15th Triennial World Congress, Barcelona, Spain

NONLINEAR SYSTEM INVERSION APPLIED TO ECOLOGICAL MONITORING

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Abstract: The aim of this work is to model and monitor the influence of abiotic (environmental) effects on a population system of several species living in the same habitat. The dynamics of the population system will described by the classical Lotka-Volterra equations. It is also supposed that abiotic effects cause changes in given (input) parameters which are observed indirectly by the biomass of certain indicator species. Then, applying system inversion, for the obtained control system with output, the unknown input will be recovered in order to monitor the environmental effects on the population system..

Keywords: — Nonlinear systems, Lotka-Volterra population system, system inversion, structural inversion.

1. INTRODUCTION

In this paper, as a starting point, the classical Lotka-Volterra model of several species is considered. Certain coefficients of this model are supposed to be influenced by abiotic effects of the environment such as pollution, meteorological changes, etc. These time-dependent coefficients are regarded as inputs (hidden parameters) of a control system. It is assumed that the biomass of certain populations (indicator species) of the community is observed. In order to recover or estimate these hidden parameters out of the observation, the system inversion, presented in (Szigeti *et al.*, 2002) will be applied.

Considering n species living in the same habitat, suppose that the biomass at the i-th species is described by x_i, a smooth function of time, its Malthus parameter is ϵ_i, and the effect of the j-th species on the i-th one is proportional to $x_i x_j$ with a coefficient Γ_{ij}. The dynamics of the population system is described by the equations

[1] This work has been supported by the Hungarian National Science Foundation (OTKA, Grant No: T 029320) and the Higher Education Research and Development Project (FKFP, Grant No: 0286199)

$$\dot{x_i} = x_i \left(\epsilon_i - \sum_{j=1}^{n} \Gamma_{ij} x_j \right), \quad (i = 1, 2, \ldots, n).$$

For vectors $a, b \in \mathbb{R}^n$ and matrices $A, B \in \mathbb{R}^{nxn}$. Introduce the operations

$$\begin{aligned} a * b &= (a_1 b_1, \ldots, a_n b_n)^T, \\ a * A &= (a_i a_{ij})_{i,j=1}^n, \\ a * A &= (a_{ij} a_j)_{i,j=1}^n, \\ A * B &= (a_{ij} b_{ij})_{i,j=1}^n. \end{aligned}$$

Then with $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)^T$ and $\Gamma = (\Gamma_{ij})_{i,j=1}^n$, the above system can be written in the form

$$\dot{x} = \epsilon * x - x * \Gamma x.$$

The study of basic properties of this model, including stability goes back to Volterra, see, e.g., (Scudo and Ziegler, 1978), also (Yodzis, 1989).

For the description of the control model, suppose that there are $m < n$ environmental factors having an effect on the population system. I of them, $u_1, u_2, \ldots, u_I$ control certain Malthus parameters in the form

$$\epsilon_k + \sum_{i=1}^{I} \eta_k^i u_i(t),$$

where for the vectors $\eta_i = (\eta_1^i, \eta_2^i, \ldots, \eta_I^i)$ $i = 1, 2, \ldots, I$, $\eta_i * \eta_j = 0$ holds whenever $i \neq j$, (each population is effected at most by one factor). Similarly, suppose that there are $J = m - I$ further environmental factors, $(v_1, v_2, \ldots, v_J)$ having an effect on the coefficients of interaction in the form

$$\Gamma_{ik} + \sum_{j=1}^{J} G_{ik}^j v_j(t),$$

where for the matrices $G_j = (G_{ik}^j)_{i,k=1}^n$, $j = 1, 2, \ldots, J$, $G^i * G^j = 0$ holds whenever $i \neq j$, and $\Gamma_{ik} = 0$ implies $G_{ik}^j = 0$ for all $j = 1, 2, \ldots, J$.

From biological point of view it is also reasonable to require that, for all $j = 1, 2, \ldots, J$, $G_{ki}^j \neq 0$ implies $G_{ik}^j \neq 0$. Now, in terms of the above notation, the corresponding control system can be rewritten in the form

$$\dot{x} = \epsilon * x - x * \Gamma x + \sum_{i=1}^{I} \eta_i u_i * x + \sum_{j=1}^{J} v_j x * G_j x. \quad (1)$$

Assume now, in order to monitor the environmental effects, that the biomass of n species is observed. Then with distinct vectors $c_1, \ldots, c_m \in \mathbb{R}^n$, the observations are given by

$$y_1 = c_1^T x, y_2 = c_2^T x, \ldots, y_m = c_m^T x,$$

or equivalently,

$$y = Cx$$

defining $C = (c_1 | c_2 | \ldots | c_m)^T$.

