15th Triennial World Congress, Barcelona, Spain

ROBUST STABILITY OF UNCERTAIN SYSTEMS VIA PARAMETER - DEPENDENT LYAPUNOV FUNCTION[1]

S.G. Savov and I.P. Popchev

Institute of Information Technologies
Bulgarian Academy of Sciences
Acad. G. Bonchev str. bl. 2, 1113 Sofia, Bulgaria
E-mail: Svetoslav666@yahoo.com

Abstract: The robust stability problem for uncertain, linear, state-space models is considered. When a fixed Lyapunov function is used to provide an admissible perturbation set, the obtained variation bounds can be too conservative. The main purpose of this investigation is to define the conditions, under which it is always possible to construct a parameter-dependent Lyapunov function for a class of uncertain systems.The contribution to robustness study is due to a new sufficient condition for robust stability. The advantages of this approach are illustrated by examples and comparison with results, obtained by known procedures is made. *Copyright © 2002 IFAC*

Keywords: linear control systems, Lyapunov stability, uncertain dynamic systems, robust stability, matrix equations.

1. INTRODUCTION

The usefulness of any system analysis or design approach is crucially dependent on the accuracy of it's mathematical model. Speaking practically, any real system is subjected to parameter variations, leading to identification errors, often called model uncertainties. It is obvious, that discrepancies between the model and the real system may result in degradation in the system's functioning. Any control system should be designed to be insensitive, i.e. a robust one, against uncertainties in the plant's model. By no doubt, the most important concern in this regard is that of robust stability.

This research considers the problem of determining admissible variation sets for an uncertain vector parameter, included in the model of a linear, dynamic system to reflect the influence of various perturbation factors and modelling inaccuracies. Lyapunov's stability theory is a key-tool for the purpose and a subjective account of some of the main results in the use of quadratic functions in robust analysis for uncertain systems is presented in (Corless, 1993). Many of the available results (Martin, 1990; Patel and Toda, 1980; Yedavalli, 1985; Zhou and Khargonekar, 1987, etc), provide norm bounded admissible perturbation sets and these are proved to be very conservative due to the symmetry of norms. Certain parameters may admit much larger perturbations, than presented by the norm bounds. Asymmetric stability bounds on the uncertain parameters are obtained, e.g. in (Gao and Antsaklis, 1993; Mansour, 1998; Wang, *et al*., 1991), showing clearly their superiority to norm-based ones. Although better, they can still be too conservative and thus the problem of asymmetric admissible perturbation sets extension is posed. The main shortcoming, shared by approaches of the kind is due to the fact, that for the analysis of an uncertain system, possibly time- varying, a fixed Lyapunov's function, or simply a Lyapunov's matrix is used. What's more - since it's choice is made arbitrarily as a rule, the obtained admissible variation sets are yet rather conservative and many actually stable systems are treated according to them as unstable ones.

The main purpose of this investigation is directed towards the question, whether it is possible to construct a Lyapunov's matrix, which depends on the uncertain parameter. In the literature, one can find some results that analyse affinely perturbed linear systems with affine parameter dependent Lyapunov's matrices (Amato, *et al*., 1997; Chockalingham, *et al*., 1995; Gahinet, *et al*., 1996). Considerable number of

[1] This research is supported by the National Science Fund of the Bulgarian Ministry of Education and Science under Contract È-805/1998.

approaches are currently in hand (see, e.g. the references in (Barmish and Kang, 1993)), but each of them has it's own demerit. Among them one should mention the results obtained in (Feron, 1995; Haddad and Bernstein, 1994, 1995; Mori and Kokame, 2000).

The contribution of this work consists mainly in defining the conditions under which such a function always exists for the class of uncertain systems considered here. This helps to achieve much more adequate reflection of the real system's nature, eliminates the arbitrary - subjective choice for a Lyapunov's matrix and finally results in defining much less conservative admissible sets.

