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OPTIMIZATION OF TARGET VALUE FOR AN INDUSTRIAL PROCESS

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Abstract: Most stochastic models for determining the optimal target value for an industrial process are developed in the extensive literature under the assumptions that the parameter values of the underlying distributions are known with certainty. In actual practice, such is simply not the case. When these models are applied to solve real-world problems, the parameters are estimated and then treated as if they were the true values. The risk associated with using estimates rather than the true parameters is called estimation risk and is often ignored. When data are limited and (or) unreliable, estimation risk may be significant, and failure to incorporate it into the model design may lead to serious errors. Its explicit consideration is important since decision rules that are optimal in the absence of uncertainty need not even be approximately optimal in the presence of such uncertainty. The aim of the present paper is to show how the invariance principle may be employed in the particular case of finding, from the statistical data, the best setting for the target value of an industrial process. The examples are given. *Copyright © 2002 IFAC*

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1. INTRODUCTION

Design of an automation system does not consist only in establishing architecture of components with the lower cost. Indeed, with an economic point of view an automation system must provide an effective control of an industrial process. This paper presents a methodology that takes into account the second aspect.

The problem considered in this paper is as follows. Individual items are produced continuously from an industrial process. Each item is checked to determine whether it satisfies a critical lower (or upper) specification limit. If it does, it is sold at the regular price; if it does not, it is sold at a reduced price. The target value of a process is generally set somewhat above the lower specification limit (or below the upper specification limit). The further the target is set from the specification limit on the safe side, the lower the proportion of rejected items. However, there is an offsetting cost (sometimes called the give-away cost) which restricts the extent to which the target value can be profitably adjusted in this direction.

For an industrial process in which items are produced continuously, suppose there is a lower specification limit h for a quality characteristic such that items with measured values less than h are rejected (for example, to be reprocessed or sold as substandard material). A target value $\mu = h + \Delta$ is to be selected so that the net income for the process is maximized.

The general problem considered here is to develop a procedure that takes process variability and production costs into account for determining the optimal value of Δ (and hence the optimal target value μ). With obvious modifications, the procedure developed here applies if the specification limit is an upper rather than a lower one. For ease of exposition, however, we choose to discuss the problem in terms of a lower specification limit.

For example, consider the situation in which the quality characteristic is the weight of packages of food. An automatic weighing machine routinely weighs all individual packages. Depending on whether a package satisfies the lower specification limit μ or not, the machine either accepts or rejects it. The rejected items are classified as substandard material and are sold at a reduced price. Since the production costs are subject to variation from day to day, it is desired to devise a relatively simple procedure that could be used routinely by production personnel for determining the best target value. Although the problem discussed in this paper concerns the weight of the product, the same approach applies to other quality characteristics.

For a particular example, Bettes (1962) solved the problem of choosing the optimal values for the target and upper specification limit. A brief description of the problem of optimal overfill is given by Grant (1972). Chiu and Wetherill (1973) reviewed the literature on the economic design of continuous inspection procedures.

In this paper, we consider the situation in which the variation of the measured value of the quality characteristic is approximated by a parametric distribution. Attention is restricted to invariant families of distributions.

In particular, the case is considered where a previously available complete or type II censored sample is from a continuous distribution with cdf $F((x-\mu)/\sigma)$ and pdf $f(x;\mu,\sigma)=\sigma^{-1} f_\bullet[(x-\mu)/\sigma]$, where $F(\cdot)$ is known but both the location (μ) and scale (σ) parameters are unknown. For such family of distributions the decision problem remains invariant under a group of transformations (a subgroup of the full affine group) which takes μ (the location parameter) and σ (the scale) into $a\mu + b$ and $a\sigma$, respectively, where b lies in the range of μ, $a > 0$. This group acts transitively on the parameter space and, consequently, the risk of any equivariant decision rule is a constant. Among the class of such decision rules there is therefore a “best” one. The effect of imposing the principle of invariance, in this case, is to reduce the class of all possible decision rules to one.

The outline of the paper is as follows. An invariant embedding technique is presented in Section 2. Formulation of the problem is given in Section 3. Section 4 is devoted to solution of the problem.

2. PRELIMINARIES

This paper is concerned with the implications of group theoretic structure for invariant performance indexes. We present an invariant embedding technique based on the constructive use of the invariance principle in mathematical statistics. This technique allows one to solve many problems of the theory of statistical inferences in a simple way.

