H_∞ TRACKING OF LINEAR SYSTEMS WITH STOCHASTIC UNCERT AINTIESAND PREVIEW

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Abstract: The problem of finite-horizon H_∞ trac king for linear contin uous time-v arying systems with stochastic parameter uncertainties is in v estigated for both, the state-feedback and the output-feedback control problems. We consider three tracking patterns which include the case where the reference signal is previewed in a fixed time-interv al ahead. In the state-feedback case a game theory approach is applied where the controller plays against nature and where necessary and sufficient conditions are found for the existence of a saddle-point equilibrium. The output-feedback control problem is solv ed as a max-min problem with the application of a bounded real lemma. A simple example demonstrates the theory. *Copyright C 2001 IFA C*

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1. INTRODUCTION

T rac king is one of the main fundamental problems in control theory ,where the system output is required to be as close as possible to an external reference signal. The H_∞ trac king control problem with preview for contin uous-time setting has been introduced by (Shaked and deSouza, 1995). This method processes the information that is gathered on the reference during the system operation and b y applying the game-theory approach it derives the optimal trac king strategy .Control with preview was also treated in (Kojima and Ishijima, 1997) for continuous-time systems, using state-space H_∞ theory for infinite-dimensional systems. The H_∞ con trol and estimation of state-multiplicative systems has been largely treated in the last decade (see Dragan and Morozan, 1997, hinriechsen and Pritchard, 1998 and the references therein). This paper solves the important problems of both, state-feedback and dynamic output-feedback con trol tracking for the state-multiplicative systems, which has never been dealt before.

Notation: We denote expectation b y $\mathcal{E}\{\cdot\}$ and by $[Q(t)]_+$, $[Q(t)]_-$ w e denote the causal and anti causal parts, respectively, of a function $Q(t)$. We pro vide all spaces $\mathcal{R}^k$, $k \geq 1$ with the usual inner product $< \cdot, \cdot >$ and with the standard Euclidean norm $|| \cdot ||$. By $||f(t)||^2_R$ w e denote the product of $f^T(t)Rf(t)$. We denote by $L^2(\Omega, \mathcal{R}^k)$ the space of square-integrable $\mathcal{R}^k-$ v alued functions on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. By $(\mathcal{F}_t)_{t>0}$ w e denote an increasing family of σ-algebras $\mathcal{F}_t \subset \mathcal{F}$. We also denote by $\tilde{L}^2([0,T); \mathcal{R}^k)$ the space of nonanticipative stoc hastic process $f(\cdot) = (f(t))_{t\in[0,T]}$ in

$\mathcal{R}^k$ with respect to $(\mathcal{F}_t)_{t\in[0,T)}$ satisfying: $||f(\cdot)||^2_{\tilde{L}_2} = \mathcal{E}\{\int_0^T ||f(t)||^2 dt\} = \int_0^T \mathcal{E}\{||f(t)||^2\}dt < \infty$. Stochastic differential equations will be interpreted to be of *Itô* type.

2. THE STATE-FEEDBACK TRACKING

Given the following linear continuous time-varying system with deterministic tracking signal of $r(t)$:

$$\begin{aligned} dx &= [A(t)x(t) + B_1(t)w(t) + B_2(t)u(t) + B_3(t)r(t)]dt \\ &\quad +F(t)x(t)d\beta(t) + G(t)u(t)d\zeta(t),\ x(0) = x_0 \\ z(t) &= C_1(t)x(t) + D_{12}(t)u(t) + D_{13}(t)r(t) \end{aligned} \quad (1)$$

where $x \in \mathcal{R}^n$ is the system state vector , $x(0)$ is any norm-bounded vector in $\mathcal{R}^n$, $w \in \tilde{L}^2([0,T);\mathcal{R}^p)$ is the exogenous disturbance signal, $z \in \mathcal{R}^q$ is the signal to be controlled and where $A(t)$, $B_1(t)$, $B_2(t)$, $B_3(t)$, $C_1(t)$, $D_{12}(t)$, $D_{13}(t)$, $F(t)$ and $G(t)$ are real known, piecewise continuous time-varying matrices of the appropriate dimensions. The variables $\beta(t)$ and $\zeta(t)$ are zero-mean real scalar Wiener processes that satisfy:

$$\mathcal{E}\{d\beta(t)\} = 0,\ \mathcal{E}\{d\zeta(t)\} = 0,\ \mathcal{E}\{d\beta(t)^2\} = dt,$$

$$\mathcal{E}\{d\zeta(t)^2\} = dt,\ \mathcal{E}\{d\beta(t)d\zeta(t)\} = \alpha dt,\ |\alpha| \le 1.$$

