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CONSERVATIVITY OF ELLIPSOIDAL STABILITY REGIONS ESTIMATES FOR INPUT SATURATED LINEAR SYSTEMS

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Abstract: This paper presents a comparative study of different existing algorithms for determining stability domains for linear systems with saturating inputs. Algorithms based on the solution of LMI problems carried out on the basis of three different saturation models, namely regions of saturation, differential inclusion, and sector modeling, are analyzed and compared in terms of their ability to provide large stability domains for the closed-loop system. The main reasons such algorithms incorporate conservativity are highlighted, this being the main contribution of this paper. Results are illustrated by means of an example.

Keywords: saturation control,linear systems, stability domains, optimization.

1. INTRODUCTION

Determining asymptotic stability regions for linear systems subject to control saturation has been studied by many authors in the last decade. The motivation for these studies comes from the fact that, in the presence of control saturation, the global stability cannot in general be ensured. Furthermore, when it is possible to compute a global stabilizing control law (see (Sussmann *et al.*, 1994; Burgat and Tarbouriech, 1996)), in general it is difficult to simultaneously guarantee good performance and robustness for the closed-loop system. On the other hand, on the ground of local stabilization, the exact determination of the basin of attraction is possible only in very particular cases. Hence it is important to determine asymptotic stability regions, in order to approximate the basin of attraction (Khalil, 1992).

The proposed methods for generating stability regions for linear systems with saturating inputs are mainly based on the concept of Lyapunov domains, i.e., domains obtained from piecewise-linear (Gomes da Silva Jr. and Tarbouriech, 1999*a*), quadratic (see, for example, (Henrion and Tarbouriech, 1999; Gomes da Silva Jr. and Tarbouriech, 1999*b*; Fong and Hsu, 2000; Hu and Lin, 2000) and references therein) and Lur'e type (Pittet *et al.*, 1997; Hindi and Boyd, 1998) Lyapunov functions. In order to take into account the nonlinear behavior of the closed-loop system and to obtain testable conditions, the saturation term should be conveniently represented. The conservativity of each approach is directly related to both the modeling method and the structure of the Lyapunov function used.

The aim of this paper is to present a comparative analysis of different techniques proposed in the literature for determining ellipsoidal domains of stability (quadratic Lyapunov function). The interest for such domains is mainly motivated by the recent developments concerning numerical algorithms and software packages for solving LMIs and convex optimization problems. If test conditions can be cast as LMI-based optimization problems where the optimization criteria can be related, directly or indirectly, to the size of the domain of stability to be computed (Gomes da Silva Jr. and Tarbouriech, 1999*b*), the problem can be easily solved. Although several methods have been proposed in this context, we can notice a lack of critical comparison between the different approaches. Furthermore, in general, the conservativity of the results is not conveniently analyzed or elucidated. This paper addresses these issues as it provides a critical analysis of some methods for computing ellipsoidal regions of asymptotic stability for systems with saturating inputs.

The paper is organized as follows. After the statement of the problem, the paper is divided in three sections that are related to different type of saturation modeling, namely: regions of saturation, differential inclusion and sector modeling. In each section the sufficient conditions to be satisfied and the corresponding algorithms for determining the ellipsoidal regions of stability are presented and discussed. Then, an example is worked out in order to provide a numerical comparison between the results obtained with the different approaches.

Notations: For two vectors x, y of $\Re^n$, the notation $x \succeq y$ means that $x_{(i)} - y_{(i)} \geq 0$, $\forall i = 1, \dots, n$. $A_{(i)}$ denotes the ith row of A. For two symmetric matrices, A and B, $A > B$ means that $A - B$ is pos. definite. $diag(x)$ denotes a diagonal matrix obtained from vector x. $1_m \triangleq [1 \dots 1]' \in \Re^m$, $0_m \triangleq [0 \dots 0]' \in \Re^m$. I_m is the m-order identity matrix. *int* denotes the interior of a set.

