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GAIN REDUCTION IN HYBRID SLIDING MODE CONTROL OF SECOND ORDER SYSTEMS [1]

Antonella Ferrara, Lorenza Magnani, Riccardo Scattolini

Dipartimento di Informatica e Sistemistica Università di Pavia - Via Ferrata 1 - 27100 Pavia - Italy
{ferrara, magnani, scatto}@conpro.unipv.it

Abstract: A hybrid sliding mode control strategy for a class of second order systems is presented in this paper. It is characterized by an event–driven gain reduction mechanism relying on a decomposition of the system state into regions. By enforcing sliding mode behaviors on a suitable set of sliding manifolds, while avoiding the generation of limit cycles, the proposed strategy proves to globally asymptotically stabilize the origin of the system state space. *Copyright 2002 IFAC*

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1. INTRODUCTION

Sliding mode control (SMC) systems, to which a large number of works has been devoted during the past two decades (Utkin, 1992; Hung *et al.*, 1993; DeCarlo *et al.*, 1998; Edwards and Spurgeon, 1998), are, intrinsically, "hybrid systems" in the sense that the control design relies on a state space decomposition through a border, the so–called sliding manifold, which is a linear or nonlinear function of the full system state, so that the control law is switched on crossing it. Yet, they do not fit the intuitive idea the researchers have of hybrid systems (Morse *et al.*, 1999), since the key point in the theory of SMC systems is to force the state trajectory not to instantaneously cross the commutation manifold as expected in classical hybrid systems, but to slide on it. Indeed, in this way, the desired dynamical features turn out to be assigned to the controlled system.

The aim of the present paper is to design and analyze a truly hybrid SMC strategy for a class of second order systems which relies on a peculiar system state decomposition into countable regions by means of a grid of conventional sliding manifolds, and a set of nested switching boundaries. Each region is a "block" in the sense used in (Caines and Wei, 1997), and a block–invariant control gain is associated with it. On the whole, the choice of the control gains corresponding to the blocks included between two switching boundaries (note that also infinity and the origin of the state space can be interpreted in this way) concurs to the attainment of the objective of either reaching a particular sliding manifold, or crossing the switching boundary closer to the origin.

As long as the state trajectory crosses a switching boundary, a gain variation is generated. More precisely, a state evolution approching the state space origin tends to determine, on crossing switching boundaries closer to the origin, a reduction of the control gain. In contrast to (Bartolini *et al.*, 1998; Gessing, 2001), where a continuous variation of the gain is generated, in the present proposal the gain reduction mechanism is event–driven and asyncronous in time. Most importantly, it does not require that the state is evolving along a sliding manifold, being active even dur-

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ing the reaching phases. The overall hybrid SMC strategy with gain reduction proves to globally asymptotically stabilize the origin of the system state space, in spite of the presence of a bounded uncertain term in the system model.

The motivation for using SMC to design a hybrid strategy mainly relies on the appreciable features of the SMC metodology, such as simplicity and robustness versus matched uncertainties and disturbancies, which are naturally inherited by the proposed control approach. Note that, the combination of SMC with hybrid control has already been investigated in (Bartolini *et al.*, 1999*b*) and (Bartolini *et al.*, 1999*a*). Yet, the control strategies proposed in such papers are characterized by a continuous adaptation of the control gain, and switching is driven by a logic relying on the decomposition of the sliding variable phase plane, rather than of the original system state space, which can appear less intuitive as far as the control laws design is concerned. Rather, the present paper can be viewed as a development of (Ferrara *et al.*, 2001) in which a hybrid SMC strategy is designed through the decomposition of the system state space into a couple of regions, and no gain reduction mechanism is implemented.

The present paper is organized as follows. The next section is devoted to the problem formulation. In Section 3 the formal description of the proposed control strategy is addressed. The analysis of its stability and convergence properties is carried out in Section 4. Finally, in Section 5, a simple example is reported.

2. PROBLEM FORMULATION

Consider the second-order nonlinear continuous-time dynamic system in controllable canonical form

$$\dot{x}(t) = f(x(t)) + gu(t) \tag{1}$$

where $x(0) = \overline{x}$, $x = [x_1 \quad x_2]' \in \Re^2$ is the state vector, $f(x) = [x_2 \quad \bar{f}(x)]' \in \Re^2$, with $\bar{f}(x(t))$ uncertain but such that

$$\left|\bar{f}(x(t))\right| < k, \tag{2}$$

k being a positive constant, is a Lipschitz vector function such that the uncontrolled system $\dot{x}(t) = f(x(t))$ has a single insulated equilibrium point at the origin, $u \in \Re^1$ is a scalar control variable which influence the state vector linearly through the constant vector $g = [0 \quad \bar{g}]'$, where, without loss of generality, $\bar{g}$ is supposed to be a positive constant.