Remark 1. If, by technical convenience, on measuring the biomass, no distinction is made between certain species, then vectors $c_1, c_2, \ldots, c_m$, may have several unit coordinates and the condition and the condition $c_i * c_j = 0$ holds whenever $i \neq j$.

The abiotic effects of the environment change inputs $u_1, u_2, \ldots, u_I, v_1, v_2, \ldots, v_J$, however, directly, only the outputs $y_1, y_2, \ldots, y_m$, are observed. Hence, in order to recover the influenced inputs, input observation system inversion is required. Both were considered in (Hou and Patton, 1998) and (Szigeti *et al.*, 2002).

Left invertibility of a linear systems and input observability are equivalent for inputs which are vanishing for $0 \leq t < t_0$. Invertibility of input affine multivariable nonlinear systems were considered, for example, in (Hirschorn, 1979), (Fliess, 1986), (Isidori, 1995).

2. INVERSION ALGORITHM

Let us consider the inversion of analytic system

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x) u_i, \quad (2)$$

$$y(x) = h(x) \quad (3)$$

with $x \in \mathbb{R}^n$, $y, u \in \mathbb{R}^m$.

For the inversion the construction of Isidori will be followed without the hypothesis that the matrix

$$A_1(x) = (L_{g_j} L_f{}^{r_i^1 - 1} h_i) \quad (4)$$

is invertible, where $r_1 = (r_1^1, r_2^1, \ldots, r_m^1)$ is the relative degree in the invertible case, how they are the orders of the derivative of h_i when first time control appear. We can suppose, that if $rank A_1(x) = d_1 < m$, then the first d_1 rows are linearly independent. Then there exists a matrix $F_1(x) \in \mathbb{R}^{(m-d_1) x m}$, of rank $m - d_1$, such that

$$F_1(x) A_1(x) = 0, \quad (5)$$

and

$$F_1(x) = (F_{ij}^1) = (f_{ij}^1(\ldots L_{g_k} L_f{}^{r_l^1 - 1} h_l(x) \ldots)),$$

where $f^1_{ij}(\alpha_{kl})$ are polynomials.

Considering derivatives of order $k < r^1_i$,

$$y_i^{(k)} = L_f{}^k h_i(x), \quad k = 0, 1, \ldots, r^1_i - 1 \qquad (6)$$

it can be proven that $L_{g_j} L_f{}^{k-1} h_i(x) = 0$.

However, for r_1, using the vectorial notation

$$y^{(r_1)} = (y_1^{(r^1_1)}, y_2^{(r^1_2)}, \ldots, y_m^{(r^1_m)})^T,$$

$$L_f{}^{r_1} h(x) = (L_f{}^{r^1_1} h_1(x), \ldots, L_f{}^{r^1_m} h_m(x))^T$$

can be proven by induction, using the hypothesis that

$$y^{(r_1)} = L_f{}^{r_1} h + A_1(x) u. \qquad (7)$$

Then, (3) and (5) imply that

$$F_1(x)(y^{(r_1)} - L_f{}^{r_1} h(x)) = 0. \qquad (8)$$

Now the output relation can be redefined taking the first d_1 components of the original output, the rest of the components are defined by (8). Then the objects

$$A_2(x, y^{(r_1)}), r_2, d_2, F_2(x, y^{(r_1)}),$$

are defined in a way analogous to the first step.

Clearly one has

$$r^1_1 = r^2_1,\ r^1_2 = r^2_2, \ldots,\ r^1_{d_1} = r^2_{d_1}.$$

Case 1. $d_1 = d_2 < m$. Then system (1) is obviously not invertible.

Case 2. $d_1 < d_2 = m$. Then the algorithm stops. From the first d_1 of (5) and the equation

$$\sum_{q=0}^{r_2} \binom{r_2}{q} * L_f^q F_1(x)(y^{(r_1+r_2-q)} - L_f^{r_1+r_2-q} h(x)) - A_2(x, y^{(r_1)}) u = 0,$$

where $q = (q_1, q_2, \ldots, q_m)^T$

$$\binom{r_2}{q} = \left(\binom{r^2_1}{q_1}, \binom{r^2_2}{q_2}, \ldots \binom{r^2_m}{q_m}\right),$$

u can be uniquely recovered, finishing the inversion algorithm.

Case 3. $d_1 < d_2 < m$. Then we proceed until the algorithm stops with cases (1) or (2). In case (2), an equation of full rank, similar to (6) is obtained. For more details of the algorithm see (Szigeti *et al.*, 2002).