2. PROBLEM STATEMENT

Consider the continuous-time linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{1}$$

where $x(t) \in \Re^n$, $u(t) \in \Re^m$, $A \in \Re^{n*n}$ and $B \in \Re^{n*m}$. Assume system (1) is in closed-loop with the saturated linear control law

$$u(t) = sat(Kx(t)) \tag{2}$$

where $sat(\cdot)$ denotes the classical symmetrical decentralized saturation function defined as follows:

$$sat(v)_{(i)} = sign(v_{(i)})\ min\{\rho_{(i)}, |v_{(i)}|\}$$

where $i = 1, ..., m$ and $\rho_{(i)}$ represents the control limit on the ith input. Due to the saturation term, the closed-loop system is nonlinear:

$$\dot{x}(t) = Ax(t) + Bsat(Kx(t)) \tag{3}$$

The polyhedral set

$$S(K, \rho) \triangleq \{x \in \Re^n \ ; \ -\rho \preceq Kx \preceq \rho\}$$

is the region of linearity of system (3). Inside this region, the control entries do not saturate and the behavior of the system is locally described by the linear model $\dot{x}(t) = (A + BK)x(t)$. We assume that the matrix K is such that all the eigenvalues of $(A + BK)$ are placed in the open left half complex plane, so that in the absence of control bounds, the closed-loop system would be *globally asymptotically stable*.

Let $P = P' > 0$ and $c > 0$ and consider the ellipsoidal set $\mathcal{E}(c) = \{x \in \Re^n \ ; \ x'Px \leq c\}$.

Definition 1. The set $\mathcal{E}(c)$ is a *region of asymptotic stability* of system (3) if: (i) the point $x = 0$ is a locally asymptotically stable equilibrium point; (ii) it is contained in the the region of attraction of the equilibrium $x = 0$.

Definition 2. The set $\mathcal{E}(c)$ is *contractive* with respect to system (3) if the function $V(x) = x'Px$ is strictly decreasing along the trajectories of (3) in $\mathcal{E}(c) - \{0\}$. In particular, if $\mathcal{E}(c)$ is contractive, then it is a region of asymptotic stability.

In particular, the problem of determining ellipsoidal regions of stability contained in region $S(K, \rho)$ is a trivial problem. The interest in this paper is in the study of conditions that allow the determination of stability regions not contained in the region of linearity and, in consequence, that take into account the nonlinear characteristic of the closed-loop system, as presented in the next sections.

3. MODELING BY REGIONS OF SATURATION

This representation consists in dividing the state space in regions called *regions of saturation*. Inside each region of saturation, the system (3) can be modeled as an affine system or, equivalently, as a system with an additive constant disturbance (Gomes da Silva Jr. and Tarbouriech, 1999*a*) (i.e. the saturated system (3) can be viewed as an hybrid system with piecewise affine dynamics (Johansson and Rantzer, 1998)).

Let $\xi \in \Re^m$ be such that each entry $\xi_{(i)}$, $i = 1, \dots, m$, takes the values 1, 0 or -1 in accordance with the saturation function (2) as follows:

$$\xi_{(i)}(t) = \begin{cases} -1 & \text{if } K_{(i)}x(t) < -\rho_{(i)} \\ 0 & \text{if } -\rho_{(i)} \leq K_{(i)}x(t) \leq \rho_{(i)} \\ 1 & \text{if } K_{(i)}x(t) > \rho_{(i)} \end{cases} \tag{4}$$

There are 3^m different vectors ξ: $\xi_j \in \Re^m$ for $j = 0, \dots, 3^m - 1$. For each vector ξ_j, the state vector belongs to a specific region called *region of saturation*. Generically, the region of saturation associated to ξ_j is denoted by

$$S(R_j, d_j) = \{x \in \Re^n \ ; \ R_j x \preceq d_j\} \tag{5}$$

where $d_j \in \Re^{l_j}$ is defined from the entries of ρ and $-\rho$, and $R_j \in \Re^{l_j * n}$ is defined from the rows of K and $-K$. We define $\xi_0 = 0_m$ so that the region associated with $j = 0$ corresponds to $S(K, \rho)$. In all other regions there is at least one control input that is saturated. If $x(t) \in S(R_j, d_j)$, defining $\bar{A}_j = A + Bdiag(1_m - |\xi_j|)K$ and $v_j = Bdiag(\xi_j)\rho$, the motion of the system (3) can be described by