The aim of the present paper is to show how the invariance principle may be employed in the particular case of finding the optimal target value for an industrial process. The technique used here is a special case of more general considerations applicable whenever the statistical problem is invariant under a group of transformations, which acts transitively on the parameter space.

Our underlying structure consists of a class of probability models $(\mathcal{X}, \mathcal{A}, \mathcal{P})$, a one-one mapping ψ taking $\mathcal{P}$ onto an index set Θ, a measurable space of actions $(\mathcal{U}, \mathcal{B})$, and a real-valued function r defined on $\Theta \times \mathcal{U}$. We assume that a group G of one-one $\mathcal{A}$ - measurable transformations acts on $\mathcal{X}$ and that it leaves the class of models $(\mathcal{X}, \mathcal{A}, \mathcal{P})$ invariant. We further assume that homomorphic images $\overline{G}$ and $\tilde{G}$ of G act on Θ and $\mathcal{U}$, respectively. ($\overline{G}$ may be induced on Θ through ψ; $\tilde{G}$ may be induced on $\mathcal{U}$ through r). We shall say that r is invariant if for every $(\theta,u) \in \Theta \times \mathcal{U}$

$$r(\overline{g}\theta, \tilde{g}u) = r(\theta,u), \quad g \in G. \tag{1}$$

Given the structure described above there are aesthetic and sometimes admissibility grounds for restricting attention to decision rules φ: $\mathcal{X} \to \mathcal{U}$ which are $(G, \tilde{G})$ equivariant in the sense that

$$\varphi(gx) = \tilde{g}\varphi(x), \quad x \in \mathcal{X}, \quad g \in G. \tag{2}$$

If $\overline{G}$ is trivial and (1), (2) hold, we say φ is G-invariant, or simply invariant (Nechval, 2000; Nechval et al., 2001).

2.1 Invariant Functions

We begin by noting that r is invariant in the sense of (1) if and only if r is a $G^\bullet$-invariant function, where $G^\bullet$ is defined on $\Theta \times U$ as follows: to each $g \in G$, with homomorphic images $\overline{g}, \tilde{g}$ in $\overline{G}, \tilde{G}$ respectively, let $g^\bullet(\theta,u) = (\overline{g}\theta, \tilde{g}u)$, $(\theta,u) \in (\Theta \times \mathcal{U})$. It is assumed that $\tilde{G}$ is a homomorphic image of $\overline{G}$.

Definition 1 (Transitivity). A transformation group $\overline{G}$ acting on a set Θ is called (uniquely) transitive if for every θ, $\vartheta\in\Theta$ there exists a (unique) $\bar{g}\in\bar{G}$ such that $\bar{g}\,\theta=\vartheta$.

When $\bar{G}$ is transitive on Θ we may index $\bar{G}$ by Θ: fix an arbitrary point $\theta\in\Theta$ and define $\bar{g}_{\theta_1}$ to be the unique $\bar{g}\in\bar{G}$ satisfying $\bar{g}\,\theta=\theta_1$. The identity of $\bar{G}$ clearly corresponds to θ. An immediate consequence is Lemma 1.

Lemma 1 (Transformation). Let $\bar{G}$ be transitive on Θ. Fix $\theta\in\Theta$ and define $\bar{g}_{\theta_1}$ as above. Then $\bar{g}_{\bar{q}\theta_1}=\overline{qg}_{\theta_1}$ for $\theta\in\Theta$, $\bar{q}\in\bar{G}$.

Proof. The identity $\bar{g}_{\bar{q}\theta_1}\theta=\bar{q}\theta_1=\overline{qg}_{\theta_1}\theta$ shows that $\bar{g}_{\bar{q}\theta_1}$ and $\overline{qg}_{\theta_1}$ both take θ into $\bar{q}\theta_1$, and the lemma follows by unique transitivity. □

Theorem 1 (Maximal Invariant). Let $\bar{G}$ be transitive on Θ. Fix a reference point $\theta_0\in\Theta$ and index $\bar{G}$ by Θ. A maximal invariant M with respect to $G^\bullet$ acting on $\Theta\times\mathscr{U}$ is defined by

$$M(\theta,u)=\tilde{g}_{\theta}^{-1}u,\quad (\theta,u)\in\Theta\times\mathscr{U}. \tag{3}$$

