

PROBABILITY-GUARANTEED ROBUST H_∞ PERFORMANCE ANALYSIS

I. Yaesh¹, S. Boyarski¹ and U. Shaked²

¹ *Advanced Systems Division, Control Dept., Israel Military Industry, Israel*

² *Dept. of EE-Systems, Tel Aviv University, Ramat Aviv 69978, Tel Aviv, Israel*
Fax: (972) 3-6407095, E-mail: shaked@eng.tau.ac.il

Abstract: This paper addresses the common engineering practice of specifying a required probability of attaining some performance level. The problem setup is that of a standard robust H_∞ performance analysis of a parameter-dependent system, except that the parameter hyper-rectangle (box) shrinks in the analysis in order to accommodate a polytopic performance goal that is better than the one attainable for the original parameter box. An affine-quadratic, multiconvex approach is applied to reduce the overdesign that is inherent in the quadratic approach. A version of the Bounded Real Lemma (BRL) in the form of Bi-Linear Matrix Inequalities (BLMIs) guarantees a minimum H_∞ -norm for a prescribed probability. These BLMIs are solved using an iterative algorithm. A uniform distribution is assumed for the system parameters, according to the uniformity principle. The probability requirement is expressed by a set of LMIs that is derived by extending an existing second-order cone method; these LMIs are to be concurrently solved with the BRL BLMIs. The proposed analysis is demonstrated via a 2-parameter example. *Copyright © 2002 IFAC*

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1. INTRODUCTION

Robust control theory is usually associated with worst-case design; namely, some performance measure (e.g., the H_∞ -norm of the closed-loop system) is required to be satisfied for all possible combinations of the uncertain parameters of the system. Worst-case H_∞ design has many variants: real bounded uncertainties were considered by (de Souza, et al., 1990), while the μ synthesis/analysis approach was developed by (Doyle, 1982). The H_∞ polytopic uncertainties approach (e.g., (Boyd, et al., 1994)) is the most popular by now; it assumes that the system matrices belong to a convex domain. Requiring quadratic stability, i.e. using a single Lyapunov function to investigate system stability and performance throughout the parameters space, one can perform analysis and synthesis by LMIs. The robust worst-case stability check (e.g., by μ analysis (Packard and Doyle, 1993) or by other extreme point methods (Kang and Barmish, 1993)) can be extremely conservative, especially for a large number of uncertain parameters. This conservatism stems from the fact that worst-case methods require the performance criteria to be satisfied for all possible parameter combinations regardless of their probability, even for those that can occur with a very low probability.

The common engineering practice, however, is to associate the required performance with some desired probability of attaining that performance. For example, a low-cost Unmanned Air Vehicle (UAV) specification might require that the standard deviation of the UAV's altitude-hold error (due to measurement errors, turbulence, gusts, aerodynamic data uncertainties, etc.) should be less than 100 feet with a 90% probability. By introducing appropriate dynamic weights, the latter specification may be formulated within a H_∞ setup: associating the required probability (0.9) with the H_∞ -norm of an augmented system (which includes the weights) enables complying with the probabilistic

specification. The above prevalent engineering practice motivates the present work, in which a probabilistic (rather than a deterministic worst-case) approach to robust H_∞ analysis is adopted. Note that the present work is not part of the recent randomized algorithms approach (Polyak and Tempo, 2001; Vidyasagar, 2001): no random samples of the uncertain system are required, and the desired probability is directly guaranteed.

When considering probability, the issue of the probability distribution of the unknown parameters of the system inevitably arises. When the parameters' joint probability density function is known, common sense dictates that it should be used in the analysis. However, in reality this is rarely the case and consequently the question of finding the worst-case distribution function (supported for specified intervals of the unknown parameters) usually arises. In (Barmish, 1994) and (Barmish and Lagoa, 1996) it was shown, using the concept of confidence degradation function and the truncation and uniformity principles, that the uniform distribution is the most conservative distribution among the common (unimodal) density functions. Therefore it is assumed throughout the present paper that the uncertain parameters are uniformly distributed.

The notions of performance (e.g., desired disturbance attenuation level, γ) and probability (of exceeding that performance, p) can be combined, in a robustness analysis context, in the following two ways:

1) Given γ , find the maximum probability p by which $\|T_{zw}\|_\infty < \gamma$ can be assured, namely

$$\text{Max}_p \left\{ \text{Prob} \left\{ \|T_{zw}\|_\infty < \gamma \right\} \geq p \right\}.$$

Here T_{zw} denotes the transference from the exogenous disturbance w to the minimized output vector z . Obviously, the prescribed γ should be smaller than the minimum γ attainable in the associated standard H_∞ problem.