The rest of the paper is organized as follows. The statement of the problem is given in section 2. Some aspects concerning linear matrix and vector equations, which are closely related to the main result, are presented in section 3. The main result is obtained in section 4. It contains a new characterization of the sets of stable and positive - definite matrices and an extension of these important results for the case considered here. Various solution's aspects are widely discussed in section 5. Two examples, illustrating the abilities of the suggested approach are solved in section 6.

2. THE ROBUST STABILITY PROBLEM

Consider the state - space model of a linear system

(1) $dx/dt = Ax, \quad x \in R^n$,

where A is a constant and stable $n \times n$ real matrix, i.e. $A \in S$, $S \equiv \{ A : \lambda \in \sigma(A) \Rightarrow Re\ \lambda < 0\}$. The set of eigenvalues of A is denoted by $\sigma(A)$. The stability problem for any system modelled by eq.(1) refers to stability of it's state coefficient matrix, which will be considered in this regard from now on. Let p, $p \geq 1$, real scalar parameters, varying in some unknown intervals (zero included), perturb s known entries of A. This corresponds to the presence of structured parametric uncertainty in a stable matrix and can be modelled as $A + \Delta$, where

(2) $$\Delta = \sum_{i=1}^{p} \alpha_i A_i \ , \ \alpha_i^- \leq \alpha_i \leq \alpha_i^+$$

The constant and known matrix A_i defines the influence structure for the i -th uncertain parameter α_i over one or more entries of A. Define the uncertain vector α as: $\alpha = (\alpha_1\ \alpha_2 \ldots \alpha_p)^T$. The robust stability problem for this class of uncertain matrices is stated as: determine an admissible vector set Ω^*, such that

(3) $\alpha \in \Omega^* \Leftrightarrow \alpha^- \leq \alpha \leq \alpha^+ \Rightarrow A + \Delta \in S$,

where α^+ and α^- are constant vectors, which should be determined. All vector inequalities are intended element - by - element.

The general scheme, followed by all approaches aimed at solving the above stated problem by using fixed Lyapunov's function, consists in the following. According to Lyapunov's stability theorem

(4) $A \in S \Leftrightarrow \tilde{A}^T P + P\tilde{A} = -Q, \ Q \in P, \ P \in P$,

$\tilde{A} = WAW^{-1}$, $rank\ W = n$, where P denotes the set of symmetric, positive definite matrices. Since A is a stable matrix, the solution P (called Lyapunov's matrix) exists uniquely, for any fixed right-hand side matrix $Q \in P$. Then, $A + \Delta \in S$, if $\tilde{\Delta}^T P + P\tilde{\Delta} < Q$. By imposing various restrictions on α, it is always possible to find an admissible vector set Ω^*, such that robust stability for the uncertain matrix is sufficiently guaranteed.

Contrary to this, the purpose here is to determine a matrix $R(\Delta) \in P$, such that:

(i) $(\tilde{A} + \tilde{\Delta})^T R(\Delta) + R(\Delta)(\tilde{A} + \tilde{\Delta}) < 0, \ \alpha \neq 0,$

(ii) $\tilde{A}^T R(0) + R(0)\tilde{A} = -Q, \ \alpha \equiv 0,$

which means, that only for $\Delta \equiv 0$, the Lyapunov's matrix $R(0) \equiv P$ is a fixed one.

3. LINEAR MATRIX AND VECTOR EQUATIONS

Consider a linear in the unknown matrix X equation

(5) $XY + Y^T X = Z, \ Y \in R^{n\times n}, \ Z \in R^{n\times n}$,

and the mapping vec: $R^{n\times n} \rightarrow R^{n^2}$ given by

(6.1) $X \rightarrow vec\ X \equiv x = (x_{ij})$,

(6.2) $Z \rightarrow vec\ Z \equiv z = (z_{ij})$.

