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FINITE-VALUED CONTROL LAW SYNTHESIS FOR NONLINEAR UNCERTAIN SYSTEMS

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Abstract: In this paper a Lyapunov technique is presented for the synthesis of controllers characterised by control signals that, for construction simplicity and/or in order to attain a better efficiency, may only assume a finite number of values. In particular, a design technique of a control law with prescribed control levels for a class of continuous-time SISO uncertain nonlinear systems is provided, which guarantees the tracking of a given reference trajectory, with a prescribed maximum error norm, a prescribed rate of convergence and a low switching frequency. *Copyright © 2002 IFAC*

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1. INTRODUCTION

The use of the traditional continuous and stationary feedback control laws does not allow solving the tracking problem for many nonlinear systems, see (Aeyels, 1985; Antsaklis, *et al.*, 1995; Itkis, 1976). Moreover, there exist various industrial processes (especially power systems) that, for construction simplicity and/or in order to attain better efficiencies, may only be controlled through control signals assuming a finite number of different values, with a relatively slow switching.

Systems that contain both continuous and discrete-valued variables or signals are called *hybrid systems*, see (Antsaklis, *et al.*, 1995). The scientific community interest about hybrid control laws synthesis is relatively recent and the recently published papers document how both theoretical and practical problems are still open, see (Sastry, 1999) and the rich bibliography therein. In (Nikitin, 1993; Nikitin, 1994; Sontag, 1990) controllers with control signals without constraints on their amplitude, but constant in prescribed intervals of time, are discussed. Vice-versa, in (Itkis, 1976) control laws are proposed with two or infinite number of levels, with an infinitely fast switching, see also (Khalil, 1996; Utkin, 1992) for a detailed discussion on variable structure control.

This paper deals with the problem of robust tracking control law synthesis for a class of hybrid systems, consisting of continuous-time SISO uncertain nonlinear plants, whose control inputs take value from a finite set. To this aim, a Lyapunov methodology is presented for the design of control laws with prescribed levels, which guarantee the tracking of a sufficiently regular trajectory with a prescribed maximum norm of the tracking error vector, a prescribed rate of convergence and good performances in terms of robustness and switching frequency. A numerical example illustrates the effectiveness of the proposed technique.

2. PROBLEM FORMULATION AND PRELIMINARY RESULTS

Throughout this paper, the following notation is used:

$$D_n(s) = \left[s \; \dot{s} \; \dots \; s^{(n-1)} \right]^T \quad \text{for each signal } s(t) \in C^{(n-1)},$$

$\|z\|_P = \sqrt{z^T P z}, \qquad z \in R^n, P \in R^{n \times n}$ symmetric and positive definite (p.d.),

$\lambda_{\max}(P)$ = maximum eigenvalue of the matrix P, supposed positive definite.

Consider the continuous-time SISO nonlinear system with uncertain parameters:

$$y^{(n)} = f(t, p, D_n(y)) + F(t, p, D_n(y))u \qquad (1)$$

where $t \in T \subseteq R$ is the time variable; $u \in U \subset R$ is the control input, which may assume a finite number l of different levels $u_i \in U$, $i = 1,2,\ldots,l$; $y \in R$ is the output to be controlled; p is the vector of m uncertain parameters ranging into a compact set $\wp \subset R^m$; F is a real scalar smooth function in its arguments, which has the following property:

$$\begin{aligned}&\forall \aleph \subset R^n,\ \aleph \text{ compact set},\\ &\exists F_{\aleph} > 0 : |F(t, p, D_n(y))| \geq F_{\aleph},\\ &\forall t \in T, \forall D_n(y) \in \aleph, \forall p \in \wp\end{aligned} \qquad (2)$$

and, analogously, f is a real scalar smooth function in its arguments, which has the following property:

$$\begin{aligned}&\forall \aleph \subset R^n,\ \aleph \text{ compact set},\\ &\exists f_{\aleph} \geq 0 : |f(t, p, D_n(y))| \leq f_{\aleph},\\ &\forall t \in T, \forall D_n(y) \in \aleph, \forall p \in \wp\end{aligned} \quad . \qquad (3)$$

Assume that $\hat{y}(\cdot)$ is the reference trajectory and that $\hat{y}(t) \in C^{(n-1)}$, with a bounded n-th derivative.