Assume that the maximum amplitude of the control variable u is varied in different regions $\Omega^i(x)$, $i = 1, \ldots, \nu$, of the state space, bounded by nested switching boundaries $\bar{\varphi}_i$, $i = 1, \ldots, \nu + 1$, defined by $\varphi_i(x) = 0$, where

$$\varphi_i(x) = x' P_i x - c_i \tag{3}$$

with $P_i = P_i' = diag\{p_{i_1}, p_{i_2}\} > 0$. Note that also infinity and the origin of the state space can be interpreted as switching boundaries, letting $\left(\sqrt{\frac{c_i}{p i_1}}, \sqrt{\frac{c_i}{p i_2}}\right)$ tend to (∞, ∞) and to $(0, 0)$, for $i = 1$ and $i = \nu + 1$, respectively. Specifically,

$$\Omega^i(x) = \{x : \varphi_{i+1}(x) > 0 \ \cap \ \varphi_i(x) < 0\} \tag{4}$$

$i = 1, \ldots, \nu + 1$, with the observation that $\Omega^0(x)$, and $\Omega^{\nu+1}(x)$ are empty set, therefore neglected in this treatment.

Moreover, the following constraint on the control amplitude is considered

$$|u_i| > |u_{i+1}|, \quad i = 1, \ldots, \nu - 1 \tag{5}$$

u_i being the value assumed by the control variable u in the region $\Omega^i(x)$. Clearly, (5) corresponds to a control gain reduction, as the state trajectory moves through regions closer to the origin of the state space, driven by the event of crossing a switching boundary.

The control problem in question is to design a hybrid control strategy, taking into account the constraint (5), so as to make the origin of the state space be a globally asymptotically stable equilibrium point of the controlled system.

3. THE HYBRID SLIDING MODE CONTROLLER

With reference to the regions Ω^i, $i = 1, \ldots, \nu$, introduce the linear functions

$$\sigma_i(x) = x_2 + \alpha_i x_1 \tag{6}$$

$\alpha_{i-1} > \alpha_i > 0$, $i = 2, \ldots, \nu$, and the corresponding sliding manifolds $\sigma_i(x) = 0$, $i = 1, \ldots, \nu$.

Then, according to the SMC theory, define the hybrid control law

$$u = -K_i sign(\sigma_i(x)), \quad K_i > 0 \tag{7}$$

when $x \in \Omega^i(x)$, $i = 1, \ldots, \nu$, where K_i has to be designed so that the reaching condition $\sigma_i(x)\dot{\sigma}_i(x) < 0$ is fulfilled in $\Omega^i(x)$.

As usual in SMC control (Utkin, 1992), the switching functions $\sigma_i(x)$, $i = 1, \ldots, \nu$, are selected so that when the state of system (1) is restricted to lay on the sliding manifolds, the system dynamics exhibits the desired behavior. A further requirement, in the present case, is determined by the constraint on the control amplitude given by (5). To comply with it, define

$$\max_x ||\bar{\varphi}_{i_x}|| \leq \mathcal{D}_i, \;\; i = 1, \ldots, \nu \qquad (8)$$

$\mathcal{D}_i$ being a positive constant, and $\bar{\varphi}_{i_x}$ being a point of the i–th switching boundary $\bar{\varphi}_i$. Note that, by virtue of the assumption of nested switching boundaries, $\mathcal{D}_i > \mathcal{D}_{i+1}$, $i = 1, \ldots, \nu$. Then, determine

$$\bar{K}_i = \frac{1}{\bar{g}}(k + \alpha_i \mathcal{D}_i), \;\; i = 1, \ldots, \nu \qquad (9)$$