2) Given $0 < p \leq 1$, find the minimum γ for which p can be assured, namely $\underset{\gamma}{\text{Min}} \{ \text{Prob} \{ \|T_{zw}\|_{\infty} < \gamma \} \geq p \}$.

When $p=1$, the standard H_{∞} -norm minimization problem is recovered.

The problem formulation in Section 2 adopts the second interpretation; that is, the best performance that can be guaranteed with a given probability is sought. However, the method developed in the sequel is applicable to both interpretations and a minor modification will produce the mechanism for solving the first problem. Sections 3 and 4 present the quadratic solution to the problem stated in Section 2. Section 3 contains the conditions for quadratically attaining the required best robust performance, which depend on the unknown vertices of a truncated parameter box. The resulting matrix inequalities turn out to be bilinear in the unknowns, and an iterative solution algorithm is proposed in which each step involves tractable LMIs. Section 4 introduces a new method for obtaining the required volume of the truncated parameter box (corresponding to the specified probability) via LMIs. These LMIs are derived using an extension of an existing second-order cone method. The LMIs of Sections 3 and 4 should be solved simultaneously to attain the best performance in the preassigned probability. Section 5 presents an affinely quadratic/multiconvex approach to the performance requirement, which relaxes that of Section 3. A two-parameter example is given in Section 6, which demonstrates the applicability of the proposed analysis method by comparison with a Monte-Carlo simulation.

The notation is fairly standard. Throughout the paper the superscript "T" stands for matrix transposition, \mathfrak{R}^n denotes the n -dimensional Euclidean space, $\mathfrak{R}^{n,m}$ is the set of all n by m real matrices, and $P > 0$ ($P \geq 0$) for $P \in \mathfrak{R}^{n,n}$ means that P is symmetric and positive definite (positive semi-definite, respectively). $\text{Prob}\{e\}$ is the probability that event (or condition) e occurs (or is satisfied). A uniform distribution of a scalar random variable between a and b will be denoted $U[a,b]$. $\{\Pi_1, \Pi_2, \dots, \Pi_l\}$ stands for the set comprising of $\Pi_1, \Pi_2, \dots, \Pi_l$. Q_{ij} are the (matrix) entries of the block matrix Q , and $\text{diag}\{Q_{ii}\}$ is a diagonal block matrix with elements Q_{ii} . T_{zw} is the disturbance-to-output transmission matrix of a linear system, and $\|G\|_{\infty}$ is the standard H_{∞} -norm of G . RHS (LHS) is the right (left)-hand side of an expression.

2. PROBLEM FORMULATION

Consider the class of uncertain linear systems that are described by the state-space equations

$$\dot{x} = A(\theta)x + B(\theta)w, \quad z = C(\theta)x + D(\theta)w \quad (1a,b)$$

where $\theta = [\theta_1, \theta_2, \dots, \theta_L]^T \in \mathfrak{R}^L$ is the uncertain parameters vector, and where the dependence of the system matrices on the parameters is affine:

$$A(\theta) = A_0 + A_1\theta_1 + \dots + A_L\theta_L = A_0 + \sum_{i=1}^L A_i\theta_i. \quad (2)$$

with similar expressions for the other three system matrices. The matrices A_0 , B_0 , C_0 , and D_0 may be thought of as those that define the nominal system; in

such a case the variables θ_i will represent the deviations of the system parameters from their nominal values. The parameters θ_i may also represent the "full" system parameters, in which case the nominal system is described by $A(\theta_{\text{nominal}})$ etc. All the matrices A_i , B_i , C_i , and D_i , $i=1, \dots, L$ are known; those that correspond to $i \geq 1$ tend to have low rank (since each parameter usually influences only few entries of a system matrix).