By imposing that:

$$\varepsilon = D_n(e), \quad e = \hat{y} - y, \qquad (4)$$

it is easy to verify that the state equation of the tracking error vector ε may be expressed in the form:

$$\dot{\varepsilon} = E\varepsilon - Bw \ , \qquad (5)$$

where:

$$E = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & 0 & 1 \\ -k_1 & -k_2 & -k_3 & \cdots & -k_n \end{bmatrix}, \qquad (6)$$

$k_i \in R, \quad i = 1,2,\ldots,n$

$$B = [0 \quad 0 \quad \ldots \quad 1]^T \qquad (7)$$

$$w = Fu + f - K^T \varepsilon - \hat{y}^{(n)}, \qquad (8)$$

$$K = [k_1 \quad k_2 \quad k_3 \quad \cdots \quad k_n]^T \ .$$

For the error system (5)-(8), the subsequent practical tracking problem may be formulated.

Problem 1. (Practical tracking problem)
Given the system (1) and the reference trajectory $\hat{y}(\cdot)$, design a control law with a finite number of levels, which guarantees that the tracking error is uniformly bounded and tends asymptotically to an arbitrarily small neighbourhood of the origin of the error space, with a rate of convergence not greater than a given $\bar{\tau}$, $\forall p \in \wp$.

In order to solve the Problem 1, the following preliminary Lemmas are introduced.

Lemma 1. Consider the system (5)-(7) and assume that the matrix E has only eingenvalues with negative real part. If the signal w satisfies the inequality:

$$vw \geq 0, \forall \varepsilon \notin S, \quad S = \left\{\varepsilon : \|\varepsilon\|_P \leq \rho, \quad \rho > 0\right\} \qquad (9)$$

where $v = B^T P \varepsilon$ and P is the solution of the Lyapunov equation

$$E^T P + PE = -Q, \qquad Q \ p.d., \qquad (10)$$

then the error ε remains uniformly bounded and converges to the hyper-ellipsoide S with a rate of convergence not greater than an exponential one characterised by a time constant:

$$\bar{\tau} = 2\lambda_{\max}\left(Q^{-1}P\right). \qquad (11)$$

Moreover, the time of convergence of ε to S is not greater than:

$$t_c = \bar{\tau} \ln\left(\frac{\|\varepsilon(t_0)\|_P}{\rho}\right), \qquad (12)$$

where t_0 is an assumed initial instant of time.

Proof. By choosing as Lyapunov function for the system (5):

$$V(\varepsilon) = \varepsilon^T P \varepsilon = \|\varepsilon\|_P^2 \qquad (13)$$

and by using (10), it is:

$$-\dot{V}(\varepsilon) = \varepsilon^T Q \varepsilon + 2vw \ , \qquad (14)$$

which, for (9) and the hypothesis of stability about E, gives:

$$\frac{\dot{V}(\varepsilon)}{V(\varepsilon)} \leq -\sup_{\varepsilon}\left(\frac{\varepsilon^T Q \varepsilon}{\varepsilon^T P \varepsilon}\right), \quad \forall \varepsilon \notin S \ . \qquad (15)$$

Since, see (Gantmacher, 1959):

$$\sup\left(\frac{\varepsilon^T Q\varepsilon}{\varepsilon^T P\varepsilon}\right)=\frac{1}{\lambda_{\max}\left(Q^{-1}P\right)}, \quad \forall\varepsilon\notin S, \qquad (16)$$

it is:

$$\begin{aligned}&\|\varepsilon(t)\|_P \le \|\varepsilon(t_0)\|_P \exp(-(t-t_0)/\bar{\tau}),\\ &\forall t,t_0\in T \mid t\ge t_0, \forall\varepsilon(t)\notin S\end{aligned} \qquad (17)$$

where $\bar{\tau}$ has the expression (11). Finally, from (9) and from (17), the (12) easily follows.