$$\dot{x}(t) = \bar{A}_j x(t) + v_j \tag{6}$$

Theorem 1. The function $V(x) = x'Px$ is a strictly decreasing Lyapunov function for the saturated system in $\mathcal{E}(c)$ if and only if the following conditions hold $\forall j, \ j = 1, \dots, 3^m - 1$:

$$\begin{array}{ll} (i) & x'[P(A+BK)+(A+BK)'P]x<0, \\ & \forall x \in S(K,\rho)\cap \mathcal{E}(c),\ x\neq 0 \\ (ii) & x'P(\bar{A}_j x+v_j)+(\bar{A}_j x+v_j)'Px<0, \\ & \forall x\in S(R_j,d_j)\cap \mathcal{E}(c),\ x\neq 0 \\ & \forall j \text{ s.t. } S(R_j,d_j)\cap int\mathcal{E}(c)\neq \emptyset \end{array} \quad (7)$$

The proof of Theorem 1 follows directly from (6). Although it provides a necessary and sufficient condition for a set $\mathcal{E}(c)$ to be contractive, it still lacks of practical benefit because the conditions (7)-(i)-(ii) are not easily solvable with the available numerical methods. We present next two sufficient conditions for (7)-(i)-(ii) that are numerically more tractable.

3.1 *Test Condition 1*

The condition below corresponds to a generalization, to multi-input systems, of the results proposed in (Fong and Hsu, 2000).

Proposition 1. If there exists nonnegative scalars γ_j and $\tau_{j(i)}, i=1,\ldots,l_j$ satisfying the following conditions $\forall j,\ j=1,\ldots,3^m-1$,

$$\begin{array}{ll} (i) & P(A+BK)+(A+BK)'P<0 \\ (ii) & \begin{bmatrix} P\bar{A}_j+\bar{A}_j'P-\gamma_j P & Pv_j-0.5R_j'T_j' \\ v_j'P-0.5T_jR_j & \gamma_j c+T_jd_j \end{bmatrix}<0 \\ & \forall j \text{ s.t. } S(R_j,d_j)\cap int\mathcal{E}(c)\neq\emptyset \end{array} \quad (8)$$

with $T_j=[\tau_{j(1)}\ \ \ldots\tau_{j(l_j)}]$, then the set $\mathcal{E}(c)$ is a region of stability for the saturated system (3).

Proof: Relation (8)-(i) implies that relation (7)-(i) is satisfied. For all regions such that $S(R_j,d_j)\cap int\mathcal{E}(c)\neq\emptyset$ there exists x satisfying $\begin{cases} x'Px-c\leq 0 \\ R_jx-d_j\leq 0 \end{cases}$. Hence, it follows that a sufficient condition for the satisfaction of (7)-(ii) is that for some nonnegative scalars γ_j and $\tau_{j(i)}, i=1,\ldots,l_j$ one verifies, for all $x\neq 0$,

$$\begin{bmatrix} x \\ 1 \end{bmatrix}' \begin{bmatrix} P\bar{A}_j+\bar{A}_j'P-\gamma_j P & Pv_j-0.5R_j'T_j' \\ v_j'P-0.5T_jR_j & \gamma_j c+T_jd_j \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix}<0$$

This conditions and, in consequence, (7)-(ii) are satisfied if (8)-(i) is satisfied, thus completing the proof. □

The result of Proposition 1 allows to verify whether a given ellipsoidal set $\mathcal{E}(c)$ is contractive or not, in which case, the condition (8)-(ii) is just an LMI feasibility test. Alternatively, given a contractive set $\mathcal{E}(c)$ one can try an homothetic expansion by interactively increasing c and testing condition (8). The condition (8) can also be used to find a contractive set $\mathcal{E}(c)$ for system (3). In this case, however, (8)-(ii) becomes a BMI since P and $\gamma_j,\ j=1,\cdots,3^{m-1}$, will both be decision variables. Note that we can set $c=1$ without loss of generality in this case. Solving BMIs usually requires employing some relaxation method (Henrion and Tarbouriech, 1999). A possible relaxation algorithm is as follows.