For a given linear transformation $T: R^{n\times n} \rightarrow R^{n^2\times n^2}$, there exists an unique matrix $M(Y)$, such that eq.(5) can be put in a vector form as:

(7) $M(Y)x = z, M(Y) \in R^{n^2\times n^2}$,

for all X. Equation (5), respectively eq.(7), has an unique solution for any right-hand side, if and only if (Horn and Johnson, 1991):

(8) $\sigma(Y) \cap \sigma(-Y) \equiv \emptyset$

The transformation of eq.(5) into eq.(7) is not practically justified in the general case. Fortunately, there exists a case, when the order of eq.(7) can be significantly decreased. Suppose, that $Z = -Z^T$ in eq.(5).It can be easily shown, that the solution matrix X is a skew-symmetric one as well, i.e.

$X = -X^T$ with entries $x_{ij} = 0$, for $i = j$ and $x_{ij} = -x_{ji}$, otherwise. The associated with matrices X and Z vectors (6.1) and (6.2) become:

$x = (0x_{12}x_{13}\ldots x_{1n} - x_{12}\,0\ldots x_{2n} - x_{13}\ldots - x_{1n}$
$-x_{2n}\ldots 0)^T$,

$z = (0\, z_{12}z_{13}\ldots z_{1n} - z_{12}\,0\ldots z_{2n} - z_{13}\ldots - z_{1n}$
$-z_{2n}\ldots 0)^T$.

For the sake of simplicity and without any loss of generality, a change in subscripts for $i < j$ is suggested, according to which the couples *(i,j)* are transformed as follows:

$$(1,2) \to 1, (1,3) \to 2, \ldots, (1,n) \to n-1,$$
$$(2,3) \to n, \ldots, (n-1,n) \to k,\ k = \tfrac{1}{2}n(n-1).$$

The solution and right-hand matrices can be presented as:

$$X = \sum_{i=1}^{k} x_i I_i, \quad Z = \sum_{i=1}^{k} z_i I_i, \quad I_i = -I_i^T,$$

where I_i is a respective matrix with only two non-zero , symmetrically positioned entries (1 and -1). Equation (5) is rewritten as:

(9) $\sum_{i=1}^{k} x_i Y_i = \sum_{i=1}^{k} z_i I_i, Y_i = I_i Y + Y^T I_i = -Y_i^T$.

This presentation of eq.(5), for $Z = -Z^T$ clearly shows, that it can be put in a vector form as:

(10) $M(Y)x = z$, $M(Y) \in R^{k\times k}$,

$$x = (x_1 x_2 \ldots x_k)^T,\ z = (z_1 z_2 \ldots z_k)^T.$$

Therefore, taking into account the specific structure of any skew-symmetric matrix helps to decrease significantly the order of eq.(7), which is important for the practical application of the present research.

4. MAIN RESULT

Recall Lyapunov's theorem (4). The next two theorems show in an alternative way the close relation between sets S and P, realized through another matrix set $S^- \equiv \{S: S^T + S < 0\}$ and make possible to characterize uniquely the sets of stable and positive definite matrices.

Theorem 4.1. A matrix $X \in S$, if and only if

(11) X=YZ, $Z \in P$, $Y \in S^-$.

Proof. Let $X \in S$. For any $Q \in P$, there exists a matrix $G \in P$, such that $XG + GX^T = -2Q.$ Then

$$XG + Q = -(XG+Q)^T = F = -F^T \Rightarrow$$
$$X = (-Q+F)G^{-1} = YZ,$$

as required. Let (11) holds for some matrix X. Therefore, $XZ^{-1} + Z^{-1}X^T < 0$, which is possible, if and only if $X \in S$.

Theorem 4.2. A matrix $Z = Z^T \in P$, if and only if

(12) $Z=YX$, $X \in S$, $Y \in S^-$.

Proof. Let $Z \in P$. Then for any matrix $Y \in S^-$,

$$ZY^{-T}Z + ZY^{-1}Z = X^TZ + ZX < 0,$$

which is possible, if and only if $X = Y^{-1}Z \in S$, or $Z=YX$ is the required presentation.