We denote $\tilde{R}(t) \stackrel{\Delta}{=} D_{12}^T D_{12}$. Our objective is to find a state-feedback control law $u(t)$ that minimizes, for the worst-case of the process disturbance $w(t)$ and the initial condition x_0, the energy of $\mathcal{E}\{z(t)\}$, with respect to the uncertain parameters, by using the available knowledge on the reference signal. We, therefore, consider, for a given scalar $\gamma > 0$, the following performance index:

$$J_E \stackrel{\Delta}{=} \mathcal{E}\{\int_0^T ||z(t)||^2 dt - \gamma^2 \int_0^T ||w(t)||^2 dt\}$$
$$+\mathcal{E}x^T(T)P_T x(T) - \gamma^2 ||x_0||^2_{R^{-1}},\ R > 0,\ P_T \ge 0. \quad (2)$$

We consider the following Riccati-type differential equation:

$$\begin{aligned} -\dot{Q} &= QA + A^T Q + \gamma^{-2} Q B_1 B_1^T Q + C_1^T C_1 \\ &\quad -\bar{S}^T \hat{R}^{-1} \bar{S} + F^T Q F, \quad Q(T) = P_T \end{aligned} \quad (3)$$

where $\hat{R} = \tilde{R} + G^T Q G$, $\bar{S} = B_2^T Q + \alpha G^T Q F + D_{12}^T C_1$.

The solution of the state-feedback tracking problem is obtained by the following theorem:

Theorem 1: *Consider the system of (1) and J_E of (2). Given $\gamma > 0$, the state-feedback tracking game possesses a saddle-point equilibrium solution iff there exists $Q(t) > 0, \forall t \in [0,T]$ that solves (3) such that $Q(0) < \gamma^2 R^{-1}$. When a solution exists, the saddle-point strategies are given by:*

$$\begin{aligned} x_0^* &= (\gamma^2 R^{-1} - Q_0)^{-1}\theta(0),\ w^* = \gamma^{-2} B_1^T (Qx + \theta), u^* = \\ &-\hat{R}^{-1}[(B_2^T Q + D_{12}^T C_1 + \alpha G^T Q F)x + D_{12}^T D_{13} r + B_2^T \theta_c], \end{aligned} \quad (4)$$

where w^*, x_0^* and u^* are the maximizing and minimizing strategies of nature and the controller, respectively, and where

$$\dot{\theta}(t) = -\bar{A}^T\theta(t) + \bar{B}_r r(t),\ t \in [0\ T],\ \theta(T) = 0, \quad (5)$$

with $\bar{A} = A - B_2\hat{R}^{-1}(D_{12}^T C_1 + \alpha G^T Q F) + (\gamma^{-2}B_1 B_1^T - B_2 \hat{R}^{-1} B_2^T)Q$, $\bar{B}_r = \bar{S}^T \hat{R}^{-1} D_{12}^T D_{13} - (QB_3 + C_1^T D_{13})$, and where $\theta_c \stackrel{\Delta}{=} [\theta(t)]_+$ (i.e the causal part of $\theta(\cdot)$) satisfies:

$$\begin{aligned} &\dot{\theta}_c(\tau) = -\bar{A}^T(\tau)\theta_c(\tau) + \bar{B}_r(\tau)r(\tau), \quad t \le \tau \le t_f, \\ &t_f = \begin{cases} t+h & if \quad t+h < T \\ T & if \quad t+h \ge T \end{cases} \quad \theta_c(t_f) = 0. \end{aligned} \quad (6)$$

The game value is then given by:

$J_E(r,u^*,w^*,x_0^*) = \bar{J}(r) + \mathcal{E}\int_0^T ||B_2^T[\theta]_-||^2_{\hat{R}_1} dt$, *where*

$\bar{J}(r) = \gamma^{-2}\mathcal{E}\int_0^T ||B_1^T\theta||^2 dt \mathcal{E}\int_0^T ||\hat{R}_1^{-1/2}(B_2^T\theta + D_{12}^T D_{13} r)||^2 dt$
$+\mathcal{E}\int_0^T ||D_{13}r||^2 dt + 2\mathcal{E}\int_0^T \theta^T B_3 r dt + \gamma^{-2}||\theta(0)||^2_{P_0}$ with $P_0 = [R^{-1} - \gamma^{-2}Q(0)]^{-1}$.

Proof The proof of the Theorem is the stochastic equivalent of the one in (Shaked and deSouza, 1995).

Next we consider the following three different tracking problems. 1) Stochastic H_∞-tracking with full preview of $r(t)$, 2) Stochastic H_∞-tracking with zero preview of $r(t)$, and 3) Stochastic H_∞ finite-fixed preview tracking of $r(t)$: In all three cases we seek a control law $u(t)$ of the form

$u(t) = H_x(t)x(t) + H_r(t)r(t)$, where $H_x(t)$ is a causal operator and where the causality of $H_r(t)$ depends on the information pattern of the reference signal. For all of the above three tracking problems we consider a related linear quadratic game in which the controller plays against nature. We, thus, consider the following game:

Find $w^*(t) \in \tilde{L}^2([0,T);\mathcal{R}^p)$, $u^*(t) \in L_2[0,T]$ and $x_0^* \in R^n$ that satisfy:

$$\begin{aligned} &J_E(r,u^*,w,x_0) \le J_E(r,u^*,w^*,x_0^*) \\ &\le J_E(r,u,w^*,x_0^*) \quad \forall r(t) \in L_2[0,T], \end{aligned} \quad (7)$$

where w^*, x_0^* and u^* are the saddle-point strategies.