Algorithm 1. :

(1) Choose $\gamma_j=\gamma, \forall j=1,\ldots,3^m-1$.
(2) Set $c=1$. Fix $\gamma_j,\ j=1,\ldots,3^m-1$, obtained in the previous step and search for P and T_j by optimizing a criterion on the size of $\mathcal{E}(c)$ subject to the LMI conditions (8)-(i)-(ii)
(3) Fix P obtained in step 2. Maximize c subject to conditions (8)-(i)-(ii) with γ_j and $T_j,\ j=1,\ldots,3^m-1$, as decision variables [1].
(4) Go to step 2 to improve the criterion on $\mathcal{E}(c)$ until a desired precision is achieved.

Note that, (P,γ_j,T_j) obtained in step 2 consists in a feasible solution for step 3 with $c=1$. Conversely (P,c,γ_j,T_j) obtained in 3 is a feasible solution for step 2 by setting $P=P/c$. Hence the convergence of the algorithm is always ensured.

Remark 1. The condition (7)-(ii) has been turned into condition (8)-(ii), which can be verified as an LMI test or, in the worse case, as a BMI. In this transformation, however, some conservativity has been introduced due to the following facts:

(i) The use of the S-procedure. Indeed, the S-procedure is only a sufficient condition in this case because there is more than a single constraint involved.

(ii) The LMI test (8)-(ii) implies that

$$\begin{bmatrix} x \\ \zeta \end{bmatrix}' \begin{bmatrix} P\bar{A}_j+\bar{A}_j'P-\gamma_j P & Pv_j-0.5R_j'T_j' \\ v_j'P-0.5T_jR_j & \gamma_j c+T_jd_j \end{bmatrix} \begin{bmatrix} x \\ \zeta \end{bmatrix}<0$$

for all $(x,\zeta)\neq 0$, while it would be enough to check the case where $\zeta=1$.

(iii) It is clear that the contractive set $\mathcal{E}(c)$ does not necessarily intersect all the regions of saturation. Moreover, only the region that does intersect the set needs to be tested. However, if the set $\mathcal{E}(c)$ is being synthesized, it is not possible to determine, a priori, whether the searched ellipsoid will intersect or not some of the regions of saturation. In this case, in Algorithm 1 the test of (8)-(ii) is performed for all regions of saturation. Hence, it can happen that condition (8)-(ii) is unnecessarily verified in some region j.

3.2 *Test Condition 2*

This condition was mainly inspired by the results presented in (Johansson and Rantzer, 1998) for generic hybrid systems.

Proposition 2. If there exists nonnegative scalars γ_j and symmetric matrices $M_j\in\Re^{l_j*l_j}$ with nonnegative entries satisfying the following conditions $\forall j,\ j=1,\ldots,3^m-1$,

$$\begin{array}{ll} (i) & (A+BK)'P+P(A+BK)<0 \\ (ii) & \begin{bmatrix} P\bar{A}_j+\bar{A}_j'P+R_j'M_jR_j-\gamma_j P & Pv_j-R_j'M_jd_j \\ v_j'P-d_j'M_jR_j & \gamma_j c+d_j'M_jd_j \end{bmatrix}<0 \\ & \forall j \text{ s.t. } S(R_j,d_j)\cap\mathcal{E}(c)\neq\emptyset \end{array} \quad (9)$$

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[1] This can be accomplished by increasing interactively c and testing (8)-(i)-(ii) as an LMI feasibility problem.

then the set $\mathcal{E}(c)$ is a region of stability for the saturated system (3).

The proof of Proposition 2 is carried out similarly as in Proposition 1. These propositions basically differ in the strategy the S-procedure is handled. Here the constraints are transformed into quadratic forms before being included in the LMIs. All remarks and the relaxation algorithm presented for Proposition 1 carry over to Proposition 2.