Let (12) holds for some matrix $Z = Z^T$. Then $X^{-T}Z + ZX^{-1} < 0$. Since $X \in S$, the unique solution G to the Lyapunov's equation $X^{-T}G + GX^{-1} = X^{-T}Z + ZX^{-1}$ is a symmetric, positive definite matrix. Obviously G = Z and consequently $Z \in P$.

These theorems play a basic role in the derivation of the main result, which is to a great extent their application for the case of an uncertain matrix robustness study, considered here.

Theorem 4.3. The uncertain matrix $A + \Delta \in S$, if

(i) $-Y \in S$, $Y = I + \Delta A^{-1}$

(ii) $FY = R = R^T$, $-F \in S^-$,

(iii) $A^T F \in S^-$.

Proof. Let (i) holds.If R, as defined in (ii), is a symmetric matrix and since $R = -F(-Y)$, according to Theorem 4.2., then it is also a positive definite one, i.e. $R \in P$. Symmetry and positive definiteness are preserved by multiplying R by A and A^T from the right and from the left, recpectively. Therefore,

$A^T F(A+\Delta) = (A+\Delta)^T F^T A = R_1 > 0,$ or $A + \Delta = (A^T F)^{-1} R_1$ .If (iii) is valid, according to Theorem 4.1. it follows, that $A + \Delta \in S$.

5. SOLUTION ASPECTS

(a1) Requirements (i), (iii), Theorem 4.3. Their satisfaction consists for the general case in the solution of a linear matrix inquality problem with respect to vectors α and *x*, i.e.

$$\alpha \in \Omega_1 \Leftrightarrow \alpha_1^- \le \alpha \le \alpha_1^+ \Rightarrow$$
$$-[I + \sum_{i=1}^{p} \alpha_i L A_i A^{-1} L^{-1}] \in S^-,$$
$$x \in \Omega_x \Leftrightarrow x^- \le x \le x^+ \Rightarrow$$

$$(A^T P + \sum_{i=1}^{p} x_i A^T I_i) \in \mathcal{S}^-,$$

where $P \in \mathcal{P}$ *is* the unique solution to eq.(4) and L is any nonsingular matrix. There exist various techniques for the purpose, e.g. (Boyd and Ghaoui, 1994; Gao and Antsaklis, 1993; Wang, *et al.* 1991, etc.)

(a2) Requirement (ii), Theorem 4.3. It is important now, to answer the question: how one can determine a matrix *F*, such that $R = R^T$ for $-F \in \mathcal{S}^-$. Let matrix *F* be chosen as $F = X + P$, where $X = -X^T$ is an unknown matrix.The condition for symmetry can be rewritten as eq.(5), where $Z = A^{-T}\Delta^T P - P\Delta A^{-1} = -Z^T$. Therefore, the solution *X* is also a skew-symmetric matrix. First of all, it should be underlined, that this choice for *F* guarantees that $-F \in \mathcal{S}^-$. Secondly, due to (i) , Theorem 4.3., condition (8) for an unique solution *X* , for any right-hand side matrix is always satisfied for $\alpha \in \Omega_1$.

(a3) Computation of matrix $M[Y(\alpha)]$ and vector $z(\alpha)$. Consider the uncertain matrix Δ in eq. (2) and eq.(9). For this special case, one has:

$$Y_i = 2I_i + \sum_{j=1}^{p} \alpha_j D_{ij} = -Y_i^T,$$

$$Z = \sum_{j=1}^{p} \alpha_j Z_j = \sum_{i=1}^{k} z_i(\alpha) I_i = -Z^T,$$

$$D_{ij} = I_i A_j A^{-1} + A^{-T} A_j^T I_i,$$

$$Z_j = A^{-T} A_j^T P - P A_j A^{-1}.$$

The constant matrices $D_{ij} = -D_{ij}^T$ and $Z_j = -Z_j^T, i = 1,2,\ldots,k$ and $j = 1,2,\ldots,p,$ can be easily computed. The respective vector form (eq.(10)) of eq.(9) is

(13) $M[Y(\alpha)]x = [2I + M(\alpha)]x = z(\alpha).$

The entries of $M(\alpha)$ and $z(\alpha)$ are some linear functions of the uncertain vector α.