All θ_i , $i=1, \dots, L$ are assumed to be scalar, mutually independent, random variables, each drawn from an unknown probability distribution supported in the finite range $[\alpha_i, \beta_i]$; that is, β_i and α_i are the (known or assumed) extremal possible values of θ_i :

$$\theta_i \in [\alpha_i, \beta_i], \quad i=1, \dots, L. \quad (3)$$

If the θ_i are interpreted as deviations from the nominal parameters, it can be assumed that all the ranges $[\alpha_i, \beta_i]$ are symmetric about 0. Thus, the parameter vector lies in a L -dimensional hyper-rectangle \mathbf{B} ("box"). The polytope \mathbf{B} has 2^L vertices ("corners"); each vertex $v_{\mathbf{B}}$ is defined by a parameter-vector whose entries are all extremal parameter values, that is $v_{\mathbf{B}} = [\theta_1, \theta_2, \dots, \theta_L]^T$, $\theta_i \in \{\alpha_i, \beta_i\}$, $i=1, \dots, L$. The set of the 2^L vertices $v_{\mathbf{B}}$ of \mathbf{B} , which plays an important role in what follows, is denoted by

$$V_{\mathbf{B}} = \{ [\theta_1, \theta_2, \dots, \theta_L]^T \mid \theta_i \in \{\alpha_i, \beta_i\}, i=1, \dots, L \}. \quad (4)$$

Note that since \mathbf{B} is a hyper-rectangle, not a general polytope, its 2^L vertices are completely defined by the L pairs $\{\alpha_i, \beta_i\}$, $i=1, \dots, L$.

Given a required probability $0 < p \leq 1$, our purpose is to find the minimal (best) disturbance attenuation level $\gamma > 0$ for which the inequality

$$\text{Prob} \{ \|T_{zw}\|_{\infty} < \gamma \} \geq p \quad (5)$$

still holds.

3. QUADRATIC SOLUTION

The core of the solution is the realization that both the required probability and the best disturbance attenuation level are attained in some truncated parameter box \mathbf{B}_T . "Many" such \mathbf{B}_T 's may satisfy the required probability, and the one that provides the best polytopic performance is sought. This yet to-be-determined diminished box is characterized by $\theta_i \in [a_i, b_i] \subseteq [\alpha_i, \beta_i]$, $i=1, \dots, L$; that is, each vertex is $v_{\mathbf{B}_T} = [\theta_1, \theta_2, \dots, \theta_L]^T$, $\theta_i \in \{a_i, b_i\}$, $i=1, \dots, L$, and the set of 2^L vertices of \mathbf{B}_T is denoted

$$V_{\mathbf{B}_T} = \{ [\theta_1, \theta_2, \dots, \theta_L]^T \mid \theta_i \in \{a_i, b_i\}, i=1, \dots, L \}. \quad (6)$$

The vertices $v_{\mathbf{B}_T}$ of \mathbf{B}_T are yet unknown; i.e., a_i and b_i are yet to be determined for all θ_i , $i=1, \dots, L$ in such a way that the probability of $\theta \in \mathbf{B}_T$ is at least p while the polytopic γ associated with \mathbf{B}_T is minimal.

The presentation of the solution begins with a discussion of the performance (disturbance attenuation) aspects. The analysis of the probability aspects is deferred to the next section.

Applying over a polytope a parameter-independent quadratic Lyapunov function $\Phi(x)=x^T P_0 x$ (where P_0 is constant), it is well known that a necessary and sufficient condition for $\|T_{zw}\|_\infty < \gamma$ throughout \mathbf{B}_T under such a function is the existence of $P_0 > 0$ that for all $\theta \in \mathbf{B}_T$ satisfies the following BRL inequality:

$$\begin{bmatrix} A(\theta)^T P_0 + P_0 A(\theta) & P_0 B(\theta) & C(\theta)^T \\ B(\theta)^T P_0 & -\gamma I & D(\theta)^T \\ C(\theta) & D(\theta) & -\gamma I \end{bmatrix} < 0. \quad (7)$$

Due to the linearity of the system (1), the affinity of the system matrices in θ_i , and the convexity of (7) for a given parameter polytope, the latter is satisfied for all $\theta \in \mathbf{B}_T$ iff it is satisfied at the 2^L vectors θ comprising $V_{\mathbf{B}_T}$ (Boyd and Yang, 1989). The unknowns in (7) are, therefore, the 2^L vectors θ defining $V_{\mathbf{B}_T}$ (in fact, the L pairs $\{a_i, b_i\}, i=1, \dots, L$), the matrix P_0 , and the minimal disturbance attenuation level γ .

Since (7) contains products of P_0 and θ it is not generally convex - it is bilinear in these variables. However, note that (7) is indeed convex in P_0, γ for fixed θ and in θ, γ for fixed P_0 (since the system matrices are affine in θ). Therefore, it is proposed to tackle (7) by the following iterative algorithm based on alternating steps of solving (7) for P_0, γ and θ, γ while minimizing γ . Note that the LMI/s that express the probability requirement (see next section) are to be simultaneously satisfied.