4. DIFFERENTIAL INCLUSION MODELING

This modeling has been successfully used for the determination of regions of stability (Gomes da Silva Jr. and Tarbouriech, 1999*b*), (Henrion and Tarbouriech, 1999) as well as for the synthesis of stabilizing control laws in presence of saturating inputs (Gomes da Silva Jr. *et al.*, 1997).

Note that the ith entry of the saturated control law defined in (2) can be also written as: $(sat(Kx(t)))_{(i)} = \alpha(x(t))_{(i)}K_{(i)}x(t)$ where

$$\alpha(x(t))_{(i)} = min\{1, \frac{\rho_{(i)}}{|K_{(i)}x(t)|}\} \tag{10}$$

The coefficient $\alpha(x(t))_{(i)}$ can be viewed as an indicator of the degree of saturation of the ith entry of the control vector. In fact, the smaller the $\alpha(x(t))_{(i)}$, the farther the state vector is from the region of linearity. Define $D(\alpha(x(t))) \triangleq diag(\alpha(x(t)))$. Thus, system (3) can be rewritten as

$$\dot{x}(t) = (A + BD(\alpha(x(t))K)x(t) \tag{11}$$

It is difficult to perform a stability test directly from (11) due to the presence of $\alpha(x(t))$. We thus proceed in deriving a testable condition for it.

4.1 *Test condition*

Let $0 < \underline{\alpha}_{(i)} \leq 1$ be a lower bound to $\alpha(t)_{(i)}$ and define the vector $\underline{\alpha} \triangleq [\underline{\alpha}_{(1)},...,\underline{\alpha}_{(m)}]'$. The vector $\underline{\alpha}$ is associated to the following region in the state space:

$$S(K,\rho^{\alpha}) = \{x \in \Re^n \; ; \; -\rho^{\alpha} \preceq Kx \preceq \rho^{\alpha}\} \tag{12}$$

where $\rho^{\alpha}_{(i)} \triangleq \frac{\rho_{(i)}}{\underline{\alpha}_{(i)}}, \forall i = 1,\dots,m$.

Consider now all the possible m-order vectors such that the ith entry takes the value 1 or $\underline{\alpha}_{(i)}$. Hence, there exists a total of 2^m different vectors. By denoting each one of these vectors by γ_j, $j = 1,\dots,2^m$, define the following matrices: $D_j(\underline{\alpha}) = D(\gamma_j) = diag(\gamma_j)$ and $A_j = A + BD_j(\underline{\alpha})K$. Note that the matrices A_j are the vertices of a convex polytope of matrices. If $x(t) \in S(K,\rho^{\alpha})$ it follows that $(A + BD(\alpha(t))K) \in \mathbf{Co}\{A_1, A_2, \dots, A_{2^m}\}$. Hence, if $x(t) \in S(K,\rho^{\alpha})$, $\dot{x}(t)$ can be determined from an appropriate convex linear combination of matrices A_j at time t, that is:

$$\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_j(x(t))A_j x(t) \tag{13}$$

with $\sum_{j=1}^{2^m} \lambda_j(x(t)) = 1, \;\; \lambda_j(x(t)) \geq 0$.

It should be pointed out that model (13) represents the saturated system only in $S(K,\rho^{\alpha})$. Actually, if $x(t) \in S(K,\rho^{\alpha})$, the polytopic model (13) can be used to determine $\dot{x}(t)$.

Proposition 3. If there exists a vector $\underline{\alpha}$ satisfying the following conditions $\forall j = 1,\dots,2^m$

$$\begin{array}{ll} (i) & P(A + BD_j(\underline{\alpha})K) + (A + BD_j(\underline{\alpha})K)'P < 0 \\ (ii) & \begin{bmatrix} P & \underline{\alpha}_{(i)}K'_{(i)} \\ \underline{\alpha}_{(i)}K_{(i)} & \rho^2_{(i)}/c \end{bmatrix} \geq 0 \;\; \forall i = 1,\dots,m \\ (iii) & 0 < \underline{\alpha}_{(i)} \leq 1 \;\; \forall i = 1,\dots,m \end{array} \tag{14}$$

then the set $\mathcal{E}(c)$ is a region of stability for the saturated system.