We arrive at the following results:

Corollary 1 Stochastic H_∞-Tracking with full preview : *The tracking signal is perfectly known over the interval $t \in [0,T]$. In this case $\theta(t)$, is as in (5) and the control law is given by: $u = K_x x + K_r r + K_\theta \theta$, where*

$$\begin{aligned} K_x &= -\hat{R}^{-1}(B_2^T Q + D_{12}^T C_1 + \alpha G^T Q F), \\ K_r &= -\hat{R}^{-1} D_{12}^T D_{13} \ and \quad K_\theta = -\hat{R}^{-1} B_2^T. \end{aligned} \quad (8)$$

In this case $J_E(r,u^*,w^*,x_0^*)$ of (7) coincides with $\bar{J}(r)$ of Theorem 1.

Corollary 2 Stochastic H_∞-Tracking with no preview : *The tracking signal is measured on line i.e at time t $r(\tau)$ is known for $\tau \le t$. In this case the control law is given by $u = K_x x + K_r r$*

and the existence of (7) is guaranteed where $J_E(r,u^*,w^*,x_0^*) = \mathcal{E}\int_0^T ||\hat{R}^{-1/2}B_2^T\theta||^2 dt + \bar{J}(r)$ *where* $\theta(\cdot)$ *satisfy (5) and* $\bar{J}(r)$ *equals that of Theorem 1.*

Corollary 3 Stochastic H_∞-Tracking with finite fixed-preview : *The tracking signal* $r(t)$ *is previewed in a known fixed interval i.e* $r(\tau)$ *is known for* $\tau \leq t+h$ *where* h *is a known preview length. Since at time* $\bar{t}$, $r(t)$ *is known for* $t \leq min(T, \bar{t}+h)$ *the following control law is obtained:* $u = K_x x + K_r r + K_\theta \theta_c$, *where* K_x, K_r *and* K_θ *are defined in (8) and* θ_c *is given by (6). The above controller achieves (7) with* $J_E(r,u^*,w^*,x_0^*) = \bar{J}(r) + \mathcal{E}\int_0^T ||\hat{R}^{-1/2}B_2^T[\theta]_-||^2 dt$ *and where* $\bar{J}(r)$ *is defined in Theorem 1.*

3. THE INFINITE-HORIZON CASE

We treat the case where the matrices of the system in (1) are all time-invariant and T tends to infinity. In this case the solution $Q(t)$ of (3), if it exists, will tend to the mean square stabilizing solution of the following equation:

$$\tilde{Q}A + A^T\tilde{Q} + \gamma^{-2}\tilde{Q}B_1B_1^T\tilde{Q} + C_1^TC_1 - \tilde{S}^T\hat{R}^{-1}\tilde{S} + F^T\tilde{Q}F = 0,$$

assuming that the pair $(\Pi C_1, A - B_2\tilde{R}^{-1}D_{12}^TC_1)$, $\Pi = I - D_{12}\tilde{R}^{-1}D_{12}^T$ is detectable (see Theorem 5.8 in Dragan *et al.*, 1992). A strict inequality is achieved from (3) for $(w(t), x_o)$ that are not identically zero, iff the left side of (3) is strictly less than zero (for the equivalence of (3) and the corresponding inequality (see Hinriechsen and Pritchard, 1998)). The latter inequality can be expressed in a LMI (Linear Matrix Inequality) form in the case where $\alpha = 0$ and $D_{12}^TC_1 = 0$. We arrive at the following theorem:

Theorem 2: *Consider the system of (1) and* J_E *of (2) with* T *goes to* ∞ *and with constant matrices,* $D_{12}^TC_1 = 0$ *and* $\alpha = 0$. *Then, for a given* $\gamma > 0$, *there exists a strategy* u^* *that satisfies* $\forall w(t) \in \tilde{L}^2([0,\infty);\mathcal{R}^p)$, $x_o \in R^n$, $J_E(r,u^*,w,x_0) < \bar{J}(r) + \mathcal{E}\int_0^\infty ||\hat{R}^{-1/2}B_2^T[\theta]_-||dt$, *where* $\bar{J}(r)$ *is given in Theorem 1, with the upper limit of the integral goes to infinity, iff there exists a positive-definite matrix* $\tilde{P} \in \mathcal{R}^{n\times n}$ *that satisfies the following LMI:*

$$\begin{bmatrix} \Upsilon_{11} & B_1 & \tilde{P}C_1^T & B_2G^T & \tilde{P}F^T \\ B_1^T & -\gamma^2 I_q & 0 & 0 & 0 \\ C_1\tilde{P} & 0 & -I & 0 & 0 \\ GB_2^T & 0 & 0 & -(\tilde{P}+G\tilde{R}^{-1}G^T) & 0 \\ F\tilde{P} & 0 & 0 & 0 & -\tilde{P} \end{bmatrix} < 0. \quad (9)$$

where $\Upsilon_{11} = A\tilde{P} + \tilde{P}A^T - B_2\tilde{R}^{-1}B_2^T$.