Algorithm 1

Step 1: Solve (7) at $V_{\mathbf{B}}$ for P_0, γ while minimizing γ .

Remark 1: This step finds the minimum γ associated with $\mathbf{B}_T = \mathbf{B}$ as in the usual (deterministic) polytopic analysis, where the performance requirement is satisfied with $p=1$.

Remark 2: When the system is not quadratically stable at $V_{\mathbf{B}}$ step 1 fails and the algorithm cannot proceed. However, the system may still be quadratically stable at some $V_{\mathbf{B}_T}$. Possible remedies are: 1) Start from some arbitrary "box" "slightly" smaller than \mathbf{B} . 2) Start from a "small" arbitrary \mathbf{B}_T that doesn't provide the required probability; Step 2 will expand it towards the required probability, if the P_0 found for the initial \mathbf{B}_T can "cope" with a larger parameter-box.

Step 2: Solve (7) for θ, γ while minimizing γ , with P_0 found in the previous step; i.e., fix the last P_0 and simultaneously solve (7) at $V_{\mathbf{B}_T}$ for $\{a_i, b_i\}, i=1, \dots, L$ and γ while minimizing γ .

Step 3: Repeat step 1, replacing $V_{\mathbf{B}}, v_{\mathbf{B}}$, and \mathbf{B} by $V_{\mathbf{B}_T}, v_{\mathbf{B}_T}$, and \mathbf{B}_T , respectively. $V_{\mathbf{B}_T}$ is defined by the $\{a_i, b_i\}, i=1, \dots, L$ found in the previous step.

Step 4 is a repetition of step 2, etc. The iterations continue in this manner until some convergence criterion is satisfied. Since the function $\gamma(V_{\mathbf{B}_T}, P_0)$ is bounded from below and monotonically non-increasing, it is locally convergent. Global convergence seems difficult to establish.

4. THE PROBABILITY LMIS

Inspired by the uniformity principle of (Barmish, 1994), which implies that the uniform distribution is the right choice for robustness analysis (see Remark 9), the uniform distribution $U[\alpha_i, \beta_i]$ is henceforth assumed for all θ_i . Thus, the probability of each θ_i being in some truncated range $[a_i, b_i] \subseteq [\alpha_i, \beta_i]$ is simply $(b_i - a_i) / (\beta_i - \alpha_i)$; and, since all θ_i were assumed mutually independent, the probability of $\theta \in \mathbf{B}_T$ is

$$\text{Prob} \left\{ \bigcap_{i=1}^L \theta_i \in [a_i, b_i] \right\} = \prod_{i=1}^L \frac{b_i - a_i}{\beta_i - \alpha_i}. \quad (8)$$

Note that this probability is merely the volume-ratio of \mathbf{B}_T and \mathbf{B} , and that only the differences $b_i - a_i$ count. The requirement to exceed some desired probability p (i.e., (5)) now takes on the form

$$\prod_{i=1}^L (b_i - a_i) \geq p \prod_{i=1}^L (\beta_i - \alpha_i) \quad (9)$$

where the RHS is completely known. Expressing (9) as an LMI for $L=1, 2$ is simple. For $L=1$ the LMI is $b_1 - a_1 \geq p(\beta_1 - \alpha_1)$. For $L=2$ inequality (9) is $(b_1 - a_1)(b_2 - a_2) \geq p(\beta_1 - \alpha_1)(\beta_2 - \alpha_2)$, and by defining its RHS as \bar{p} ($\bar{p} > 0$) this inequality can be directly cast (using Schur complements) in the LMI

$$\begin{bmatrix} b_1 - a_1 & \sqrt{\bar{p}} \\ \sqrt{\bar{p}} & b_2 - a_2 \end{bmatrix} \geq 0. \quad (10)$$

For $L > 2$, however, representing (9) as an LMI in all $b_i - a_i$ is not straightforward. The LMI representation of (10) is obtained using an extension of the multi three-dimensional second-order cones representation of the geometrical mean function described in (Nestrov and Nemirovskii, 1994). The method given there applies to $L=2^q$ only, where q is any positive integer; this is a significant limitation in practical situations. The theorem below removes the above restriction - it applies to any positive integer L - and presents a clear, recursive way of constructing the desired LMIs. The theorem addresses the following problem, which is nothing but a reformulation of (9) with $y_i = b_i - a_i$ and with $\bar{p} = p \prod_{i=1}^L (\beta_i - \alpha_i)$:

Problem Π_0 : Given a scalar $\bar{p} > 0$ and any positive integer L , find L scalars $y_i, y_i > 0, i=1, \dots, L$ such that

$$\prod_{i=1}^L y_i \geq \bar{p}. \quad (11)$$

Π_0 is meaningful only when y_i are constrained also otherwise; and, since in our case the other constraints are in LMI form, it is desirable to obtain a characterization of Π_0 in LMI form.