Proof. For each $(\theta,u)\in(\Theta\times\mathscr{U})$ and $\bar{g}\in\bar{G}$

$$\begin{aligned}M(\bar{g}\theta,\tilde{g}u)&=(\tilde{g}_{\bar{g}\theta}^{-1})\tilde{g}u=(\tilde{g}\tilde{g}_{\theta})^{-1}\tilde{g}u\\&=\tilde{g}_{\theta}^{-1}\tilde{g}^{-1}\tilde{g}u=\tilde{g}_{\theta}^{-1}u=M(\theta,u)\end{aligned} \tag{4}$$

by Lemma 1 and the structure preserving properties of homomorphisms. Thus M is $G^\bullet$-invariant. To see that M is maximal, let $M(\theta_1,u_1)=M(\theta_2,u_2)$. Then $\tilde{g}_{\theta_1}^{-1}u_1=\tilde{g}_{\theta_2}^{-1}u_2$ or $u_1=\tilde{g}\,u_2$, where $\tilde{g}=\tilde{g}_{\theta_1}\tilde{g}_{\theta_2}^{-1}$. Since $\theta_1=\bar{g}_{\theta_1}\theta_0=\bar{g}_{\theta_1}\bar{g}_{\theta_2}^{-1}\theta_2=\bar{g}\theta_2$, $(\theta_1,u_1)=g^\bullet(\theta_2,u_2)$ for some $g^\bullet\in G^\bullet$, and the proof is complete. □

Corollary 1 (Invariant Embedding). An invariant function, $r(\theta,u)$, can be transformed as follows:

$$r(\theta,u)=r(\bar{g}_{\hat{\theta}}^{-1}\theta,\tilde{g}_{\hat{\theta}}^{-1}u)=\ddot{r}(v,\eta), \tag{5}$$

where $v=v(\theta,\hat{\theta})$ is a function (it is called a pivotal quantity) such that the distribution of v does not depend on θ; $\eta=\eta(u,\hat{\theta})$ is an ancillary factor; $\hat{\theta}$ is the maximum likelihood estimator of θ (or the sufficient statistic for θ).

Corollary 2 (Best Invariant Decision Rule). If $r(\theta,u)$ is an invariant loss function, the best invariant decision rule is given by

$$\varphi^*(x)=u^*=\eta^{-1}(\eta^*,\hat{\theta}), \tag{6}$$

where

$$\eta^*=\arg\inf_{\eta}E_{\eta}\{\ddot{r}(v,\eta)\}. \tag{7}$$

Corollary 3 (Risk). A risk function (performance index)

$$R(\theta,\varphi(x))=E_{\theta}\{r(\theta,\varphi(x))\}=E_{\eta_\circ}\{\ddot{r}(v_\circ,\eta_\circ)\} \tag{8}$$

is constant on orbits when an invariant decision rule $\varphi(x)$ is used, where $v_\circ=v_\circ(\theta,x)$ is a function whose distribution does not depend on θ; $\eta_\circ=\eta_\circ(u,x)$ is an ancillary factor.

For instance, consider the problem of estimating the location-scale parameter of a distribution belonging to a family generated by a continuous cdf F: $\mathscr{P}=\{P_\theta$: $F((x-\mu)/\sigma)$, $x\in R$, $\theta\in\Theta\}$, $\Theta=\{(\mu,\sigma)$: $\mu,\sigma\in R$, $\sigma>0\}=\mathscr{U}$. The group G of location and scale changes leaves the class of models invariant. Since $\bar{G}$ induced on Θ by $P_\theta\to\theta$ is uniquely transitive, we may apply Theorem 1 and obtain invariant loss functions of the form

$$r(\theta,\varphi(x))=r[(\varphi_1(x)-\mu)/\sigma,\varphi_2(x)/\sigma], \tag{9}$$

where

$$\theta=(\mu,\sigma)\text{ and }\varphi(x)=(\varphi_1(x),\varphi_2(x)). \tag{10}$$

Let $\hat{\theta}=(\hat{\theta}_1,\hat{\theta}_2)$ and $u=(u_1,u_2)$, then

$$r(\theta,u)=\ddot{r}(v,\eta)=\ddot{r}(v_1+\eta_1 v_2,\eta_2 v_2), \tag{11}$$

where $v=(v_1,v_2)$, $v_1=(\hat{\theta}_1-\mu)/\sigma$, $v_2=\hat{\theta}_2/\sigma$; $\eta=(\eta_1,\eta_2)$, $\eta_1=(u_1-\hat{\theta}_1)/\hat{\theta}_2$, $\eta_2=u_2/\hat{\theta}_2$.