Comments(1).The Lyapunov's matrix, by means of which robustness is studied in this case is

$$R(\Delta) = FY = (X + P)(I + \Delta A^{-1}) \in \mathcal{P}.$$

When Lyapunov's stability theorem (4) is applied for $A + \Delta$ and $W = A$, one can easily verify, that

$$(A + A\Delta A^{-1})^T R(\Delta) + R(\Delta)(A + A\Delta A^{-1}) =$$
$$(I + \Delta A^{-1})^T V (I + \Delta A^{-1}) < 0,$$

where $V = A^T(X + P) + (X^T + P)A < 0,$ in accordance with Theorem 4.3. For all $\Delta \neq 0, R(\Delta)$ depends on the uncertain part. The only case when $R(\Delta)$ is a fixed, constant matrix is for $\Delta = 0$, since *X=0* and $R(0) = P$.

(2). Requirement (iii), Theorem 4.3. imposes a restriction on the solution vector *x*. When eq.(13) is taken into consideration, one has to solve the vector inequality

$$2x^- \le 2x = z(\alpha) - M(\alpha)x \le 2x^{+}.$$

The solution to it, $\alpha \in \Omega_2$ is obtained by checking all extreme cases (so called corner vectors) for α and $x \in \Omega_x$.

The solution to the overall problem (3) is given by $\alpha \in \Omega^* \equiv \Omega_1 \cap \Omega_2$.

(3). Matrix $M(Y)$ does not depend on the choice for matrix *Q*, respectively matrix *P*. Therefore, when a particular case is studied, changes may occur only in vector $z(\alpha)$.

(4). In the special case, when $L\Delta = T^\Delta LA,$ where T^Δ denotes an upper (lower) triangular matrix, for some nonsingular matrix *L*, problem (i), Theorem 4.3. has an exact solution. It can be prooved, that *M(Y)* is also a triangular matrix, but due to lack of space, this is omited.

6. EXAMPLES

6.1. Consider a second order uncertain system

$$A + \Delta = \begin{bmatrix} -1+\alpha_1 & -1+\alpha_2 \\ 1-\alpha_3 & 0 \end{bmatrix},$$

$A + \Delta \in \mathcal{S} \Leftrightarrow \alpha_1 < 1, \alpha_2 < 1, \alpha_3 < 1$.

Matrix *Y* is computed as

$$Y = \begin{bmatrix} 1-\alpha_2 & \alpha_1 - \alpha_2 \\ 0 & 1-\alpha_3 \end{bmatrix},$$

$-Y \in \mathcal{S} \Leftrightarrow \alpha_2 < 1, \alpha_3 < 1$. Let $Q = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ in eq.(4). The scalar solution (k=1) to eq.(11) is

$$x = (-2\alpha_1 + \alpha_2 + \alpha_3)(2 - \alpha_2 - \alpha_3)^{-1}.$$

Requirement (iii), Theorem 4.3. is met, if $x > -1$. The admissible subset Ω_2 can be defined easily as $\alpha_1 < 1.$ The final solution to this example is given by $\alpha_1 < 1, \alpha_2 < 1$ and $\alpha_3 < 1,$ which is just the exact one.

6.2. Consider a third order uncertain system

$$A+\Delta=\begin{bmatrix}-1+\alpha_1 & 0 & -1\\ 0 & -3+\alpha_2 & 0\\ -1 & -2 & -4\end{bmatrix},$$

$A+\Delta\in S \Leftrightarrow \alpha_1 < 1.75$ and $\alpha_2 < 3$.

$$For\ L=\begin{bmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}=L^T,\ LL^T=I\ and$$

$Y=I+A^{-1}\Delta,$ one has

$$LYL^T=\begin{bmatrix}1-0.333\alpha_2 & 0 & 0\\ -0.0952\alpha_2 & 1-0.5714\alpha_1 & 0\\ -0.0952\alpha_2 & 0.1429\alpha_1 & 1\end{bmatrix},$$

$-Y\in S\Leftrightarrow \alpha_1 < 1.75, \alpha_2 < 3$.