The next lemma offers a one-step reduction of Π_0 ; the complete solution of Π_0 will consist of a series of successive applications of this lemma.

Lemma 1 *Problem Π_0 is equivalent to the following auxiliary problem Π_1 : find m_1 positive scalars $s_{1,j}, j=1, \dots, m_1$ (the first subscript of $s_{1,j}$ and the subscript of m_1 stand for Π_1) such that*

$$\prod_{j=1}^{m_1} s_{1,j} \geq \bar{p}^{1/2}. \quad (12)$$

When L is even, $m_1=L/2$ and

$$y_{2j-1} y_{2j} \geq s_{1,j}^2, \quad j=1, \dots, m_1. \quad (13)$$

When L is odd, $m_1=(L-1)/2+1$ and

$$y_{2j-1} y_{2j} \geq s_{1,j}^2, \quad j=1, \dots, m_1 - 1 \quad (14a)$$

$$y_L \geq s_{1,m_1}^2. \quad (14b)$$

Note that each of the inequalities of (13) or (14a) can be put in the simple scalar 2x2 LMI form (see (10))

$$\begin{bmatrix} y_{2j-1} & s_{1,j} \\ s_{1,j} & y_{2j} \end{bmatrix} \geq 0 \quad (15)$$

and so can also (14b), by regarding 1 as a second factor on its LHS. The lemma replaces (11) by (12), adding a set of m_1 LMIs requiring m_1 auxiliary unknowns. Note that the number of factors in the LHS of (12) is about half the number of factors in the LHS of (11). Now, since (12) has the same structure as (11), the same one-step reduction lemma can be applied to (12), creating problem Π_2 which replaces (12) by a similar inequality (however with $\bar{p}^{1/4}$ on its RHS). This requires m_2 additional unknowns $s_{2,j}$, $j=1, \dots, m_2$ and m_2 additional LMIs like (15), e.g.

$$\begin{bmatrix} s_{1,2j-1} & s_{2,j} \\ s_{2,j} & s_{1,2j} \end{bmatrix} \geq 0. \quad (16)$$

This successive reduction process is exhausted when arriving at an expression similar to (12) that has only two unknowns on its LHS, and hence can itself be expressed as a LMI similar to (16). The above reasoning is summarized in the following theorem.

Theorem 1 *Problem Π_0 is equivalent to the series of auxiliary problems Π_k , $k=1, \dots, l$, where l is the smallest positive integer such that $2^{l+1} \geq L$. Each problem Π_k is consecutively obtained from its predecessor Π_{k-1} by using Lemma 1.*

Theorem 1 states, in other words, that in order to solve (11) for y_i one should simultaneously solve the collection of LMIs (of the type (15) or (16)), created by a successive reduction of (11) according to Lemma 1, for y_i and all the auxiliary unknowns defined in the process. Note that the probability LMIs are in fact formulated with $b_i - a_i$; a_i and b_i link them to $V_{\mathbf{B}_T}$ of the BRL inequalities.

Remark 3: The last auxiliary problem, Π_l , necessarily addresses

$$s_{l,1} s_{l,2} \geq \bar{p}^{1/(2^l)}. \quad (17)$$

Remark 4: In Π_k , $k=1, \dots, l$, \bar{p} explicitly appears only once, in Π_l ; and, since l is known, the RHS of (17) can be redefined as $(\tilde{p})^2$ - thus making the above collection of LMIs linear in the probability variable \tilde{p} (see (13), (15)). This fact enables conversion of the proposed solution for the performance maximization problem into a solution for the probability maximization problem merely by minimizing $-\tilde{p}$ instead of γ .

Remark 5: When $L=2^q$ and q is a positive integer, the above theorem is equivalent to the second order cones representation of the geometrical mean function given in (Nestrov and Nemirovskii, 1994).

Now that both components comprising the quadratic solution to the problem presented in Section 2 have been introduced (the BRL inequalities of Section 3 and the probability LMIs of the current section), some concluding remarks on the above solution are in order.