♦

With regard to the time constant $\bar{\tau}$, it may be rendered dependent only on the eigenvalues of the matrix E through a suitable choice of matrix Q in (10), as pointed out by the following:

Lemma 2. If the roots $\lambda_i, i=1,2,\ldots,n$, of the characteristic polynomial of the matrix E:

$$a(\lambda)=\lambda^n+k_n\lambda^{n-1}+\ldots+k_1=\prod_{i=1}^{n}(\lambda-\lambda_i) \qquad (18)$$

are real, distinct and negative, and the matrix Q in (10) is chosen as:

$$Q=-2\left(Z^T\right)^{-1}\Lambda Z^{-1}, \qquad (19)$$

with the matrix Z such that:

$$Z^{-1}EZ=diag(\lambda_1,\lambda_2,\ldots,\lambda_n), \qquad (20)$$

then:

$$P=\left(ZZ^T\right)^{-1}=\begin{bmatrix} n & \sum_{i=1}^{n}\lambda_i & \cdots & \sum_{i=1}^{n}\lambda_i^{n-1}\\ \sum_{i=1}^{n}\lambda_i & \sum_{i=1}^{n}\lambda_i^2 & \cdots & \sum_{i=1}^{n}\lambda_i^{n}\\ \vdots & \vdots & \cdots & \vdots\\ \sum_{i=1}^{n}\lambda_i^{n-1} & \sum_{i=1}^{n}\lambda_i^{n} & \cdots & \sum_{i=1}^{n}\lambda_i^{2n-2}\end{bmatrix}^{-1} \qquad (21)$$

and

$$\tau=2\lambda_{\max}\left(Q^{-1}P\right)=-\frac{1}{\max\limits_{i=1,2,\ldots,n}\{\lambda_i\}}. \qquad (22)$$

Proof. The proof is omitted due to paper space constraints.

♦

3. CONTROL LAW SYNTHESIS

The following theorem resolves the practical tracking problem of Section 2.

Theorem 1. Consider the error system (5)-(8), associated to the system (1) and to the reference trajectory $\hat{y}$, and assume that the components of the vector K are chosen such that the matrix E is Hurwitz. The finite-valued control law:

$$u(t,\varepsilon):T\times R^n\to U \qquad (23)$$

defined as follows, $\forall t\in T$:

- if $\varepsilon\notin S$, u equal to one of the levels $u_i\in U$ such that:

$$u_i\ge\max_{p\in\wp}\left[\frac{-f\left(t,p(t),D_n(\hat{y}(t))-\varepsilon\right)+K^T\varepsilon+\hat{y}^{(n)}(t)}{F\left(t,p(t),D_n(\hat{y}(t))-\varepsilon\right)}\right],$$
$$\text{if } vF>0,\ v=B^T P\varepsilon \qquad (24)$$

or

$$u_i<\min_{p\in\wp}\left[\frac{-f\left(t,p(t),D_n(\hat{y}(t))-\varepsilon\right)+K^T\varepsilon+\hat{y}^{(n)}(t)}{F\left(t,p(t),D_n(\hat{y}(t))-\varepsilon\right)}\right],$$
$$\text{if } vF<0,\ v=B^T P\varepsilon \qquad (25)$$

- if $\varepsilon\in S$, u equal to any level $u_i\in U$,

provides a signal w in (8) that satisfies the condition (9) of the Lemma 1 and that hence resolves the problem of practical tracking.

Proof. The proof directly follows from Lemma 1 and the expression of w in (8), by the light of the assumptions (2) and (3), and the compactness of $\wp$.

♦

Clearly, the convergence of $\|\varepsilon\|_P$ also implies the convergence of $e=\hat{y}-y$. If a specific bound on e is imposed, the following theorem may be used.

Theorem 2. In order to obtain that the tracking error $e=\hat{y}-y$ fulfils the condition $|e|\le\bar{e}$, the value of ρ in the control law has to be equal to:

$$\rho=\frac{\bar{e}}{\sqrt{p_{11i}}}, \qquad (26)$$

where p_{11i} is the (1,1) element of the matrix P^{-1}.

Proof. Let $\bar{\varepsilon}$ be the tracking error vector when $e=\bar{e}$. Taking into account that $\|\varepsilon\|_P^2=\varepsilon^T P\varepsilon$, with the intention of having $|e|\le\bar{e}$, it is necessary that:

$$2P\bar{\varepsilon} = \sigma \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \sigma \in R, \tag{27}$$

and hence:

$$\bar{\varepsilon} = \frac{1}{2} \begin{pmatrix} p_{11i} \\ ? \\ \vdots \\ ? \end{pmatrix} \sigma = \begin{pmatrix} \bar{e} \\ ? \\ \vdots \\ ? \end{pmatrix}, \tag{28}$$

where the question marks denote elements which, for the purpose of the proof, do not need to be specified.