An invariant embedding technique, which is used for determining the optimal target value for an industrial process, is based on the result of Corollary 1.

3. PROBLEM STATEMENT

Let X be the measured value of the quality characteristic, h be the lower specification limit, and $\mu = h + \Delta$ be the target value. In industrial applications, the optimal value of Δ is ordinarily greater than zero but negative values are possible. We will return to this point later.

For the sake of simplicity, we shall consider the situation in which the quality characteristic data can be measured on a continuous scale. We restrict attention to the case where these quality characteristic values constitute independent observations from a distribution belonging to invariant family. In particular, we consider a distribution belonging to location-scale family generated by a continuous cdf F: $\mathscr{P}=\{P_\theta: F((x-\mu)/\sigma), x\in R, \theta\in\Theta\}$, $\Theta=\{(\mu,\sigma): \mu,\sigma\in R, \sigma>0\}$, which is indexed by the vector parameter $\theta=(\mu,\sigma)$, where μ and σ (>0) are respectively parameters of location and scale. The group G of location and scale changes leaves the class of models invariant. The purpose in restricting attention to such families of distributions is that for such families the decision problem is invariant, and if the estimators (decision rules) are equivariant (i.e. the group of location and scale changes leaves the decision problem invariant), then any comparison of estimation procedures is independent of the true values of any unknown parameters. The common distributions used in industrial problems are the normal, exponential, Weibull, and gamma distributions.

Let us assume that the net income from a single item is given by

$$r(\sigma,\Delta)=\begin{cases} c_1-c(X-h)/\sigma & \text{if} \quad X\geq h \\ c_2 & \text{if} \quad X<h \end{cases}$$

$$=\begin{cases} c_1-c\left(Z+\dfrac{\Delta}{\sigma}\right) & \text{if} \quad Z\geq -\dfrac{\Delta}{\sigma} \\ c_2 & \text{if} \quad Z<-\dfrac{\Delta}{\sigma}, \end{cases} \tag{12}$$

where

$$Z=\frac{X-\mu}{\sigma} \tag{13}$$

is a pivotal quantity, $c_1>c_2$. The expected net income per item is

$$E_\sigma\{r(\sigma,\Delta)\}=c_1\int_h^\infty f(x;\mu,\sigma)dx$$

$$-\frac{c}{\sigma}\int_h^\infty (x-h)f(x;\mu,\sigma)dx+c_2\int_{-\infty}^h f(x;\mu,\sigma)dx$$

$$=c_1\int_{-\Delta/\sigma}^\infty f^\bullet(z)dz-c\int_{-\Delta/\sigma}^\infty (z+\frac{\Delta}{\sigma})f^\bullet(z)dz$$

$$+c_2\int_{-\infty}^{-\Delta/\sigma} f^\bullet(z)dz, \tag{14}$$

where f(.) is a probability density function belonging to location-scale family, $f^\bullet(.)$ is the probability density function of Z. The first term on the right-hand side of equation (14) is the income from the accepted items, the second term is the give-away cost and the third term is the income from the rejected items.

The problem is to maximize equation (14) with respect to Δ.

4. PROBLEM SOLUTION

4.1 Complete Information

Let us assume that the parameter σ is known. Differentiating equation (14) with respect to Δ, we have

$$E'_\sigma\{r(\sigma,\Delta)\}=$$

$$\frac{1}{\sigma}\left[(c_1-c_2)f^\bullet\left(-\frac{\Delta}{\sigma}\right)-c\left(1-F^\bullet\left(-\frac{\Delta}{\sigma}\right)\right)\right]. \tag{15}$$

Setting equation (15) equal to zero, we have

$$\frac{f^\bullet\left(-\dfrac{\Delta}{\sigma}\right)}{1-F^\bullet\left(-\dfrac{\Delta}{\sigma}\right)}=\frac{c}{c_1-c_2}. \tag{16}$$

Denote the solution of equation (16) by Δ_0. If the second derivative

$$E''_\sigma\{r(\sigma,\Delta)\}=$$

$$\frac{1}{\sigma}\left[(c_1-c_2)f^{\bullet\prime}\left(-\frac{\Delta}{\sigma}\right)-c\frac{1}{\sigma}f^\bullet\left(-\frac{\Delta}{\sigma}\right)\right] \tag{17}$$

with $\Delta = \Delta_0$ is less than zero, that is, if

$$\frac{f^{\bullet\prime}\left(-\dfrac{\Delta}{\sigma}\right)}{\dfrac{1}{\sigma}f^\bullet\left(-\dfrac{\Delta}{\sigma}\right)}<\frac{c}{c_1-c_2}, \tag{18}$$

then $\Delta_0 = \Delta^*$ (the optimal value). In other words, if inequality (18) is true, the solution of equation (16) gives the best value of Δ.

