorem 1, the system dynamics given by (53)–(58) and $\mathbf{S} = \mathbf{0}$, are state strictly passive from $\mathbf{u}_c \mapsto \mathbf{v}_c$ with $\mathbf{u}_c = [\mathbf{u}_1, \ldots, \mathbf{u}_N]^T$ and $\mathbf{v}_c = [\mathbf{v}_1, \ldots, \mathbf{v}_N]^T$ where $\mathbf{u}$ and $\mathbf{v}$ are defined in (14).

Proof. Consider the energy functions given in (15)–(16) for $x \in [x_{i-1}^+, x_i^-]$. The total energy function for the cascaded system can then be written as $V(x,t) = \sum_{i=1}^{N}(V_{ki} + V_{pi})$. The time derivative of $V(x,t)$ along the solution of the system is found to be

$$\dot{V} = \mathbf{u}_c^T\mathbf{v}_c + \sum_{i=1}^{N}\int_{x_{i-1}^+}^{x_i^-}\mathbf{H}^T\mathbf{r}_t dx \quad (59)$$

By (59) it can be concluded that the system (53)–(58) preserves its state strictly passive property in a cascaded form. ■

4.2 Transversal Boundary Control of the Cascaded Cable Dynamics

At this point, let the dynamics of each cable element in a cascaded cable system have the same structure as the one given in (9). The initial and boundary conditions are as given in (54)–(58).

Corollary 4. In the view of Theorem 2, let the boundary functions in (56)–(58) be chosen as

$$\mathbf{u}_0(t) = \kappa_0^+\mathbf{r}_t\left(x_0^+\right) \quad (60)$$

$$\mathbf{u}_i(t) = -\kappa_i\mathbf{r}_t(x_i) \quad (61)$$

$$\mathbf{u}_N(t) = -\kappa_N^-\mathbf{r}_t\left(x_N^-\right) \quad (62)$$

where κ_0^+, κ_i and κ_N^- are some nonnegative constants for $i = 1, \ldots, N-1$. Then the energy stored in the cascaded cable system (53), in the absence of nonvanishing perturbations, $\mathbf{S} = \mathbf{0}$, decays asymptotically to zero along its trajectories.

Proof. Consider the energy function $V_{0i}(x,t)$ for a cable element i as given in (19)–(21). The total energy storage function for the total system is then

$$V(x,t) = \sum_{i=1}^{N} V_{0i}(x,t) \tag{63}$$

for $i = 1, \ldots, N$. The time derivative of $V_{0i}(x,t)$ along the solution of the system gives

$$\begin{aligned} \dot{V}(x,t) = & -T_0^{+}\mathbf{r}_x^T\left(x_0^{+}\right)\mathbf{r}_t\left(x_0^{+}\right) \\ & + \sum_{i=1}^{N-1}\left[T_i^{-}\mathbf{r}_x^T\left(x_i^{-}\right) - T_i^{+}\mathbf{r}_x^T\left(x_i^{+}\right)\right]\mathbf{r}_t\left(x_i\right) \\ & + T_N^{-}\mathbf{r}_x^T\left(x_N^{-}\right)\mathbf{r}_t\left(x_N^{-}\right) \\ & + \sum_{i=1}^{N}\int_{x_{i-1}^{+}}^{x_i^{-}} \mathbf{H}^T\mathbf{r}_t dx \end{aligned} \tag{64}$$

where $T\left(x_i^{-}\right) = T_i^{-}$ and $T\left(x_i^{+}\right) = T_i^{+}$. Substituting the boundary control functions in (60)–(62) into (64) gives

$$\begin{aligned} \dot{V}(x,t) = & -\kappa_0^{+}\left\|\mathbf{r}_t\left(x_0^{+}\right)\right\|^2 - \sum_{i=1}^{N-1}\kappa_i\left\|\mathbf{r}_t\left(x_i\right)\right\|^2 \\ & -\kappa_N^{-}\left\|\mathbf{r}_t\left(x_N^{-}\right)\right\|^2 + \sum_{i=1}^{N}\int_{x_{i-1}^{+}}^{x_i^{-}}\mathbf{H}^T\mathbf{r}_t dx \\ & < 0 \end{aligned}$$

which shows that the energy stored in the system in (53), decays asymptotically to zero along its trajectories. ■

5. CONCLUSIONS

A cable model and an energy-dissipating controller for a long towed cable have been presented. The cable is divided into elements connected through transversal actuators. Each cable element is modeled in a three-dimensional string-like form where the cable is allowed to possess nonlinear tension. The passivity properties of the cable element were established. Exponential energy decay in the coupled transversal motion of the cable element is achieved by boundary control in the presence of vanishing perturbations. When subjected to nonvanishing perturbations, the energy growth in the cable is shown to be upper bounded. Similarly, the passivity properties of a cascaded cable system have been established. Using Lyapunov's direct method, asymptotic energy decay is established in the cascaded system with passive controllers applied at the connection points.

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