Proof: The inequality that is obtained from (3) for $\alpha = 0$ and $D_{12}^TC_1 = 0$ is

$$\tilde{Q}A + A^T\tilde{Q} + \gamma^{-2}\tilde{Q}B_1B_1^T\tilde{Q} + C_1^TC_1 - \tilde{S}^T\hat{R}^{-1}\tilde{S} + F^T\tilde{Q}F < 0,$$

where $\tilde{S} = B_2^T\tilde{Q}$. Denoting $\tilde{P} = \tilde{Q}^{-1}$, we multiply the latter inequality by $\tilde{P}$ from both sides and obtain, using the matrix inversion lemma and the identity: $\alpha[I+\beta\alpha]^{-1} = [I+\alpha\beta]^{-1}\alpha$, the following inequality:

$$A\tilde{P} + \tilde{P}A^T + \gamma^{-2}B_1B_1^T + \tilde{P}C_1^TC_1\tilde{P} - B_2\tilde{R}^{-1}B_2^T$$
$$+B_2G^T[\tilde{P} + G\tilde{R}^{-1}G^T]^{-1}GB_2^T + \tilde{P}F^T\tilde{P}^{-1}F\tilde{P} < 0.$$

By using Schur's complement formula,we obtain the LMI of Theorem 2.

4. OUTPUT-FEEDBACK TRACKING

We consider the system of (1) where $G = 0$ and where the system output is

$$dy(t) = [C_2(t)x(t) + D_{21}(t)w(t)]dt + H(t)x(t)d\zeta(t) + n(t)dt\ ,$$

where $y(t) \in \mathcal{R}^z$ and where $\zeta(t)$ satisfies $\mathcal{E}\{d\zeta(t)^2\} = dt$, $\mathcal{E}\{d\beta(t)d\zeta(t)\} = 0$. Like in the state-feedback case we seek a control law $u(t)$, based on the information of the reference signal $r(t)$ that minimizes the tracking error between the the system output and the tracking trajectory, for the worst case of the initial condition x_0, the process disturbances $w(t)$, and the measurement noise $n(t)$. We, therefore, consider the following performance index:

$$J_O(r,u,w,n,x_0) \ = \ J_E(r,u,w,x_0) \ - \ \gamma^2\mathcal{E}\int_0^T ||n(t)||^2 dt,$$

where J_E is given in (2). Similarly to the state-feedback case we solve the problem for the above three tracking patterns. The problem is solved along the lines of the standard solution where use is made of the state-feedback solution of section 3, thus arriving to an estimation problem to which we apply a Bounded Real Lemma (BRL) for tracking systems, which is partially derived from the state-feedback solution. We first bring the following BRL solution:

4.1) Bounded Real Lemma for tracking systems: We consider the following system:

$$dx = [A(t)x(t) + B_3(t)r(t)]dt + F(t)x(t)d\beta(t))$$
$$+B_1(t)w(t)dt,\ x(0) = x_0,\ z(t) = C_1(t)x(t) + D_{13}(t)r(t), \quad (10)$$

which is obtained from (1) by setting $B_2(t) \equiv 0$ and $D_{12}(t) \equiv 0$. We consider the following index of performance:

$$J_B \triangleq \mathcal{E}\int_0^T ||z(t)||^2 dt - \gamma^2\mathcal{E}\int_0^T ||w(t)||^2 dt - \gamma^2||x_0||^2_{R^{-1}},$$

where $R > 0$. We arrive at the following theorem:

Theorem 3: *Consider the system of (10) and the above J_B. Given $\gamma > 0$, J_B satisfies $J_B \le \tilde{J}(r,\epsilon)$ $\forall w(t) \in \tilde{L}^2([0,\infty);\mathcal{R}^p)$, $x_o \in R^n$, where* $\tilde{J}(r,\epsilon) = \mathcal{E}\int_0^T ||D_{13}r||^2 dt + \gamma^{-2}\mathcal{E}\int_0^T ||B_1^T\tilde{\theta}||^2 dt$ $+2\mathcal{E}\{\int_0^T \tilde{\theta}^T B_3 r dt + ||\tilde{\theta}(0)||^2_{\epsilon^{-1}}\}$, *iff there exists $\tilde{Q}(t) > 0, \forall t \in [0,T]$ that solves the following Riccati-type equation:*

$$-\dot{\tilde{Q}} = \tilde{Q}A + A^T\tilde{Q} + \gamma^{-2}\tilde{Q}B_1B_1^T\tilde{Q} + C_1^TC_1 + F^T\tilde{Q}F, \quad \tilde{Q}(0) = \gamma^2 R^{-1} - \epsilon I, \tag{11}$$

for some $\epsilon > 0$, where

$$\dot{\tilde{\theta}}(t) = -\hat{A}^T\tilde{\theta}(t) + \hat{B}_r r(t), \quad t \in [0\ T],\ \tilde{\theta}(T) = 0, \tag{12}$$

and where $\hat{A} = A + \gamma^{-2}B_1B_1^T\tilde{Q}$, $\hat{B}_r = -[\tilde{Q}B_3 + C_1^TD_{13}]$.