For $Q=(A^TA)^{1/2}$ in eq.(4), the vector form (eq.(10)) of eq.(9) is

$$\begin{bmatrix}2-0.5714\alpha_1-0.3333\alpha_2 & 0.1905\alpha_2 & -0.1429\alpha_1\\ 0 & 2-0.5714\alpha_1 & 0\\ 0 & -0.0952\alpha_2 & 2-0.3333\alpha_2\end{bmatrix}x=$$

$$\begin{bmatrix}-0.0161\alpha_1+0.0573\alpha_2\\ 0.0673\alpha_1\\ 0.0876\alpha_2\end{bmatrix}$$

It was found, that requirement (iii), Theorem 4.3. is guaranteed, if

$$(-0.9\quad -0.1178\quad -0.6)^T\le x\le(0.8333\quad 0.1178\quad 0.3)^T.$$

When the admissible extreme values for x are substituted, the subset Ω_2 is defined from the inequalities:

$$0.503\alpha_1+0.357\alpha_2\le 1.6666;$$
$$0.553\alpha_1+0.265\alpha_2\le 1.8;$$
$$0.1346\alpha_1\le 0.2356;\quad 0.199\alpha_2\le 0.6;$$
$$0.1236\alpha_2\le 1.2.$$

The results obtained here are comparised with those get by other two approaches, using fixed Lyapunov's matrix *P*. The comparison is done for one and the same matrix *P*.

The uncertain state matrix $A+\Delta\in S$, if:

(i) parameter dependent Lyapunov's matrix

$$-\infty<\alpha_1<1.75;\quad -\infty<\alpha_1\le 1.18;$$
$$-\infty<\alpha_2\le 2.2;\quad -\infty<\alpha_2<3;$$

$$-\infty<\alpha_1\le 1.6;\quad -\infty<\alpha_1\le 1.54;$$
$$-\infty<\alpha_2\le 2.41;\quad -\infty<\alpha_2\le 2.5;$$

(ii) fixed Lyapunov's matrix (Mansour, 1988)

$$-4198\le\alpha_1\le 1.73;\quad -45\le\alpha_1\le 0;$$
$$-85\le\alpha_2\le 0;\quad -25985\le\alpha_2\le 2.83;$$

$$-3600\le\alpha_1\le 1.6;\quad -2998\le\alpha_1\le 0.1;$$
$$-3400\le\alpha_2\le 1;\quad -24591\le\alpha_2\le 2.5;$$

(iii) fixed Lyapunov's matrix (Gao and Antsaklis, 1993)

$$-3333\le\alpha_1\le 1.73;\quad -57\le\alpha_1\le 0.047;$$
$$-17\le\alpha_2\le 0.02;\quad -2500\le\alpha_2\le 2.7;$$

$$-3062\le\alpha_1\le 1.6;\quad -330\le\alpha_1\le 0.1724$$
$$-204\le\alpha_2\le 0.023;\quad -2252\le\alpha_2\le 2.5.$$

Although, as expected, the results obtained by the suggested here approach are considerably better, it should be stressed on the fact, that when such a powerful approach (Mansour, 1998) is applied, upper positive bounds on α_2 and α_1 cannot be established in the first two cases, respectively.

7. CONCLUSION

The robust stability problem for a class of uncertain, linear, dynamic systems is considered in this research. The main purpose consists in the construction of a Lyapunov function (matrix), which depends on the uncetain part and thus reflects much more adequately it's nature in comparison with the case when a fixed one is used. It is shown, that whenever a stable matrix is perturbed by an additive uncertain matrix with entries varying in some unknown intervals, it is always possible to get a solution by this approach.