Remark 6: Note that the probability LMIs need be solved (simultaneously with the performance inequalities) only in the even-numbered steps of Algorithm 1; in the odd-numbered steps $V_{\mathbf{B}_T}$ is not changed.

Remark 7: Running the above mechanism with several values of $0 < p \leq 1$ enables one to sketch the overall tradeoff between performance and probability for a given system.

Remark 8: The quadratic approach (same P_0) is quite restrictive, especially when “many” parameters are involved or when their ranges are “large” (causing significantly different system behavior over $V_{\mathbf{B}}$).

Remark 9: The solution is inherently conservative. The first conservatism source is the choice of uniform distribution, in the spirit of the uniformity principle of (Barmish, 1994) and (Barmish and Lagoa, 1996), which states that

$$\text{Min}_{f \in F} \text{Prob} \{ \theta^f \in \Theta_{\text{good}} \} = \text{Prob} \{ \theta^u \in \Theta_{\text{good}} \} \quad (18)$$

where, roughly speaking, Θ is the parameters space, Θ_{good} is the part of Θ where the system’s performance is considered good, θ^u is the random parameters vector uniformly distributed over Θ , and θ^f is that vector distributed according to any distribution f belonging to the family F of unimodal distributions that are non-increasing with the distance from the mode. In simple words, the uniform distribution provides the lowest probability of the system being “good”. The second conservatism source is the simply-connected nature of the \mathbf{B}_T ’s considered; any polytopic approach would ignore the fact that in the complementary part(s) of \mathbf{B} , i.e. $\mathbf{B} - \mathbf{B}_T$, there may well be other (separate) areas in which the desired γ is also attained. Thus, the true probability of attaining γ is greater than that represented by the largest simply connected polytope \mathbf{B}_T .

The following section improves the above solution by relaxing its strictly quadratic element; it is replaced by an affinely quadratic approach.

5. AFFINELY QUADRATIC SOLUTION

The concepts of Affine Quadratic Stability (AQS), AQ H_∞ Performance (AQP), and multiconvexity have been introduced in (Gahinet, *et al.*, (GAC) 1996). These robust stability/performance tests extend the standard notions of quadratic stability and performance by introducing a Lyapunov function $\Phi(x, \theta) = x^T P(\theta) x$ with affine dependence of $P(\theta)$ on the uncertain system’s parameter vector θ . Sufficient conditions for AQS/ AQP in the form of a finite set of tractable LMIs is obtained by adding a “multiconvexity” constraint on the above Lyapunov function. The resulting AQS LMI test is less conservative than quadratic stability and compares favorably with μ analysis.

AQP is defined as follows (the parameter time-dependence aspects of (GAC, 1996) have been omitted):

Definition: The system (1)-(2) has affine quadratic H_∞ performance γ (AQP) in \mathbf{B}_T if there exist $L+1$ symmetric matrices P_0, \dots, P_L such that

$$P(\theta) := P_0 + \theta_1 P_1 + \dots + \theta_L P_L > 0 \quad (19)$$

$$M(\theta) :=$$

$$\begin{bmatrix} A(\theta)^T P(\theta) + P(\theta) A(\theta) & P(\theta) B(\theta) & C(\theta)^T \\ B(\theta)^T P(\theta) & -\gamma I & D(\theta)^T \\ C(\theta) & D(\theta) & -\gamma I \end{bmatrix} \quad (20a)$$

$$M(\theta) < 0 \quad (20b)$$

are satisfied for all $\theta \in \mathbf{B}_T$. In such case, the system is asymptotically stable and $\|z\|_{L_2} < \gamma \|w\|_{L_2}$ for all $\theta \in \mathbf{B}_T$ and for all L_2 -bounded input w (provided that $x(0) = 0$).

Adding a multiconvexity requirement enables obtaining a tractable sufficient test for AQP in the form of a finite set of LMIs:

Theorem 2 *The system (1)-(2) has AQP γ in \mathbf{B}_T if $A(\theta_{mean})$ is stable and there exist $L+1$ symmetric matrices P_0, \dots, P_L which satisfy*

$$\begin{bmatrix} A_i^T P_i + P_i A_i & P_i B_i \\ B_i^T P_i & 0 \end{bmatrix} \geq 0, \quad i = 1, \dots, L \quad (21)$$

$$M(\theta) < 0 \quad \forall \theta \in V_{\mathbf{B}_T} \quad (22)$$

where θ_{mean} is defined by $\theta_i = (a_i + b_i)/2$, $i = 1, \dots, L$ and $M(\theta)$ is defined in (20a),(19).