Proof: The solution of the BRL does not acquire saddle-point strategies (since $u(t)$ is no longer an adversary). It can, however, be readily derived based on the first part of the sufficiency proof of Theorem 1 where we set $B_2(t) \equiv 0$ and $D_{12}(t) \equiv 0$, and where we take $P_T = 0$. We obtain the following index of performance :

$$J_B = \gamma^2||x_0 - \hat{x}_0||^2_{\tilde{P}_0^{-1}} - \gamma^2\mathcal{E}\int_0^T ||w - \gamma^{-2}B_1^T(\tilde{Q}x + \tilde{\theta})||^2 dt,$$

The neccesity follows from the fact that for $r(t) \equiv 0$, one gets $\tilde{J}(r,\epsilon) = -x_0^T\epsilon x_0$ and thus the existence of $\tilde{Q} > 0$ that solves (11) is the necessary condition in the stochastic BRL (Dragan *et al.*, 1992). By taking small enough values of ϵ in $\tilde{Q}(0)$ the neccesity proof still holds. We note that the choice of $\epsilon > 0$ in $\tilde{Q}(0)$ of (11) reflects on both, the above cost value of $\tilde{J}(r,\epsilon)$ and the minimum achievable γ. If one chooses $0 < \epsilon << 1$ then, the cost of $\tilde{J}(r,\epsilon)$ increases while the solution of (11) is easier to achieve, which results in a smaller γ. The choice of large ϵ, on the other, hand causes the reverse effect, which leads to a larger γ.

Due to the special structure of the stochastic uncertainty in the system of (1) together with $dy(t)$, the solution of the output-feedback control problem, can not be obtained by applying a saddle point strategies but rather as a max-min problem. We consider the system of (1) together with $dy(t)$ and we assume that (3) has a solution $Q(t) > 0$ over $[0,T]$. Using an expression which is similar to $J_E(r,u,w,x)$ in the state-feedback case (see equation A.5 in Shaked and deSouza, 1995), the index of performance turns to be:

$$J_O(r,u,w,n,x_0) = -\gamma^2||x_0 - \hat{x}_0||^2_{P_0^{-1}} - \gamma^2\int_0^T ||n(t)||^2 dt,$$
$$+\mathcal{E}\int_0^T ||[u + \hat{R}^{-1}\bar{S}^Tx + \hat{R}^{-1}(B_2^T\theta + D_{12}^TD_{13}r)]||^2_{\hat{R}} dt + \bar{J}(r)$$
$$-\gamma^2\mathcal{E}\int_0^T ||w - \gamma^{-2}B_1^T(Qx + \theta)||^2 dt$$

where $\bar{J}(r)$ is defined in Theorem 1 and where we take $G \equiv 0$ in both $\hat{R}$ and $\bar{S}$ following (3). We also note that in the full preview case $[\theta(t)]_+ = \theta(t)$. We define

$$\bar{w}(t) = w(t) - w^*(t), \quad \bar{u}(t) = u(t) + \hat{R}^{-1}[D_{12}^TD_{13}r + B_2^T\theta], \tag{13}$$

where $w^*(t)$ is defined in (4). We obtain:

$$J_O(r,u,w,n,x_0) = -\gamma^2||x_0 - \hat{x}_0||^2_{P_0^{-1}} - \gamma^2\mathcal{E}\int_0^T ||\bar{w}||^2 dt$$
$$+\mathcal{E}\int_0^T ||\hat{R}^{1/2}[\bar{u} + \hat{C}_1x]||^2 dt + \bar{J}(r) - \gamma^2\mathcal{E}\int_0^T ||n(t)||^2 dt,$$

with $\hat{C}_1 = \hat{R}^{-1}[B_2^TQ + D_{12}^TC_1]$ and where $P_0 = [R^{-1} - \gamma^{-2}Q(0)]^{-1}$. We seek a controller of the form

$$\bar{u}(t) = -\hat{C}_1(t)\hat{x}(t).$$

We, therefore, re-formulate the state equation ofthe system and we obtain:

$$dx = [\bar{\bar{A}}(t)x(t) + B_1(t)\bar{w}(t) + B_2(t)\bar{u}(t) + \bar{r}(t)]dt + F(t)x(t)d\beta(t), \tag{14}$$

where

$$\bar{\bar{A}}(t) = A + \gamma^{-2}B_1B_1^TQ,\ \bar{r}(t) = [B_3 - B_2\hat{R}^{-1}D_{12}^TD_{13}]r - [B_2\hat{R}^{-1}B_2^T - \gamma^{-2}B_1B_1^T]\theta. \tag{15}$$