If algorithm stops at the i^{th} step, then the full rank condition for $A_i(x, y, \dot{y}, \ldots) = 0$, can be replaced by a differential-algebraic condition of polynomial form

$$P_i(u, \dot{u}, \ldots y, \dot{y}, \ldots) = 0,$$

using Diop's state elimination, see (Diop, 1991). The specific algebraic structure of systems (1) is very useful, in the computation of the general algorithm, however it is tedious enough to describe in a general way. Invertibility, or more specifically, that certain input formally appears in an output, after a given number of derivatives, can be characterized by the diagraph of the food web. Hence, it seems that further research would be interesting to introduce the structural invertibility of a control systems, and to characterize by graph methods.

Conditions on $\eta_i * \eta_j$ and $G^i * G^j$ are structural ones.

3. EXAMPLE

As an illustration a five-species system is considered in which, for the sake of simplicity intra-specific competitions are excluded. The parameter Γ_{ij} figuring in the model are suppose to be positive, and two types of interaction occur.

There are five predator-pray pairs: (2,1), (3,1), (3,2), (4,3) and (5,4), and there is amensalism (a one-way negative effect) between species 5 and 3. The first two Malthus parameters are affected by environmental control factors and, in an structured way, interaction parameters are influenced by two abiotic control parameters. The biomass of the first three species is observed. Under these conditions the control system reads as follows:

$$\begin{aligned}
\dot{x_1} &= x_1(\epsilon_1 - \Gamma_{12}x_2 - \Gamma_{13}x_3) + \\
&\quad x_1(\eta_1 u - (\Gamma_{12}x_2 + \Gamma_{13}x_3)v_1), \\
\dot{x_2} &= x_2(\epsilon_2 + \Gamma_{21}x_1 - \Gamma_{23}x_3) + \\
&\quad x_2(\eta_2 u + (\Gamma_{21}x_1 - \Gamma_{23}x_3)v_1), \\
\dot{x_3} &= x_3(\epsilon_3 + \Gamma_{31}x_1 + \Gamma_{32}x_2 - \Gamma_{34}x_4 - \Gamma_{35}x_5) + \\
&\quad x_3(\eta_3 u + (\Gamma_{31}x_1 + \Gamma_{32}x_2)v_1), \\
\dot{x_4} &= x_4(\epsilon_4 + \Gamma_{43}x_3 - \Gamma_{45}x_5) - \Gamma_{45}x_4x_5v_2, \\
\dot{x_5} &= x_5(\epsilon_5 + \Gamma_{54}x_4) + \Gamma_{54}x_4x_5v_2. \qquad (9)
\end{aligned}$$

$$y_1 = x_1,\ y_2 = x_2,\ y_3 = x_3,$$

From the output derivatives, the equations

$$\begin{aligned}\dot{y_1} &= x_1(\epsilon_1 - \Gamma_{12}x_2 - \Gamma_{13}x_3) + \\ &\quad x_1(\eta_1 u - (\Gamma_{12}x_2 + \Gamma_{13}x_3)v_1), \\ \dot{y_2} &= x_2(\epsilon_2 + \Gamma_{21}x_1 - \Gamma_{23}x_3) + \\ &\quad x_2(\eta_2 u + (\Gamma_{21}x_1 - \Gamma_{23}x_3)v_1), \\ \dot{y_3} &= x_3(\epsilon_3 + \Gamma_{31}x_1 + \Gamma_{32}x_2 - \Gamma_{34}x_4 - \Gamma_{35}x_5) + \\ &\quad x_3(\eta_3 u + (\Gamma_{31}x_1 + \Gamma_{32}x_2)v_1) \qquad (10)\end{aligned}$$

are obtained.

The resolubility of this equations in the indeterminates x_4, u, v_1, or x_5, u, v_1, or $\Gamma_{34}x_4 - \Gamma_{35}x_5, u, v_1$, is necessary to the invertibility of the system. However, each of those is equivalent to the following determinant condition

$$Det \begin{vmatrix} 0 & \eta_1 x_1 & -x_1(\Gamma_{12}x_2 + \Gamma_{13}x_3) \\ 0 & \eta_2 x_2 & x_2(\Gamma_{21}x_1 - \Gamma_{23}x_3) \\ x_3 & \eta_3 x_3 & x_3(\Gamma_{31}x_1 + \Gamma_{32}x_2) \end{vmatrix} \neq 0,$$

that is,

$$x_1x_2x_3(\eta_1\Gamma_{21}x_1 + \eta_2\Gamma_{12}x_2 + (\eta_2\Gamma_{13} - \eta_1\Gamma_{23})x_3)$$

is different of zero.