The term ‘‘multiconvexity’’ refers to the inequalities (21), or to $A_i^T P_i + P_i A_i \geq 0$ in the case of AQS, which impose convexity for each θ_i separately; this is less demanding than convexity with respect to the vector θ .

Replacing the quadratic approach presented in Section 3 by an affinely quadratic approach consists of replacing (7) by (21),(22), where the discussion following (7) concerning the appearance of the unknowns applies also here, except that the single unknown matrix P_0 is now replaced by the set of unknown matrices P_0, \dots, P_L included in $P(\theta)$. This fact compounds the computational difficulty relative to the quadratic case, since even for fixed matrices P_0, \dots, P_L (corresponding to the even-numbered steps of Algorithm 1) the inequality (22) is not convex in θ, γ because of the products $P(\theta)A(\theta)$ and $P(\theta)B(\theta)$. Note that the actual unknowns are the L pairs $\{a_i, b_i\}$, $i = 1, \dots, L$.

It is proposed to numerically solve the above AQP formulation by an iterative algorithm denoted **Algorithm 2**, which is identical to Algorithm 1 except for the following modifications:

1. Solve the array of LMIs (21),(22) instead of an array of (7).
2. In the odd-numbered steps solve for P_0, \dots, P_L, γ rather than for P_0, γ .
3. In the even-numbered steps fix P_0, \dots, P_L obtained in the previous step.

4. In the even-numbered steps, where (the minimal) γ and $\{a_i, b_i\}$, $i = 1, \dots, L$ are sought, ‘‘convexify’’ the search for $\{a_i, b_i\}$ by fixing all their cross-terms (like $a_i a_j$, $i \neq j$) at their values from the previous step. To spell out this essentially simple remedy to the non-convexity of (22) in θ , consider the (1,1) entry of (22): note that $A(\theta)^T P(\theta) + P(\theta) A(\theta) < 0$ can be cast as

$$A_0^T P_0 + P_0 A_0 + \sum_j (Q_{0j} + Q_{j0}) \theta_j + \theta^T Q \theta < 0 \quad (23a)$$

or

$$\begin{bmatrix} R & \theta^T \\ \theta & -(\text{diag}\{Q_{ii}\})^{-1} \end{bmatrix} < 0, \quad i = 1, \dots, L \quad (23b)$$

where Q , W and Q_{0j}, Q_{j0} are defined by

$$Q_{ii} = A_i^T P_i + P_i A_i \geq 0 \quad (24a)$$

$$Q_{ij} = A_i^T P_j + P_j A_i, \quad i \neq j \quad (24b)$$

$$R = A_0^T P_0 + P_0 A_0 + \sum_j (Q_{0j} + Q_{j0}) \theta_j + \sum_{i \neq j} (Q_{ij} + Q_{ji}) \bar{\theta}_i \bar{\theta}_j \quad (25)$$

and $\bar{\theta}_i$ denotes θ_i of the previous step. (In (24b), the indexes of Q_{ij} can assume the value 0 to define Q_{0j}, Q_{j0} .) Currently, there is no proof of convergence for this algorithm; nevertheless, it has produced encouraging results.

Remark 10: A strengthened version of (24a) is used, that allows taking the inverses of Q_{ii} in (23b). (The latter are generally non-definite because of the low rank A_i 's.) The inequality (24a) is modified to be a strict inequality and Q_{ij} is added to its LHS, where $\mu_i > 0$, $i = 1, \dots, L$ are additional optimization parameters; in (25) the term $(\sum \mu_i \omega_i^2) I$, $\omega_i \in \{a_i, b_i\}$ is added. See (GAC, 1996), Sec. V.

6. EXAMPLE

Consider the system (1)-(2) with $L = 2$ where

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -0.1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},$$

$$\alpha_1 = -0.09, \quad \beta_1 = 0.09, \quad \alpha_2 = -0.06, \quad \beta_2 = 0.06,$$

$$B = [0 \ 1]^T, \quad C = [1 \ 0], \quad \text{and } D = 0.$$

The four Bode plots of the plant at $V_{\mathbf{B}}$ are considerably different from each other; indeed, Algorithm 1 is not effective. The minimum γ computed over $V_{\mathbf{B}}$ by the AQP approach (using the Matlab[®] LMI Toolbox) is 103.1. The resulting minimum γ over $V_{\mathbf{B}_T}$ for $p = 0.8$, computed by algorithm 2, is 39.95; the corresponding parameter-box volume ratio is 0.8, as required. Comparison with Monte-Carlo simulation (Ray and Stengel, 1993) reveals the inherent conservatism in the proposed method (Figure 1): for 90% of the parameter vectors θ in \mathbf{B} (rather than 80%) the standard minimum γ is lower than 39.95. In Figure 2 the tradeoff between the minimal γ and $1-p$ is presented, as computed by several applications of algorithm 2 and as derived from the Monte-Carlo results. Note the tradeoff-line shape and the algorithm's safety margin.