From (28) follows:

$$\sigma = \frac{2\bar{e}}{p_{11i}} \tag{29}$$

and therefore:

$$\bar{\varepsilon} = P^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \frac{\bar{e}}{p_{11i}} \;. \tag{30}$$

After all:

$$\rho^2 = \bar{\varepsilon}^T P \bar{\varepsilon} = \begin{pmatrix} 1 & 0 & \cdots & 0 \end{pmatrix} P^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \frac{\bar{e}^2}{p_{11i}^2}, \tag{31}$$

which gives the (26).

♦

Remark 1. Note that by virtue of (2), (3), (11), (17), (24) and (25), the needed maximum and minimum levels of u depend on the reference trajectory $\hat{y}(t)$ and its first n derivatives, on $\varepsilon(t_0)$ and on the required velocity of convergence.

Remark 2. The error bound ρ may be chosen as low as one would desire. Nevertheless, if the value of ρ is decreased, the error ε tends to exit from the region S in less time. This implies that the average switching frequency of the control increases till it becomes prohibitive from a realisation point of view.

Remark 3. When $\varepsilon \in S$, in order to reduce the switching frequency among the control levels, it may be chosen as control u the closest level, among the available ones, to the nominal control value calculated on the base of the nominal knowledge of the parameters p. In the case where this information is not available, the last level assumed on the boundary of S may be selected and initially, if $\varepsilon(t_0) \in S$, the null level.

Remark 4. In principle, the conditions (24)-(25) of the Theorem 1 are fulfilled by using, of the available levels, the maximum and the minimum levels only. The intermediate levels are not indispensable for the reference trajectory tracking, but they are useful for alleviating the average switching frequency. Indeed, by adopting the strategy of selecting the closest level which satisfies the (24)-(25), the escaping velocity of ε from S diminishes and therefore also the switching frequency.

Remark 5. On the basis of Lemma 2, if the vector K in (24)-(25) is suitably chosen, it is possible to obtain any prescribed maximum rate of convergence $\bar{\tau}$ of ε.

Remark 6. If the system (1) is linear:

$$y^{(n)} = -p_n y^{(n-1)} - \ldots - p_1 y + p_0 u \tag{32}$$

and $\wp \subseteq R^{n+1}$ is an hyper-rectangle, then the maximum and minimum values in (24) and (25) are attained in correspondence of one of the vertices of $\wp$, see (Celentano, *et al.*, 1993).

Remark 7. In practice, the control law (24)-(25) may be easily implemented by digital computers. To this end it is necessary to have at one's disposal the signals y and its first n-1 derivatives and the sign of F. Nevertheless, because of the inequalities in Theorem 1, with the increase of the sampling period and of the I/O and elaboration delays, the term vw in the Lyapunov function derivative $-\dot{V}$, defined in equation (14), diminishes $\forall \varepsilon \notin S$, in general, and consequently the actual rate of convergence of the tracking error vector increases until the system becomes unstable.

Remark 8. If the matrix P is chosen as in Lemma 2, taking into account the (21), the value of ρ in (26), which allows having $|e| \le \bar{e}$, becomes:

$$\rho = \frac{\bar{e}}{\sqrt{n}} \;. \tag{33}$$

4. EXAMPLE

Consider the uncertain system:

$$\ddot{y} = -p_2 \dot{y} - p_1 y + p_0 u \tag{34}$$

with $y(t_0) = -1$, $\dot{y}(t_0) = 0$, $t_0 = 0$, and the reference trajectory:

$$\hat{y}(t) = \cos(t). \tag{35}$$

By choosing $\lambda_1 = -1, \lambda_2 = -2$, it is consequently:

$$\bar{\tau} = 1\,\text{sec}\,,$$

$$P = \begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{5}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{2} \end{pmatrix},$$

$$v = \frac{3}{4}\varepsilon_1 + \frac{1}{2}\varepsilon_2\,,$$

$$\rho = \frac{\bar{e}}{\sqrt{2}}\,,$$

$$S = \left\{ \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} : \frac{5}{4}\varepsilon_1^2 + \frac{3}{2}\varepsilon_1\varepsilon_2 + \frac{1}{2}\varepsilon_2^2 \le \frac{\bar{e}^2}{2} \right\},$$

and finally, by virtue of Theorem 1:

- if $\varepsilon \notin S$, u equal to one of the levels $u_i \in U$ such that:

$$u_i \ge \max_{p \in \wp} \left[\frac{\ddot{\hat{y}} + p_2\dot{\hat{y}} + p_1\hat{y} + (k_1 - p_1)\varepsilon_1 + (k_2 - p_2)\varepsilon_2}{p_0} \right]$$
$$\text{if } vp_0 > 0 \qquad (36)$$

or

$$u_i < \min_{p \in \wp} \left[\frac{\ddot{\hat{y}} + p_2\dot{\hat{y}} + p_1\hat{y} + (k_1 - p_1)\varepsilon_1 + (k_2 - p_2)\varepsilon_2}{p_0} \right]$$
$$\text{if } vp_0 < 0 \qquad (37)$$

- if $\varepsilon \in S$, u equal to any level $u_i \in U$ (see Remark 3).

Suppose that the parameters p_0, p_1 and p_2 are unitary and known. In the hypothesis that:

$$\bar{e} = 0.2,$$
$$U = \{-1.5, -1, -0.5, 0, +0.5, +1, +1.5\},$$

the reference trajectory and the obtained control u and output y are reported in Fig. 1 . The error phase-plane trajectory is shown in Fig. 2 .

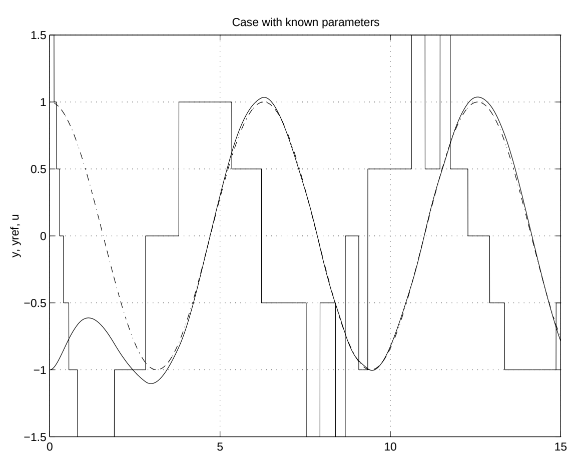

Fig. 1. The output y (solid line), the reference trajectory $\hat{y}$ (dash-dotted line), and the finite-valued control u.

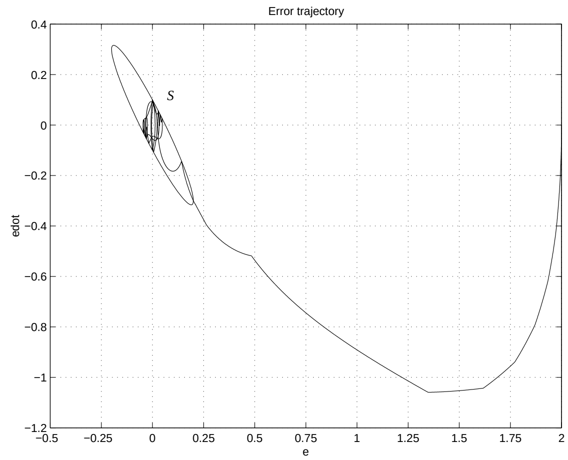

Fig. 2. The error trajectory in the phase plane.

Vice versa, supposing that a realisation of the uncertain parameters is:

$$p_0(t) = 1 + 0.1\cos\left(\frac{1}{2}t\right)$$
$$p_1(t) = 1 + 0.1\sin\left(\frac{1}{3}t\right) \qquad (38)$$
$$p_2(t) = 1 + 0.1\sin\left(\frac{1}{4}t\right)$$

and that the control law designer only knows an estimate of the belonging set $\wp$, in the hypothesis that:

case a

$$\bar{e} = 0.2,$$
$$U = \{-1.5, -1.0, -0.5, 0, +0.5, +1.0, +1.5\},$$
$$p_2 \in [0.9, 1.1],\ p_1 \in [0.9, 1.1],\ p_0 \in [0.9, 1.1],$$

the control u and the output y that are obtained are reported in Fig. 3.