We consider the following Luenberger-type state observer:

$$d\hat{x}(t) = \bar{\bar{A}}\hat{x}(t)dt + L[d\bar{y} + !\hat{C}_2\hat{x}(t)dt] + g(t)dt,\ \hat{x}(0) = 0,\quad \hat{z}(t) = \hat{C}_1\hat{x}(t), \quad \hat{C}_2 = C_2 + \gamma^{-2}D_{21}B_1^TQ \tag{16}$$

where

$$\bar{y} = y - \gamma^{-1}D_{21}B_1^T\theta, \quad g(t) = B_2\bar{u}(t) + \bar{r}(t). \tag{17}$$

We note that $\bar{y} = \hat{C}_2x(t) + Hx(t)\zeta(t) + D_{21}\bar{w} + \eta(t)$. Denoting $e(t) = x(t) - \hat{x}(t)$ and using the latter we obtain: $de(t) = [\bar{\bar{A}} - L\hat{C}_2]e(t)dt + \hat{B}\hat{w}(t)dt + [Fd\beta(t) - LHd\zeta(t)]x(t)$, where we define

$$\hat{w}(t) \stackrel{\Delta}{=} [\bar{w}^T(t) \quad n^T(t)]^T, \quad \hat{B} = [\bar{B}_1 - LD_{21} \quad -L].$$

Defining $\xi(t) = [x^T(t) \quad e^T(t)]^T$, $\tilde{r}(t) = [r^T(t) \quad \theta^T(t)]^T$, we obtain

$$d\xi(t) = [\tilde{A}dt + \tilde{F}d\beta(t) + \tilde{H}d\zeta(t)]\xi(t) + \tilde{B}_1\hat{w}(t)dt + \tilde{B}_3\tilde{r}(t)dt,\ \xi^T(0) = [x^T(0)\ \ x^T(0)]^T,\ \tilde{z}(t) = \tilde{C}_1\xi(t), \tag{18}$$

where

$$\tilde{A} = \begin{bmatrix} \bar{\bar{A}} - B_2\hat{C}_1 & B_2\hat{C}_1 \\ 0 & \bar{\bar{A}} - L\hat{C}_2 \end{bmatrix},\ \tilde{B}_1 = \begin{bmatrix} B_1 & 0 \\ B_1 - LD_{21} & -L \end{bmatrix},$$

$$\tilde{F} = \begin{bmatrix} F & 0 \\ F & 0 \end{bmatrix},\ \tilde{H} = \begin{bmatrix} 0 & 0 \\ -LH & 0 \end{bmatrix},\ \tilde{C}_1 = [0\ \ \hat{C}_1],$$

$$\tilde{B}_3 = \begin{bmatrix} B_3 - B_2\hat{R}^{-1}D_{12}^TD_{13} & B_2\hat{R}^{-1}B_2^T - \gamma^{-2}B_1B_1^T \\ 0 & 0 \end{bmatrix}. \tag{19}$$

Applying the results of Theorem 3 to the system of (18) with the matrices of (19), we obtain the following Riccati-type equation:

$$-\dot{\hat{P}} = \hat{P}\tilde{A} + \tilde{A}^T\hat{P} + \gamma^{-2}\hat{P}\tilde{B}_1\tilde{B}_1^T\hat{P} + \tilde{H}^T\hat{P}\tilde{H} + \tilde{F}^T\hat{P}\tilde{F}$$

$$+\tilde{C}_1^T\tilde{C}_1,\ \hat{P}(0)=\begin{bmatrix}\hat{P}_{0,11} & -0.5\gamma^2\rho I\\ -0.5\gamma^2\rho I & 0.5\gamma^2\rho I\end{bmatrix}, \quad (20)$$

where $\hat{P}_{0,11} = \gamma^2 R^{-1} - Q(0) - \epsilon I + 0.5\gamma^2\rho I$ and with $\epsilon > 0,\ \rho >> 1$. The initial condition of (20) is derived from the fact that the initial condition of (18) corresponds to the case where a large weight of say, $\rho >> 1$, is imposed on $\hat{x}(0)$ to force nature to select $e(0) = x(0)$ (i.e $\hat{x}(0) = 0$) (see Shaked and Supline, 2001). In the case where the augmented state-vector is chosen as $\xi(t) = [x^T(t) \quad \hat{x}^T(t)]^T$ the initial condition of $\hat{P}_0$ of (20) would satisfy, following (11),

$$\hat{P}(0)=\begin{bmatrix}\gamma^2R^{-1}-Q(0) & 0\\ 0 & \gamma^2\rho I\end{bmatrix}+\begin{bmatrix}-\epsilon I & 0\\ 0 & -0.5\gamma^2\rho I\end{bmatrix},$$

where $\gamma^2 R^{-1} - Q(0)$ is the initial weight and where the factor of 0.5 in $-0.5\gamma^2\rho I$ is arbitrarily chosen such the $(2,2)$ block of $\hat{P}(0)$ is positive definite. The above $\hat{P}(0)$ can be readily transformed to account for the augmented state-vector of $\xi(t) = [x^T(t) \quad e^T(t)]^T$ by the pre- and post-multiplication of the above matrices, with Υ^T and Υ, respectively, where $\Upsilon \triangleq \begin{bmatrix} I & 0\\ I & -I\end{bmatrix}$, the result of which is the initial condition of (20).