As it is well known, see e.g. (Branicky, 1988), a hybrid strategy where the controller switches between different control laws can result in an overall unstable closed–loop system even if each control law is designed so as to guarantee stability. So, the proposed hybrid SMC strategy does not guarantee by itself the global asymptotic stability of the origin of the controlled system state space, but some further conditions on the gains K_i must be imposed. To this end, first observe that each region $\Omega^i(x)$, can be partitioned into eight different blocks $\Omega^i_{a,b,c}(x)$ where

$$a = \begin{cases} 1 & if \; x_2 > 0 \\ -1 & if \; x_2 < 0 \end{cases}$$
$$b = \begin{cases} 1 & if \; sign(\sigma_i(x)) > 0 \\ -1 & if \; sign(\sigma_i(x)) < 0 \end{cases}$$
$$c = \begin{cases} 1 & if \; sign(\sigma_{i+1}(x)) > 0 \\ -1 & if \; sign(\sigma_{i+1}(x)) < 0 \end{cases}$$

$i = 1, \ldots, \nu - 1$. Note that, in view of the choice of the α_i's, $\Omega^i_{1,1,-1}(x)$ and $\Omega^i_{-1,-1,1}(x)$ are always empty.

Associated with $\Omega^i_{a,b,c}(x)$, $i = 1, \ldots, \nu$, it is also possible to define the $\Delta^i_{a,b,c}(x)$, $i = 1, \ldots, \nu$, vicinity of the switching boundaries $\bar{\varphi}_i$ inside the blocks $\Omega^i_{a,b,c}(x)$ as follows

$$\Delta^i_{a,b,c}(x) = \{x \in \Omega^i_{a,b,c}(x) : (|x_2| > \delta_1) \cap (\|x - \bar{\varphi}_i\| < \delta_2)\}$$

for $i = 1, \ldots, \nu$, where δ_1 and δ_2 are arbitrarily small positive constants. Moreover, denote with

$$\Delta^i(x) = \bigcup \Delta^i_{a,b,c}(x)$$

$a = 1, b = 1, c = 1, i = 1, \ldots, \nu$, the $\Delta^i(x)$ vicinity of the switching boundaries.

Now let $\tilde{K}_i$, $i = 1, \ldots, \nu$, be positive values such that

$$\tilde{K}_i > \frac{|x' P_i f(x)|}{|x' P_i g|}, \;\; \forall \, x \in \Delta^i(x) \qquad (10)$$

$i = 1, \ldots, \nu$. Finally, assume that the control gains K_i, $i = 1, \ldots, \nu$, of the hybrid control law (7) are chosen as follows

$$K_i \geq \max\left\{\bar{K}_i, \tilde{K}_i\right\}, \;\; i = 1, \ldots, \nu \qquad (11)$$

4. STABILITY AND CONVERGENCE ANALYSIS

The stability of the origin of the closed-loop system (1), (7), (11) is now investigated by analyzing the behavior of the state trajectories in the vicinities $\Delta^i(x)$ of the switching boundaries. For the sake of clarity, the analysis is carried out in the particular case of $\nu = 2$, so that $||\bar{\varphi}_{i_x}|| \to \infty$, for $i = 1$, $||\bar{\varphi}_{i_x}|| \to 0$, for $i = \nu + 1 = 3$, and $\bar{\varphi} := \bar{\varphi}_i$, for $i = 2$. Note, however, that the following results can be easily extended to the case $\nu > 2$.

A reaching condition is analyzed with reference to $\bar{\varphi}$ in order to establish which parts of it exerts an attractive or repulsive action on the controlled state trajectories. To this end, note that in view of definition (3), system (1), and the hybrid control law (7), in $\Delta^i(x)$, $i = 1, 2$, it results that

$$\dot{\varphi}(x) = 2x'Pf(x) - 2x'Pg_iK_i sign(\sigma_i(x)) \qquad (12)$$

$P := P_2$. Moreover, in $\Delta^1_{a,b,c}(x)$, one has $\varphi(x) > 0$ and, in view of (10), (11),

$$sign(x'Pg_1K_1 sign(\sigma_1(x))) = sign(ab) \qquad (13)$$

and

$$sign(\varphi(x)\dot{\varphi}(x)) = -sign(ab) \qquad (14)$$

In contrast, in $\Delta^2_{a,b,c}(x)$, one has $\varphi(x) < 0$,

$$sign(x'Pg_2K_2 sign(\sigma_2(x))) = sign(ac) \qquad (15)$$

and

$$sign(\varphi(x)\dot{\varphi}(x)) = sign(ac) \qquad (16)$$

Three different cases can occur:

case 1 : when $ab = -1$ and $ac = -1$, $\varphi(x)\dot{\varphi}(x) > 0$ in $\Delta^1_{a,b,c}(x)$, while $\varphi(x)\dot{\varphi}(x) < 0$ in $\Delta^2_{a,b,c}(x)$, so that the state trajectories move from $\Delta^2_{a,b,c}(x)$ to $\Delta^1_{a,b,c}(x)$;