The main contribution to system's robustness study is due to Theorem 4.3., which extends the important results get by Theorem 4.1. and 4.2. for the class of systems considered here. The applicability of this approach and it's superiority over some available ones is illustrated by two examples.It is believed, that the philosophy of the present approach can be used to define necessary and sufficient condition for robust stability for a nominal state matrix, influenced by a given structured perturbation uncertainty in terms of a parameter-dependent Lyapunov matrix.

REFERENCES

Amato, F., M.Corless, M.Mattei and R.Setola(1997). A multivariable stability margin in the presence of time-varying bounded rate gains. *Int. Journ. Robust and Non-Linear Contr.*, 7, pp. 127-143.

Barmish, B., H.Kang (1993). A survey of extreme point results for robustness of control systems. *Automatica*, 29, pp.13-35.

Boyd, S., E.Ghaoui, E.Feron and V. Blakrishnan (1994). Linear matrix inequalities in system and control theoty. *SIAM Stidies in Applied Mathematics*. *SIAM*, 15.

Chockalingham, G., S.Dasgupta, B.Anderson and M. Fu (1995). Lyapunov functions for uncertain systems with applications to the stability of time varying systems. In: P*roc.of Conf. Decision and Contr*., San Antonio, USA.

Corless, M (1993). Robust analysis and design with quadratic Lypunov function. In: P*roc.of 12-th IFAC Congr*., Sydney, Australia.

Feron, E., P.Apkarian and P.Gahinet (1995). S-procedure for the analysis of control systems with parametric uncertainties via parameter dependent Lyapunov functions. In: *Proc.of 3-rd European Contr. Conf.* Rome, Italy.

Gahinet, P., P.Apkarian and M.Chilai (1996). Affine parameter dependent Lyapunov functions and real parameter uncertainty. *IEEE Trans.Automat.Contr*., 41, pp.141-146.

Gao, Z. and P. Antsaklis (1993). Explicit asymmetric bounds for robust stability of continuous and discrete-time systems. *IEEE Trans.Automat.Contr*., 38, pp. 332-335.

Haddad, W. and D. Bernstein (1994). Parameter dependent Lyapunov functions and the discrete- time Popov criterion for robust analysis. *Automatica*, 30, pp. 1015-1021.

Haddad, W. and D. Bernstein (1995). Parameter dependent Lyapunov functions and the Popov criterion in robust analysis and synthesis. *IEEE Trans. Automat.Contr*., 40, pp. 536-543.

Horn, R. and C. Johnson (1991). *Topics in matrix analysis.* Cambridge University Press, Cambridge, USA.

Mansour, M. (1988). Sufficient conditions for the symmetric stability of interval matrices *Int.J.Contr*., 47, pp. 1973-1974.

Martin, J.(1990). State space measures of robustness of pole locations for structured and unstructured perturbations. In: Proc. of *SIAM Conf. on Linear Algebra*, San Francisco, CA.

Mori, T. and H. Kokame (2000). A parameter dependent Lyapunov function for a polytope of matrices. *IEEE Trans. Automat. Contr*., 45, pp. 1516-1519.

Patel, R. and M. Toda (1980). Quantitative measure of robustness for multivariable system. In: Proc. of Joint. Automat. Contr. Conf., San Francisco, CA.Wang, M., E. Lee and D. Boley (1991). Matrix pencil and matrix measure methods for robust stability in real parameter spaces. In: P*roc. of 30-th Conf.on Decision and Contr*., Brighton, England.

Yedavalli, R. (1985). Improved measures of stability for linear state-space models. *IEEE Trans. Automat. Contr*., 30, pp. 557-579..

Zhou, K. and P. Khargonekar (1987). Stability robustness bounds for linear state-space models with structured uncertainty. *IEEE Trans.Automat Contr.*, 32, pp.621-623.