The solution of the (20) involves the simultaneous solution of both $\hat{P}(t)$ and the filter gain L and can not be obtained readily due to mixed terms of the latter variables in (20). Considering, however, the monotonicity of $\hat{P}$ with respect to a free semi-positive definite term in (20) (Hinriechsen and Pritchard, 1998), the solution to the above Riccati-type equation can be obtained by solving the following Differential Linear Matrix Inequality (DLMI) :

$$\begin{bmatrix}\dot{\hat{P}}+\tilde{A}^T\hat{P}+\hat{P}\tilde{A}+\tilde{F}^T\hat{P}\tilde{F} & \hat{P}\tilde{B}_1 & \tilde{C}_1^T & \tilde{H}^T\hat{P}\\ \tilde{B}_1^T P & -\gamma^2 I_p & 0 & 0\\ \tilde{C}_1 & 0 & -I_q & 0\\ \hat{P}\tilde{H} & 0 & 0 & -\hat{P}\end{bmatrix}\le 0, (21)$$

where $\hat{P} > 0$ and with $\hat{P}(0)$ of (20) and where we require that $trace\{P(\tau)\}$ be minimized at each time instant $\tau \in [0,\ T]$.

Recently, novel methods for solving DLMIs has been introduced in (Shaked and Supline, 2001). Applying the method of (Shaked and Supline, 2001), the above DLMI can be solved by discretizing the time interval $[0,\ T]$ into equally spaced time instances resulting in the following discretisized DLMI :

$$\begin{bmatrix}\Psi_{11} & \hat{P}_k\tilde{B}_{1,k} & \tilde{C}_{1,k}^T & \tilde{H}_k^T\hat{P}_k\\ \tilde{B}_{1,k}^T\hat{P}_k & -\gamma^2\tilde{\varepsilon}^{-1}I_p & 0 & 0\\ \tilde{C}_{1,k} & 0 & -\tilde{\varepsilon}^{-1}I_q & 0\\ \hat{P}_k\tilde{H}_k & 0 & 0 & -\tilde{\varepsilon}^{-1}\hat{P}_k\end{bmatrix}\le 0, \quad (22)$$

for $k = 0, 1, .., N-1$ and where

$\hat{P}_{k+1} - \hat{P}_k + \tilde{\varepsilon}(\tilde{A}_k^T\hat{P}_k + \hat{P}_k\tilde{A}_k) + \tilde{\varepsilon}\tilde{F}_k^T\hat{P}_k\tilde{F}_k,\ \tilde{A}_k = \tilde{A}(t_k),\ \tilde{B}_{1,k} = \tilde{B}_1(t_k),\ \tilde{C}_{1,k} = \tilde{C}_1(t_k),\ \tilde{H}_k = \tilde{H}(t_k)$, and $\tilde{F}_k = \tilde{F}(t_k)$ with $\{t_i,\ i = 0,..N-1,\ t_N = T,\ t_0 = 0\}$ and

$$t_{i+1} - t_i \triangleq \tilde{\varepsilon} = N^{-1}T,\ i = 0, ...N-1. \quad (23)$$

The discretized estimation problem thus becomes one of finding, at each $k \in [0,\ N-1]$, $\hat{P}_{k+1} > 0$ of minimum trace that satisfies (22).

The latter DLMI is initiated with the initial condition of (20) at the instance $k = 0$ and a solution for both, the filter gain L_k and $\hat{P}_{k+1}$ (i.e $\hat{P}_1$ and L_0) is sought for, under the minimum trace requirement of $\hat{P}_{k+1}$. The latter procedure repeats itself by a forward iteration up to $k = N-1$, where N is chosen (and therefore $1/\tilde{\varepsilon}$) to be large enough to allow for a smooth solution (see also Shaked and Supline, 2001). We summarize the above results, for the full preview case, by the following theorem:

Theorem 4: *Consider the system of (1) with $dy(t)$ and J_O. Given $\gamma > 0$ and $\epsilon > 0$, the output-feedback tracking control problem, where $r(t)$ is known a priori for all $t \le T$ (the full preview case), possesses a solution iff there exist $Q(t) > 0,\ \forall t \in [0,T]$ that solves (3) such that $Q(0) < \gamma^2 R^{-1}$, and $\hat{P}(t)$ that solves (20) $\forall t \in [0,T]$ starting from the initial condition of (20), where R is defined in (2). If a solution to (3) and (20) exist we obtain the following control law:*

$$u_{of}(t) = -\hat{C}_1(t)\hat{x}(t) \quad (24)$$

where $\hat{x}(t)$ is obtained by solving for (16).