See (Gomes da Silva Jr. *et al.*, 1997), (Gomes da Silva Jr. and Tarbouriech, 1999*b*) for a proof. Similarly to Propositions 1 and 2, the sufficient condition stated in Proposition 3 allows both to test if a given $\mathcal{E}(c)$ is contractive and to determine a contractive set based on some geometric criteria. In the first case, since P and c are given, conditions (14)-(i)-(ii)-(iii) can be easily tested as an LMI feasibility problem in $\underline{\alpha}$. In the second case, P and $\underline{\alpha}$ appear as decision variables[2] and condition (14)-(i) become a BMI whereas (14)-(ii)-(iii) are LMIs. A possible relaxation scheme in this case is as follows (see (Gomes da Silva Jr. and Tarbouriech, 1999*b*) and (Henrion and Tarbouriech, 1999) for more details), whose convergence is guaranteed similarly as in Algorithm 1.

Algorithm 2. :

(1) Choose $\underline{\alpha}$.
(2) Set $c = 1$. Fix $\underline{\alpha}$ obtained in the previous step, and search for P by optimizing a criterion on the size of $\mathcal{E}(c)$ subject to the LMI constraints given by (14)-(i)-(ii)-(ii)
(3) Fix P obtained in step 2. Minimize $\mu = \frac{1}{c}$ subject to LMI constraints given by (14)-(i)-(iii) with $\underline{\alpha}$ as a decision variable.
(4) Go to step 2 until a desired precision is achieved..

Remark 2. The conservativity of the condition given by Proposition 3 is due to the modeling of the behavior of the saturated system by a differential inclusion. In fact, (14)-(i) is a necessary and sufficient condition for the quadratic stability of the polytopic system $\dot{x}(t) = \sum_{j=1}^{2^m} \lambda_j(t)A_j x(t), \forall\lambda_j(t)$ such that $\sum_{j=1}^{2^m} \lambda_j(t) = 1, \;\; \lambda_j(t) \geq 0$. Notice, however, that the trajectories of this system includes all trajectories of the saturated system (3), but the converse is not necessarily true.

5. SECTOR MODELING

This modeling has been used for the determination of regions of stability as well as for the computation of stabilizing and performing control law in presence

[2] We can consider $c = 1$, without loss of generality.

of saturating inputs (Pittet *et al.*, 1997), (Hindi and Boyd, 1998).

Let us first define the nonlinearity ψ_s as follows:

$$\psi_s(Kx(t)) = sat(Kx(t)) - \Omega_{min}Kx(t) \qquad (15)$$

From this definition, the closed-loop system (3) equivalently reads:

$$\dot{x}(t) = (A + B\Omega_{min}K)x(t) + B\psi_s(Kx(t)) \qquad (16)$$

where $\Omega_{min} \in \Re^{m*m}$ is a positive diagonal matrix chosen such that matrix $A + B\Omega_{min}K$ is Hurwitz. By definition, the nonlinearity $\psi_s(Kx(t))$ is decentralized and satisfies the sector condition (Khalil, 1992), (Pittet *et al.*, 1997)

$$\psi_s(Kx)'[\psi_s(Kx) - (\Omega_{max} - \Omega_{min})Kx] \leq 0 \qquad (17)$$

$\forall x \in S(K, \rho^{\Omega_{min}}) = \{x \in \Re^n \ ; \ -\rho^{\Omega_{min}} \preceq Kx \preceq \rho^{\Omega_{min}}\}$, $\rho_{(i)}^{\Omega_{min}} = \frac{\rho_{(i)}}{\Omega_{min(i,i)}}, i = 1, ..., m$ with $\Omega_{max} - \Omega_{min} > 0$.

Since (16) involves ψ_s, it is difficult to perform a stability test directly from it. In the sequel we develop a testable sufficient condition for the stability of (16).