The optimal value of Δ is such that equation (16) and inequality (18) are satisfied simultaneously.

For example, if the random variable X follows a normal distribution with the probability density function

$$f(x;\mu,\sigma)=\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right),$$

$$x\in(-\infty,\infty);\ \mu\in(-\infty,\infty),\ \sigma>0, \tag{19}$$

inequality (18) is reduced to

$$-\frac{\Delta}{\sigma} < \frac{c}{c_1 - c_2}. \tag{20}$$

If $\Delta_0 \geq 0$, the inequality (20) holds because $\sigma > 0$ (hence the left-hand side is negative) and the right-hand side is always greater than zero.

Let us suppose that X is a Weibull variate with the probability density function $f(x;\mu,\sigma,\alpha)$ and distribution function $F(x; \mu,\sigma,\alpha)$, where:

$$f(x;\mu,\sigma,\alpha) = \frac{\alpha}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{\alpha-1} \exp\left[-\left(\frac{x-\mu}{\sigma}\right)^{\alpha}\right],$$

$$\sigma,\ \alpha > 0, \quad x \geq \mu. \tag{21}$$

$$F(x;\mu,\sigma,\alpha) = 1 - \exp\left[-\left(\frac{x-\mu}{\sigma}\right)^{\alpha}\right], \tag{22}$$

The parameter α is a shape parameter and it gives a variety of shapes to $f(x;\mu,\sigma,\alpha)$. It is assumed that this parameter is known. Thus we have a simple and a flexible family of density functions. It can be shown that for this distribution inequality (18) is true if $\Delta_0 < 0$ and $\alpha > 1$.

Numerical Example 1. We shall now illustrate the method using the example introduced in Section 1. The items here are packages of food. The weight marked on each package is one pound, and this is the lower specification limit. The selling price of an accepted package is 1.5$ and the cost of excess material is 1$ per pound. A rejected package is sold for 0.5$. It is assumed that X (the measured value of the weight) is a Weibull variate with the probability density function given by (21). The standard deviation of the process is approximately 0.01 and the shape parameter α is equal to 3. We have, then, for this process,

$$\begin{aligned} c_1 &= 1.5\$ \\ c &= 1\$ \\ c_2 &= 0.5\$ \\ \sigma &= 0.01 \\ h &= 1. \end{aligned} \tag{23}$$

Then equation (16) is reduced to

$$\alpha\left(-\frac{\Delta}{\sigma}\right)^{\alpha-1} = \frac{c}{c_1 - c_2}. \tag{24}$$

Hence,

$$\Delta = -\sigma\left(\frac{1}{\alpha}\frac{c}{c_1 - c_2}\right)^{1/(\alpha-1)}, \tag{25}$$

and therefore

$$\Delta^* = -0.00577, \tag{26}$$

so that, according to this analysis, the optimal target value is

$$\mu = h + \Delta^* = 0.99423 \text{ pounds}. \tag{27}$$

4.2 Incomplete Information

More often than not the parameter $\sigma > 0$ is unknown. We shall assume that there is obtainable from some informative experiment (a random sample of observations $X^n=(X_1, \ldots ,X_n)$ under a preassigned target value μ) a sufficient statistic S_2 for σ with density function $q(s_2;\sigma)$ of the form

$$q(s_2;\sigma) = \sigma^{-1} f(s_2/\sigma). \tag{28}$$

It is required to find the best invariant estimator of Δ on the basis of the data sample $X^n=(X_1, \ldots ,X_n)$ relative to the expected net income function $E_\sigma\{r(\sigma,\Delta)\}$ (see (14)).

We are thus assuming that for the family of density functions an induced invariance holds under the group G of transformations: $S_2 \to aS_2$ $(a > 0)$. The family of density functions satisfying the above conditions is, of course, the limited one of normal, negative exponential, Weibull and gamma (with known index) density functions. The structure of the problem is, however, more clearly seen within the general framework.