In the case where $r(t)$ is measured on line, or with preview $h > 0$, we note that w(t)which is not restricted by causality constrains, will be identical to the one in the case of the full preview. We obtain the following:

Corollary 4 *H_∞ Output-feedback tracking with fixed-finite preview of $r(t)$: In this case we obtain: $u_{of}(t) = -\hat{C}_1[\hat{x}]_+$, where*

$$d[\hat{x}]_+ = [\bar{\hat{A}} + L\hat{C}_2]\hat{x}dt + Ld[\bar{y}]_+ + [g(t)]_+dt,$$
$$[g(t)]_+ = B_2\bar{u} + [B_3 - B_2\hat{R}^{-1}D_{12}^TD_{13}]r - [B_2\hat{R}^{-1}B_2^T - \gamma^{-2}B_1B_1^T][\theta]_+,\quad d[\bar{y}]_+ = dy - \gamma^{-1}D_{21}B_1^T[\theta]_+.$$

Corollary 5 *H_∞ Output-feedback tracking with no preview of $r(t)$: In this case $[\theta(t)]_+ = 0$ and we obtain: $u_{of}(t) = -\hat{C}_1[\hat{x}]_+$, where*

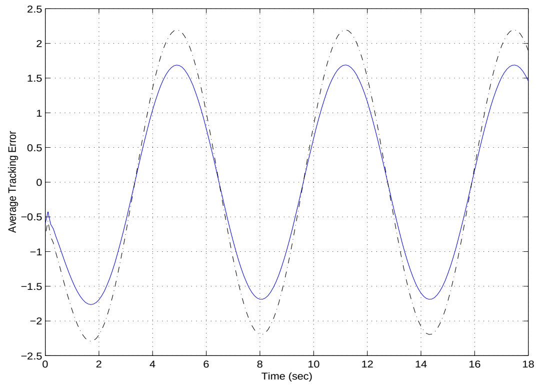

Fig. 1. Comparison between the tracking errors obtained in the standard solution (dashed lines) and by the new method (solid lines) for $r = sin(t)$, measured on line.

$$d[\hat{x}]_+ = [\bar{\hat{A}} + L\hat{C}_2]\hat{x}dt + Ly + [g(t)]_+ dt,$$
$$[g(t)]_+ = B_2\bar{u} + [B_3 - B_2\hat{R}^{-1}D_{12}^T D_{13}]r.$$

5. EXAMPLE

We consider the system of (1) with the following objective function:
$J = lim_{T\to\infty}\mathcal{E}\int_0^T ||Cx - r||^2 + 0.01||u||^2 - \gamma^2||w||^2 d\tau$
where there is an access to the states of the system, where

$$A = \begin{bmatrix} 0 & 1 \\ -1 & -0.4 \end{bmatrix},\ F = \begin{bmatrix} 0 & 0 \\ 0 & -0.1 \end{bmatrix},\ B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$,\ B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ B_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } C = [-0.5\ \ 0.4]$$

and where $G = 0$. The case of $h = 0$ can be solved using the stochastic solution of (El Ghaoui, 1995) where r_k is considered as a disturbance. The disturbance vector w_k becomes the augmented disturbance vector $\tilde{w}_k \triangleq [w_k^T\ r_k^T]\ .^T$ Using the notation of the standard problem, we define

$$B_1 = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix},\ D_{11} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} \text{ and } D_{12} = \begin{bmatrix} 0 \\ .1 \end{bmatrix}.$$

We obtain a minimum of $\gamma = 2.07$ for the latter solution. Using the results of Theorem 2, we obtain $\gamma_{min} = 1.06$. We compared the two solutions for $\gamma = 2.1$ and we obtained for the standard solution the control law $u(t) = K_x * x(t)$ where $K_x = [-500.65\ \ -53.01]$. For our solution, using Corollary 2, the resulting control law is: $u(t) = [-16.10 \quad -14.62]x(t), K_r = 0$. In Figure 1 the average tracking error $(Cx(t) - r(t))$, with respect to the statistics of the multiplicative noise, is depicted as a function of time for $r = sin(1 * t)$. The improvement achieved by our new method, in this frequency, is clearly visible.

6. CONCLUSIONS

In this paper we solve the problem of tracking signals with preview in the presence of Wiener-type stochastic parameter uncertainties in the system state-space model. A saddle-point tracking strategy is obtained, for the state-feedback case, which is based on the measurement of the system state and the previewed reference signal. The game value depends on the reference signal and is usually greater than zero. The infinite-horizon state-feedback tracking solution has been readily obtained, for a given $\gamma > 0$, by solving a single LMI. The output-feedback tracking control was solved along the lines of the standard solution, where the problem is re cast into an estimation problem. The later solution was carried by applying the BRL, found for state-multiplicative systems, to systems with reference inputs. It is shown via the example that the tracking error is considerably reduced, in the state-feedback case, using a preview in comparison with an alternative stochastic design where the tracking signal is taken as a disturbance.

7. REFERENCES

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