case 2 : when $ab = 1$ and $ac = 1$, $\varphi(x)\dot{\varphi}(x) < 0$ in $\Delta^1_{a,b,c}(x)$, while $\varphi(x)\dot{\varphi}(x) > 0$ in $\Delta^2_{a,b,c}(x)$, so that the state trajectories move from $\Delta^1_{a,b,c}(x)$ to $\Delta^2_{a,b,c}(x)$;

case 3 : when $ab = -1$ and $ac = 1$, $\varphi(x)\dot{\varphi}(x) > 0$ both in $\Delta^1_{a,b,c}(x)$ and in $\Delta^2_{a,b,c}(x)$. Hence, the state trajectories cannot go through the switching boundary $\bar{\varphi}$, which is "repulsive" both in $\Delta^1_{a,b,c}(x)$ and in $\Delta^2_{a,b,c}(x)$.

The following results can be proved in a row.

Proposition 1. The trajectories of the hybrid closed-loop system do not have any limit cycle.

Proof: First observe that limit cycles entirely included into Ω^1 or Ω^2 cannot exist in view of the globally stabilizing property of the SMC control law (7) in Ω^1 and Ω^2, respectively, guaranteed by the choice of the gain (11). Hence, a limit cycle, if any, should belong to both Ω^1 and Ω^2. In particular, it should exit from Ω^2 through $\Delta^2_{-1,1,1}(x)$ and exit from Ω^1 in $\Delta^1_{1,1,1}(x)$ (limit cycles of type 1). Alternatively, it should exit from Ω^2 through $\Delta^2_{1,-1,-1}(x)$ and exit from Ω^1 in $\Delta^1_{-1,-1,-1}(x)$ (limit cycles of type 2). Any other possibility is forbidden, since it would imply that the closed trajectory intersects the sliding manifold $\sigma_1(x) = 0$ in Ω^1, but in this case the trajectory would follow this sliding manifold until it reaches $\bar{\varphi}$. As for the presence of limit cycles of type 1, note that every trajectory starting in Ω^2 and passing in Ω^1 through $\Delta^2_{-1,1,1}(x)$ should cross in Ω^1 a level line of the Lyapunov function $\frac{1}{2}\sigma_1^2$ in the forbidden direction. The same arguments can be used to show the unfeasibility of the limit cycles of type 2. △

Proposition 2. Any trajectory moving from Ω^2 to Ω^1, reaches in Ω^1 the sliding manifold $\sigma_1(x) = 0$.

Proof: This is a direct consequence of the fact that every trajectory starting in Ω^2 and passing in Ω^1 through $\Delta^2_{-1,1,1}(x)$ should cross in Ω^1 a level line of the Lyapunov function $\frac{1}{2}\sigma_1^2$ in the forbidden direction. △

Proposition 3. The origin of the state space is a globally asymptotically stable equilibrium point for system (1) controlled by the hybrid SMC strategy (7), (11).

Proof: Assume first that $x(0) \in \Omega^2$. Then, two cases are possible.

- A1 The trajectory starting from $x(0)$ reaches the sliding manifold $\sigma_2(x) = 0$ and goes to the origin with the dynamics imposed by the choice of α_2.
- A2 The trajectory leaves Ω^2 and enters Ω^1. In view of Proposition 2, it reaches the sliding manifold $\sigma_1(x) = 0$; then, the case B1 below holds.

When $x(0) \in \Omega^1$, one of the following two cases holds.

- B1 The trajectory starting from $x(0)$ reaches the sliding manifold $\sigma_1(x) = 0$; since, by assumption, $\alpha_1 > \alpha_2$, the trajectory enters in Ω^2 and reaches the sliding manifold $\sigma_2(x) = 0$ (case A1). Note that in this last case, it cannot exit Ω^2 without passing through $\sigma_2(x) = 0$.
- B2 The trajectory enters Ω^2 and one of the cases A1–A2 applies.