Since the expected net income function $E\{r(\sigma,\Delta)\}$ is invariant under the group G of scale changes, the technique of invariant embedding (see Section 2) allows one to transform $E\{r(\sigma,\Delta)\}$ as follows:

$$E\{r(\sigma,\Delta)\} = \ddot{r}(V_2,\eta_2)$$

$$= c_1\int_{-\eta_2 V_2}^{\infty} f^{\bullet}(z)dz - c\int_{-\eta_2 V_2}^{\infty} (z+\eta_2 V_2) f^{\bullet}(z)dz$$

$$+ c_2\int_{-\infty}^{-\eta_2 V_2} f^{\bullet}(z)dz, \tag{29}$$

where

$$V_2 = \frac{S_2}{\sigma}, \tag{30}$$

$$\eta_2 = \frac{\Delta}{S_2}. \tag{31}$$

We choose η_2 such that the function

$$R(\eta_2) = E\{\ddot{r}(V_2,\eta_2)\} \tag{32}$$

is maximized.

The distribution of V_2, which does not depend on an unknown parameter σ, can be obtained from (28). Thus, the unknown parameter σ is eliminated from the problem.

The best invariant estimator (BIE) of Δ is given by

$$\Delta_{BIE} = \eta_2^* S_2, \tag{33}$$

where

$$\eta_2^* = \arg\max_{\eta_2} R(\eta_2). \tag{34}$$

Differentiating equation (32) with respect to η_2, we have

$$R'(\eta_2) =$$

$$E\left\{(c_1 - c_2)V_2 f^{\bullet}(-\eta_2 V_2) - cV_2[1 - F^{\bullet}(-\eta_2 V_2)]\right\}. \tag{35}$$

Setting equation (35) equal to zero, we have

$$\frac{E\left\{V_2 f^{\bullet}(-\eta_2 V_2)\right\}}{E\left\{V_2[1 - F^{\bullet}(-\eta_2 V_2)]\right\}} = \frac{c}{c_1 - c_2}. \tag{36}$$

Denote the solution of equation (36) by η_2°. If the second derivative

$$R''\{\eta_2\} =$$

$$E\left\{(c_1 - c_2)V_2 f^{\bullet\prime}(-\eta_2 V_2) - cV_2^2 f^{\bullet}(-\eta_2 V_2)\right\} \tag{37}$$

with $\eta_2 = \eta_2^{\circ}$ is less than zero, that is, if

$$\frac{E\left\{V_2 f^{\bullet\prime}(-\eta_2 V_2)\right\}}{E\left\{V_2^2 f^{\bullet}(-\eta_2 V_2)\right\}} < \frac{c}{c_1 - c_2}, \tag{38}$$

then $\eta_2^{\circ} = \eta_2^*$ (the optimal value). In other words, if inequality (38) is true, the solution of equation (36) gives the best value of η_2.

The optimal value of η_2 is such that equation (36) and inequality (38) are satisfied simultaneously.

Numerical Example 1 (Continued). It is assumed that the parameter σ>0 of the Weibull distribution is unknown. Let $X^n=(X_1, \dots ,X_n)$ be a random sample of observations of the measured value of the weight of packages of food under a preassigned target value μ and

$$S_2^{\alpha} = \sum_{i=1}^{n} (X_i - \mu)^{\alpha}. \tag{39}$$

It can be justified by using the factorization theorem that S_2^{α} is a sufficient statistic for σ. The sampling distribution of this statistic is given by

$$q(s_2^{\alpha}; \sigma) = \frac{1}{\Gamma(n)\sigma^{\alpha n}} [s_2^{\alpha}]^{n-1} \exp(-s_2^{\alpha}/\sigma^{\alpha}),$$
$$s_2^{\alpha} \geq 0. \tag{40}$$

Using the general results of this paper, we can define an optimal value (η_2^*) of η_2 which satisfies the equation

$$\frac{\Gamma(n+1)}{\Gamma(n+1/\alpha)} \frac{\alpha(-\eta_2)^{\alpha-1}}{(1-\eta_2)^{(1-1/\alpha)}} = \frac{c}{c_1 - c_2}. \tag{41}$$

Then the best invariant estimator (BIE) of Δ is given by (33), so that, according to this analysis, the optimal target value is

$$\mu = h + \Delta_{BIE} = h + \eta_2^* S_2. \tag{42}$$

5. CONCLUSIONS

The results obtained in this paper agree with the simulation results, which confirm the validity of the theoretical predictions of performance of the suggested approach.

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