5.1 *Test Condition*

Let us consider that Ω_{max} is a given diagonal matrix such that $\Omega_{max} \geq I_m$.

Proposition 4. If there exist a diagonal positive matrix Ω_{min} and a positive scalar ε satisfying

$$\begin{bmatrix} (A + B\Omega_{min}K)'P + P(A + B\Omega_{min}K) + \varepsilon P & * \\ B'P + (\Omega_{max} - \Omega_{min})K & -2I_m \end{bmatrix} < 0$$
$$\begin{bmatrix} P & * \\ \Omega_{min(i,i)}K_{(i)} & \frac{\rho_{(i)}^2}{c} \end{bmatrix} \geq 0, \ \forall i = 1, ..., m \qquad (18)$$
$$0 < \Omega_{min(i,i)} \leq 1, \ \forall i = 1, ..., m$$

where $*$ is obtained from the symmetry of the matrices, then the set $\mathcal{E}(c)$ is a region of stability for the saturated system (3).

A proof can be found in (Pittet *et al.*, 1997). The underlining idea in Proposition 4 is the application of the circle criterion (Khalil, 1992) to system (16), which also is the main source of conservativity in this method. Indeed, (18) ensures that $\mathcal{E}(c)$ is a contractive set for the systems

$$\dot{x}(t) = (A + B\Omega_{min}K)x(t) + B\psi(t, Kx(t)) \qquad (19)$$

where ψ is any nonlinearity satisfying the sector condition $\psi(t, Kx)'[\psi(t, Kx) - (\Omega_{max} - \Omega_{min})Kx] \leq 0$, $\forall x \in S(K, \rho^{\Omega_{min}})$, $\forall t \geq 0$. It is clear that the class of nonlinearities ψ includes ψ_s. However, it also includes many other nonlinearities which are not related to the saturated system (16).

Compared to Proposition 3, the stability test in Proposition 4 is done via one inequality described in (18)-(i) instead of 2^m inequalities described in (14)-(i). This means less computational burden.

As opposed to all the other test conditions, (18) cannot be normalized in c. As a result, any choice of c leads to different results, which may be overly conservative. A possible solution to this problem is to include c in a multiobjective criterion on the size of $\mathcal{E}(c)$. For instance, we can choose a criterion in the form $(f(P) + \eta\frac{1}{c})$ where $f(P)$ is a term accounting for the influence of P on the size of $\mathcal{E}(c)$ and η is a positive weighting constant. Then, different criteria can be tried out in order to find the best solution.

Similarly to the other cases, condition (18) becomes a BMI when we are interested in synthesizing an contractive ellipsoidal set. In this case, we can use the following relaxation algorithm, whose convergence can be deduced similarly as in Algorithm 1:

Algorithm 3. :

(1) Choose $0 < \Omega_{min} < I$ and $0 < \varepsilon$.
(2) Fix Ω_{min} and ε obtained in the previous step, and search for P by minimizing $(f(P) + \eta\frac{1}{c})$ subject to the LMI constraints given by (18)-(i)-(ii)-(iii).
(3) Fix P obtained in step 2. Minimize $\mu = \frac{1}{c}$ subject to LMI constraints given by (18)-(i)-(ii)-(iii) with ε, Ω_{min} and $\mu = \frac{1}{c}$ as decision variables.
(4) Go to step 2 until a desired precision is achieved..

6. NUMERICAL EXAMPLE

The goal is to compare the effectiveness of the algorithms in synthesizing large stability domains and to verify the actual effect of the conservative steps involved in each algorithm. We solve the problem of finding an ellipsoidal asymptotic stability domain by applying each of the methods described in the paper. For each method, we search for the best possible ellipsoid. All the results are plotted to allow a visual comparison of the size of the stability regions obtained with each method.