That, with the other necessary condition, that $\Gamma_{45}x_4x_5 \neq 0$ or $\Gamma_{54}x_4x_5 \neq 0$, equivalently, $(\ \Gamma_{45}^2 + \Gamma_{54}^2)x_4x_5 \neq 0$, already is sufficient to the invertibility, to: that is

$$\begin{aligned}&(\Gamma_{45}^2 + \Gamma_{54}^2)x_1x_2x_3x_4x_5(\eta_1\Gamma_{21}x_1 + \\ &\eta_2\Gamma_{12}x_2 + (\eta_2\Gamma_{13} - \eta_1\Gamma_{23})x_3) \neq 0. \qquad (11)\end{aligned}$$

$(\Gamma_{45}^2 + \Gamma_{54}^2)x_4x_5 \neq 0$, means, that control v_2 there exists effectively.

The condition $0 < x_1, < x_2, \dots, x_5$ imply that the product is not zero. The positiveness means that an ecosystem of 5 (and non less), species is modelled. The unique real condition is

$$\eta_1\Gamma_{21}y_1 + \eta_2\Gamma_{12}y_2 + (\eta_2\Gamma_{13} - \eta_1\Gamma_{23})y_3) \neq 0,$$

which automatically holds for certain families of parameters, for example, if

$\eta_2\Gamma_{21} > 0, \eta_2\Gamma_{12} > 0, \eta_2\Gamma_{13} - \eta_1\Gamma_{23} > 0.$

are fulfilled. Let us define new outputs by mixing of (1),(2),(3):

$$\begin{aligned}&\eta_2x_2\dot{y_1} - \eta_1x_1\dot{y_2} = x_1x_2(\eta_2\epsilon_1 - \eta_1\epsilon_2 - \\ &(\eta_1\Gamma_{21}x_1 + \eta_2\Gamma_{12}x_2 + (\eta_2\Gamma_{13} - \eta_1\Gamma_{23})x_3) - \\ &(\eta_1\Gamma_{21}x_1 + \eta_2\Gamma_{12}x_2 + \eta_2\Gamma_{13} - \eta_1\Gamma_{23})x_3),\end{aligned}$$

$$\begin{aligned}&\eta_3x_3\dot{y_1} - \eta_1x_1\dot{y_3} = x_1x_3(\eta_3\epsilon_1 - \eta_1\epsilon_3 - \\ &(\eta_1\Gamma_{31}x_1 + (\eta_1\Gamma_{32} + \eta_3\Gamma_{12})x_2 + \eta_3\Gamma_{13})x_3) - \\ &(\eta_1\Gamma_{31}x_1 + (\eta_1\Gamma_{32} + \eta_3)\Gamma_{12})x_2 + \eta_3\Gamma_{13})x_{43})u) - \\ &(\eta_1\Gamma_{31}x_1 + (\eta_1\Gamma_{32} + (\eta_3\Gamma_{12})x_2 + \eta_3\Gamma_{13})x_3)v_1 + \\ &(\eta_1\Gamma_{34}x_4 + \eta_1\Gamma_{35})x_5).\end{aligned}$$

From (1) and (2) define the mixing of the last two output relations:

$$\begin{aligned}&(\eta_2x_2\dot{y_1} - \eta_1x_1\dot{y_2})(\eta_1\Gamma_{31}x_1 + (\eta_1\Gamma_{32} + (\eta_3\Gamma_{12})x_2 \\ &+\eta_3\Gamma_{13})x_3)x_3 - (\eta_3x_3\dot{y_1} - \eta_1x_1\dot{y_3})(\eta_1\Gamma_{21}x_1 + \\ &(\eta_2\Gamma_{12}x_2 + (\eta_2\Gamma_{13} - \eta_1\Gamma_{23})x_3)x_2 = x_1x_2x_3 \\ &(\eta_2\epsilon_1 - \eta_1\epsilon_2(\eta_1\Gamma_{31}x_1 + (\eta_1\Gamma_{32} + \eta_3\Gamma_{12}x_2 + \\ &(\eta_3\Gamma_{13})x_3)x_3) + (\eta_1\epsilon_3 - \eta_3\epsilon_1)\eta_1\Gamma_{21}x_1 + \eta_2\Gamma_{12}x_2 \\ &+(\eta_2\Gamma_{13} - \eta_1\Gamma_{23})x_3) - (\eta_1\Gamma_{21}x_1 + \eta_2\Gamma_{12}x_2 + \\ &(\eta_1\Gamma_{34}x_4 + \eta_1\Gamma_{35}x_5)(\eta_2\Gamma_{13} - (\eta_1\Gamma_{23}x_3)).\end{aligned}$$