7. CONCLUSIONS

The problem of computing the minimum H_∞ -norm of an affinely parameter-dependent polytopic system with a given probability has been considered. Adopting the uniform distribution as the most conservative p.d.f., as implied by the uniformity principle, a solution to this problem has been derived in terms of sets of BLMI and LMIs. The BLMI comprise of a version of the BRL using the notion of AQP, and are combined with the so-called probability LMIs which are derived by extending an existing second-order cone method. An iterative algorithm has been proposed to solve the BLMI.

The results of the proposed AQP algorithm have been illustrated via a lightly damped system with two uncertain parameters. The algorithm easily converges locally, and its results are encouraging. A substantial (60%) decrease of the robust disturbance attenuation level has been gained at the cost of only 20% decrease in the probability. The overall tradeoff between the robust performance and its probability has also been computed. A corresponding tradeoff found by a Monte-Carlo simulation showed, as expected, some remaining conservatism; it may be partly removed by using recent results on bi-quadratic performance. Note that a solution to the dual problem of probability maximization for a given disturbance attenuation level is easily obtained by a minute modification of the proposed algorithm.

The results of the present paper can be extended in more than one direction. Allowing correlation between the uncertain parameters, non-uniform p.d.f.'s and nonlinear dependence on the system parameters can be useful in realistic cases. Finally, usage of the proposed methods for controller synthesis is obviously an important area for further research.

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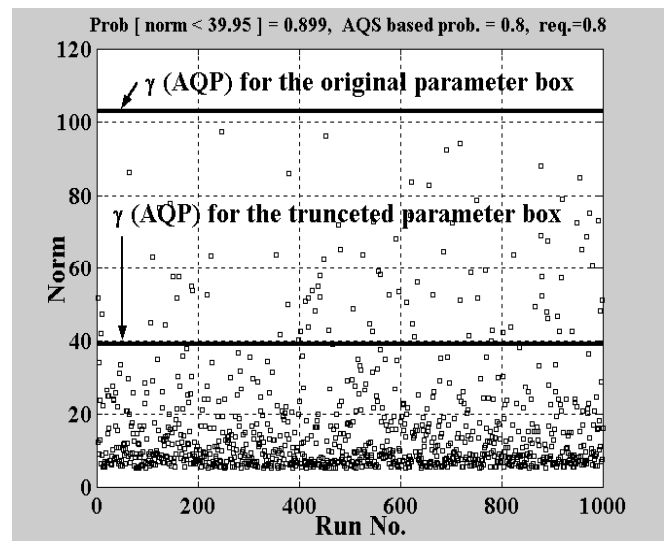


Fig. 1 Comparison between Monte-Carlo analysis and Algorithm 2 results.

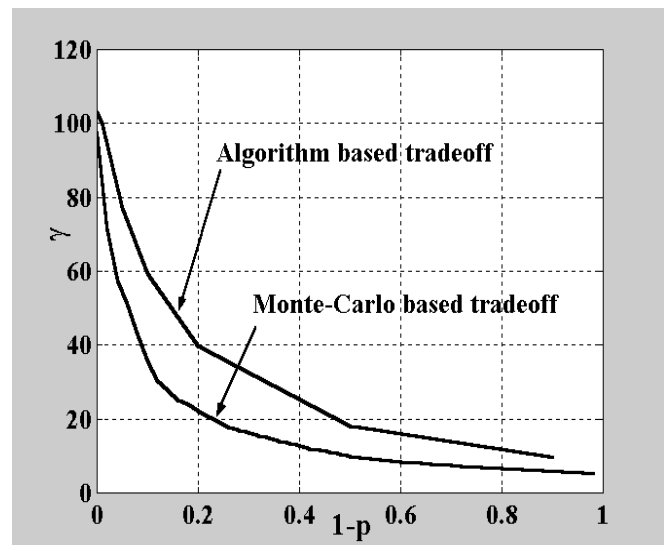


Fig. 2 Tradeoff between performance and probability: Algorithm 2 and Monte-Carlo results.