Consider the multi-input second order linear system:

$$A = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -3 \end{bmatrix} \ ; \ B = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho = \begin{bmatrix} 5 \\ 2 \end{bmatrix} \ ; \ K = \begin{bmatrix} -0.7283 & -0.0338 \\ -0.0135 & -1.3583 \end{bmatrix}$$

For the 1st condition of the regions of saturation approach, we have applied Algorithm 1 considering as criterion the maximization of the minor axis of the ellipsoidal region (i.e. minimization of the greater eigenvalue of P). The optimal value was obtained for $\gamma = 0.25$ which is $P = 10^{-3}\begin{bmatrix} 0.5886 & 0.0023 \\ 0.0023 & 0.2800 \end{bmatrix}; \ c = 1$

No solution was found with 2nd condition of the regions of saturation approach.

For the polytopic modeling approach, we also considered the maximization of the minor axis of the ellipsoidal region in the application of Algorithm 2. Initializing $\underline{\alpha} = 1_m$, we obtained $\underline{\alpha} = [0.0275 \quad 0.0034]$ and

$$P = 10^{-4}\begin{bmatrix} 0.1608 & 0.0001 \\ 0.0001 & 0.1592 \end{bmatrix} \ ; \ c = 1$$

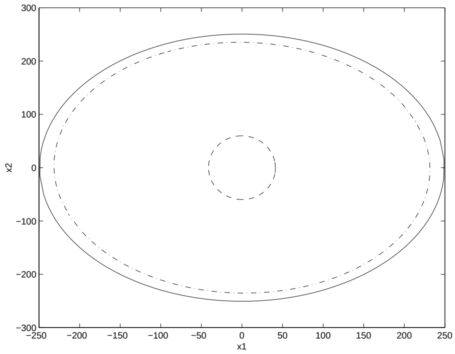

Fig. 1. Regions of saturation 1st condition (dashed); polytopic approach (solid); sector approach (dash-dotted)

In the sector modeling approach, we have considered in step 2 of Algorithm 3, the criterion $100\mu + \sigma_{max}(P)$ to be minimized with $\mu = \frac{1}{c}$. Considering $\Omega_{max} = I_2$ and starting with $\Omega_{min} = I_2$ and $\varepsilon = 0$ we obtained $c = 6.9782 \times 10^3$, $\varepsilon = 9.1709 \times 10^{-5}$ and the following

$$P = \begin{bmatrix} 0.1302 & 0.0020 \\ 0.0020 & 0.1261 \end{bmatrix}; \ \Omega_{min} = \begin{bmatrix} 0.0296 & 0 \\ 0 & 0.0055 \end{bmatrix}$$

Figure 1 depicts the ellipsoids obtained with the different approaches. Note that the ellipsoid obtained with the polytopic approach is slightly bigger than the one obtained with the sector approach. It is important to notice that the system has two other equilibrium points at $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \pm \begin{bmatrix} 257.931 \\ -7.931 \end{bmatrix}$. The set obtained with the regions of saturation 1st condition is significantly smaller than the sets obtained with the other approaches.

7. CONCLUDING REMARKS

Three diferent methods to synthesize ellipsoidal regions of asymptotic stability for linear systems with input saturation have been compared. All the methods were given in terms of LMI/BMI tests.

The main sources of conservativeness can be summarized as follows. (i) *Regions of saturation modeling*. Key steps in the development of the test conditions, including the use of the S-procedure. (ii) *Differential inclusion modeling*. Stability is ensured for the differential inclusion. Even though it encompasses the dynamics of the saturated system, it also includes dynamics that are not inherent to the saturated system. (iii) *Sector modeling*. The circle criteria guarantees stability for all nonlinearities satisfying the sector condition rather than only for the saturation nonlinearity.

In the problem of synthesizing a contractive ellipsoid, the test conditions are BMIs for all methods considered, leading to a difficult problem to solve. The comparative results provided have been obtained by means of relaxation methods.

The sector modeling leads to the smallest number of LMIs. A single LMI plays the role of 2^m LMIs in the differential inclusion modeling and 3^m LMIs in the modeling by regions of saturation. On the other hand, this scheme does not allow normalization of the level set of the Lyapunov function, as opposed to all other test conditions presented. This fact may have sensible implications in the conservativeness of the results.

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