We finally have to prove that the overall state trajectory goes to the origin in finite time. To this end, by virtue of the choice of the control law (7), and, in particular, of the control gains in (11), classical results of the theory of SMC guarantee a finite run time to reach the sliding manifolds $\sigma_i(x) = 0$, $i = 1, 2$, and to follow them up to the origin (Utkin, 1992). △

5. SIMULATION EXAMPLE

As an example, the proposed hybrid SMC strategy with event–driven reduction of the control gain is applied to the following system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_2 - 3sin(x_1) + u \end{cases} \qquad (17)$$

Three regions Ω^i, $i = 1, \ldots, 3$, are delimited by the switching boundaries

$$\begin{array}{l} \bar{\varphi}_2 : x_1^2 + 4x_2^2 - 144 = 0 \\ \bar{\varphi}_3 : x_1^2 + 4x_2^2 - 16 = 0 \end{array} \qquad (18)$$

apart from ∞ and 0.

Associated with the Ω^i's, the following sliding manifolds

$$\begin{array}{l} \sigma_1(x) = x_2 + 4x_1 = 0 \\ \sigma_2(x) = x_2 + 2x_1 = 0 \\ \sigma_3(x) = x_2 + 1x_1 = 0 \end{array} \qquad (19)$$

and the corresponding control gains $K_1 = 70$, $K_2 = 50$, $K_3 = 10$ are selected. The state trajectory of the controlled system, starting from $x(0) = [-6\ 3]'$ and moving to the origin of the state space through a sequence of "reaching" and "sliding" phases, is shown in Fig. 1. In Fig. 2, the evolution of the σ_i's versus time is depicted, while the switched control signal u with gain reduction is illustrated in Fig. 3.

6. CONCLUSIONS

A hybrid SMC strategy has been presented in the paper. Sliding mode behaviors are suitably generated so that they have finite duration when they occur on sliding manifolds which are separated from the origin by switching boundaries. Alternatively, they asymptotically steer the controlled system state trajectory to the origin of the state space. As a result, the proposed hybrid

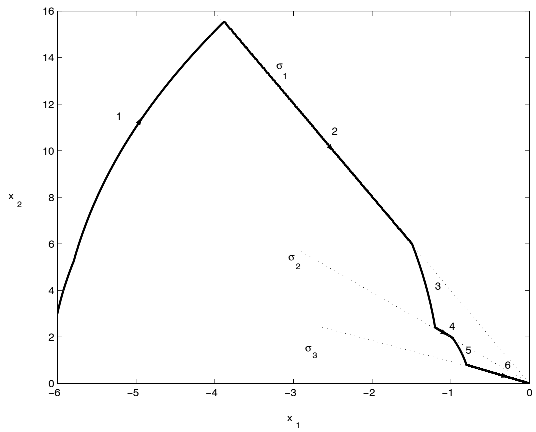

Fig. 1. The state trajectory of the controlled system

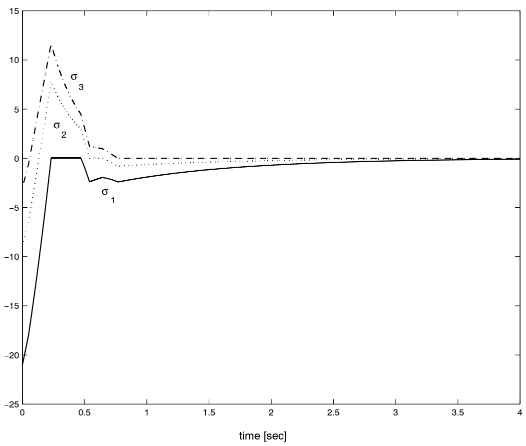

Fig. 2. The evolution of the σ_i's

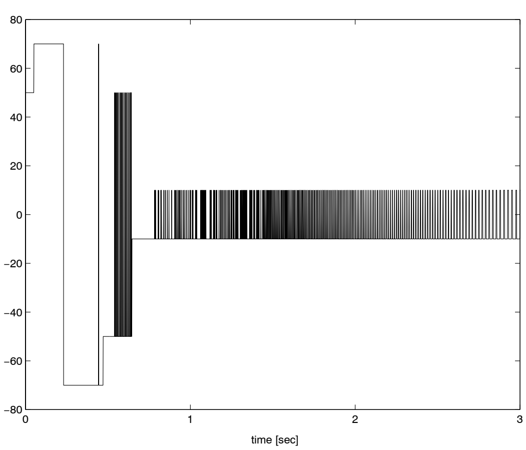

Fig. 3. The switched control with gain reduction

SMC strategy proves to globally asymptotically stabilize the origin of the system state space. This positive stability result is attained in spite of a control gain reduction, which is driven by the event of crossing a switching boundary, as the state trajectory moves through regions closer to the origin of the state space.

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