Denoting by $q(x_1, x_2, x_3)$ the fraction

$$\frac{\eta_1\Gamma_{31}x_1 + (\eta_1\Gamma_{32} + \eta_3\Gamma_{12})x_2 + \eta_3\Gamma_{13})x_3}{\eta_1\Gamma_{21}x_1 + \eta_2\Gamma_{12}x_2 + (\eta_2\Gamma_{13} - \eta_1\Gamma_{23})x_3},$$

the following relation

$$\begin{aligned}&\Gamma_34x_4 + \Gamma_35x_5 = \frac{\dot{y}_1}{\eta_1x_1}(\eta_3 - \eta_2q(x_1, x_2, x_3)) + \\ &\frac{\dot{y}_2}{x_1}q(x_1, x_2, x_3) - \frac{\dot{y}_3}{x_3} + \frac{\eta_1\epsilon_3 - \eta_3\epsilon_1}{\eta_1} + \\ &\frac{\eta_2\epsilon_1 - \eta_1\epsilon_2}{\eta_1}q(x_1, x_2, x_3). \qquad (12)\end{aligned}$$

is obtained

The last equation is an equation for the combination $\Gamma_34x_4 + \Gamma_35x_5$ in terms of the states x_1, x_2, x_3 and the output derivatives $\dot{y}_1, \dot{y}_2, x\dot{y}_3$, or, in terms of the outputs and their derivatives (all sates x_1, x_2, x_3 can be replaced by y_1, y_2, y_3. Hence, by differentiation of (12) in the expression $\Gamma_34\dot{x}_4 + \Gamma_35\dot{x}_5$ inputs u, v_1, v_2 will appear, obtaining the third equation, together with arbitrary two of (10) , to invert system (9), under the invertibility condition, given in (11).

The first 3 states are directly observed, while x_4, or, x_5, or $\Gamma_34x_4 + \Gamma_35x_5$ can be computed from the last mixed output relation. One state can not be computed simultaneously from the observed outputs, in general a state observer is required for one state variable.

4. CONCLUSIONS

Certain population communities influenced by abiotic effects of the environment were modelled in terms of the classical Lotka-Volterra model. By considering the changes of the model parameters as inputs, and the observation of the biomass of the indicator species as outputs, an algebraic control system with output was obtained as the mathematical description of the biological problem. Then, powerful nonlinear technics, as system inversion was applied in order to recover the changing parameters representing the abiotic effects .

The applied inversion method, in spite, that the algorithm of the left inversion, in general is not unique, always works with polynomials as a consequence that the mixing matrices $F_1(x), F_2(x, y, \dot{y}, \ldots)$ are polynomials of the Lie derivatives $L_{g_j} {L_f}^{r_i^1 - 1} h_i$ and ones of their products, and the output derivatives, and that the the original system is algebraic. Therefore, the algorithm can be implemented by a symbolic computational package such as MAPLE, MATHEMATICA, etc.

The present paper has been aimed at a methodological development of population system monitoring of the authors is make simulations results within a project based on the data of fish populations of the Bolsena lake in Italy, which will be published elsewhere.

5. REFERENCES

Diop, S. (1991). Elimination in control theory. *Math. Cont. Syst. Sign.* **4**, 17–32.

Fliess, M. (1986). A note on the invertibility of nonlinear input-output differential systems. *Sys. Cont. Letters.* **8**, 147–151.

Hirschorn, R. M. (1979). Invertibility of multivariable nonlinear control systems. *IEEE Trans. Aut. Contr.* **AC-24**, 855–865.

Hou, M. and R. J. Patton (1998). Input observability and input reconstruction. *Automatica* **34**, 789–794.

Isidori, A. (1995). *Nonlinear Control Systems.* third edition ed.. Springer Verlag.

Scudo, F. and Y. R. Ziegler (1978). *The Golden Age of Theoretical Ecology.* Springer Verlag. Berlin-Heidelberg-New York.

Szigeti, F., J. Bokor and A. Edelmayer (2002). Input reconstruction by means of system inversion: application to fault detection and isolation. In: *Proc. of 15th IFAC World Congress, Barcelona, Spain.* Vol. Submitted.

Yodzis, P. (1989). *Introduction to Theoretical Ecology.* Harper & Row. New York.