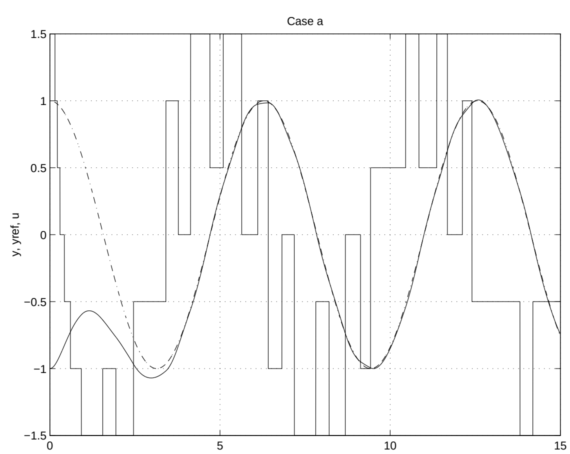

Fig. 3. The output y (solid line), the reference trajectory $\hat{y}$ (dash-dotted line), and the finite-valued control u.

case b (a reduced number of control levels)

$$\bar{e} = 0.2,$$
$$U = \{-1.5, 0, +1.5\},$$
$$p_2 \in [0.9, 1.1],\ p_1 \in [0.9, 1.1],\ p_0 \in [0.9, 1.1],$$

the control u and the output y that are obtained are reported in Fig. 4.

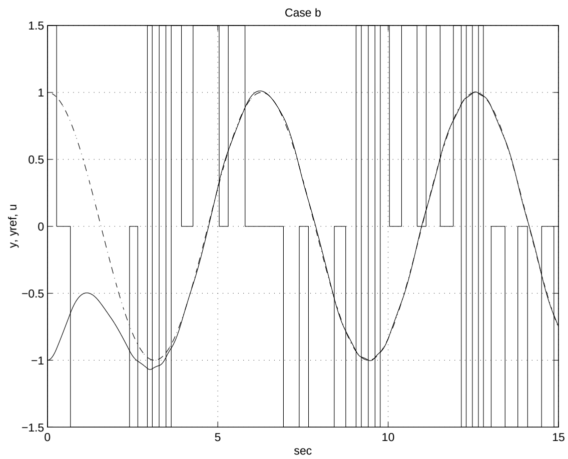

Fig. 4. The output y (solid line), the reference trajectory $\hat{y}$ (dash-dotted line), and the finite-valued control u.

case c (a reduced maximum error, larger control levels and larger amount of uncertainty on the parameters)

$$\bar{e} = 0.1,$$
$$U = \{-2, 0, 2\},$$
$$p_2 \in [0.7, 1.3],\ p_1 \in [0.7, 1.3],\ p_0 \in [0.7, 1.3],$$

the control u and the output y that are obtained are reported in Fig. 5.

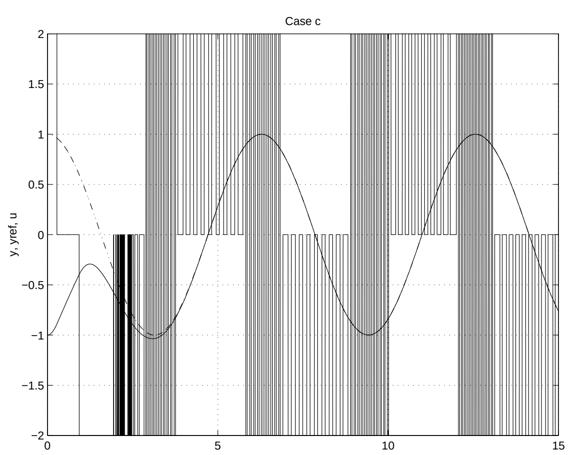

Fig. 5. The output y (solid line), the reference trajectory $\hat{y}$ (dash-dotted line), and the finite-valued control u.

5. CONCLUSIONS

In this paper a Lyapunov methodology has been presented for the design of control laws with prescribed finite values for a class of uncertain nonlinear SISO systems. In particular, the necessary theoretical results have been provided for the solution of a practical tracking problem, guaranteeing the tracking of a sufficiently regular trajectory, with a prescribed maximum tracking error, a prescribed convergence velocity and good performances in terms of robustness and switching frequency. A numerical example has illustrated the applicability of the technique.

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