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OUTPUT FEEDBACK REGULARIZATION OF NON-LINEAR DAE SYSTEMS

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Abstract: This work considers non-linear differential algebraic equation (DAE) systems whose state space depends on the manipulated inputs. An ODE representation of such systems cannot be derived independently of the controller design. An output feedback compensator is derived, which results in a modified DAE system whose state space does not depend on the new inputs and can be used for output feedback controller synthesis. Its application is illustrated in the context of control of a high-purity distillation column.

Keywords: DAEs; singular systems; high-purity distillation; non-linear control

1. INTRODUCTION

Differential-algebraic-equation (DAE) systems arise naturally as dynamic models of a wide range of engineering applications (Newcomb, 1981; McClamroch, 1986; Muller, 1997; Kumar and Daoutidis, 1999*a*). It is by now well-established that DAEs behave fundamentally differently from ODEs. The notion of index (Brenan *et al.*, 1996) provides a measure of that difference: a distinguishing feature of high-index DAEs is the presence of underlying constraints that restrict the solution to a lower dimensional space and require the specification of initial conditions on this space to obtain smooth solutions. The possible dependence of this state-space on forcing inputs poses additional conceptual and technical problems when these inputs are viewed as manipulated inputs for control purposes. Motivated by the above, the numerical analysis (Brenan *et al.*, 1996) and control of linear (Campbell, 1982; Lewis, 1986; Dai, 1989; Rhem and Allgower, 2001) and non-linear (McClamroch, 1990; Kumar and Daoutidis, 1999*a*) DAEs have attracted a lot of attention.

The state feedback control of a broad class of non-linear DAEs has been addressed in (Kumar and Daoutidis, 1999*a*) on the basis of an equivalent ODE description (state-space realization) of the DAE system if the underlying state space does not depend on the manipulated inputs (such systems have been termed *regular*), whereas if it does (such systems have been termed *non-regular*), this step is preceded by a state feedback modification to make the underlying state space independent of the inputs. In the present work, we lay the foundation for addressing the output feedback control of non-linear DAE systems. In the case of regular systems, this problem can be addressed on the basis of an equivalent ODE description, similar to the state feedback problem. However, in the case of non-regular systems, this approach necessitates the derivation of an output feedback compensator which results in a regular DAE system, suitable for the derivation of an underlying ODE system which can be used as the basis for output feedback controller synthesis. In what follows, we address the derivation of such a regularizing compensator for a broad class of non-linear DAEs that arise in practice, and we illustrate its application in the context of a two-point control problem in a

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high-purity distillation column with large internal flowrates.

2. PRELIMINARIES

We consider non-linear DAE systems that have the following semi-explicit description:

$$\begin{aligned} \dot{x} &= f(x) + b(x)z + g(x)u \\ 0 &= k(x) + l(x)z + c(x)u \\ y &= Hx \end{aligned} \tag{1}$$

where $x \in \mathcal{X} \subset \mathbb{R}^n$ is the vector of differential variables, $z \in \mathcal{Z} \subset \mathbb{R}^p$ is the vector of algebraic variables ($\mathcal{X}, \mathcal{Z}$ are open sets of dimensions n, p, respectively), $u \in \mathbb{R}^m$ is the vector of manipulated inputs, $f(x), k(x)$ are analytic vector fields of dimensions n, p, respectively, $b(x), g(x), l(x), c(x)$ are smooth matrices of appropriate dimensions and $H \in \mathbb{R}^{m \times n}$ is a constant matrix. Given that typically the controlled outputs y are subsets of the state variables x and for simplicity of the subsequent development, we assume that $y_i = x_i$, for $1 \leq i \leq m$. We assume that the system is high-index (i.e., rank$[l(x)]= p_1 < p$) and rank$[l(x) \;\; c(x)]= m_1 \leq p$, with $m_1 \geq p_1$. The goal is to design a feedback compensator using only the measurements of the outputs y to obtain a modified DAE system for which the constraints are independent of the new inputs.

In (Kumar and Daoutidis, 1999a), an algorithm was described which aims at increasing the rank of the coefficient matrix for z, $l(x)$, without introducing derivatives of the inputs u. It involves, in each iteration, row operations on the algebraic equations to identify the underlying constraints on x, of which a minimal number involve the inputs u. The constraints involving the inputs u are retained, while those that are independent of u are differentiated to obtain the algebraic equations for the succeeding iteration. For regular systems, the procedure converges with a final set of algebraic equations that can be solved for z in terms of x and u. On the other hand, for non-regular systems, the procedure converges with a final set of algebraic equations that is still singular with respect to z, i.e. it cannot be solved for z, but also explicitly identifies those constraints in x which involve u in a non-singular fashion.

When applied to the system of the form in Eq.1, the algorithm, after s iterations, yields the following DAE system:

$$\begin{aligned} \dot{x} &= f(x) + b(x)z + g(x)u \\ 0 &= \begin{bmatrix} \bar{k}(x) \\ \hat{k}(x) \end{bmatrix} + \begin{bmatrix} \bar{l}(x) \\ 0 \end{bmatrix} z + \begin{bmatrix} \bar{c}(x) \\ \hat{c}(x) \end{bmatrix} u \\ 0 &= \mathbf{k}(x) \\ y &= Hx \end{aligned} \tag{2}$$

where $\bar{l}(x)$ is a full row rank $p_{s+1} \times p$ matrix (we have $p_{s+1} < p$ because the DAE system is assumed to be non-regular), $\hat{c}(x)$ is a full row rank $(p - p_{s+1}) \times m$ matrix, and $\mathbf{k}(x) = 0$ are constraints among the differential variables x identified by the algorithm. The DAE system in Eq.2 is equivalent to the original DAE system in Eq.1 in the sense that for consistent initial conditions $(x(0), z(0))$ and smooth inputs $u(t)$, both systems have the same smooth solution $(x(t), z(t))$.

3. DESIGN OF THE OUTPUT FEEDBACK COMPENSATOR

For the new DAE system in Eq.2, an output feedback compensator will now be designed to modify the constraints $\hat{k}(x) + \hat{c}(x)u = 0$ that involve the inputs u, with the objective of obtaining a modified DAE with new inputs v that do not appear in any of the underlying constraints.

Given that the constraints $\hat{k}(x) + \hat{c}(x)u = 0$ have to be differentiated at least once to obtain a set of algebraic equations solvable in z, the algebraic variables $z(t)$ are functions of the differential variables $x(t)$, the inputs $u(t)$ and at least one of their derivatives. Thus, any causal feedback law for u must be independent of the algebraic variables z. Moreover, a static output feedback compensator of the form

$$u = \mathcal{F}(y, v)$$

will not achieve the desired objective, since the resulting feedback modified constraints would still involve the new inputs v. These reasons motivate the choice of a dynamic output feedback compensator.

Given that $\hat{c}(x)$ is a full row rank matrix, without loss of generality, the manipulated inputs u can be arranged in such a way that:

$$\hat{c}(x) = [\hat{c}_1(x) \; \hat{c}_2(x)] \tag{3}$$

where $\hat{c}_1(x) \in \mathbb{R}^{(p-p_{s+1}) \times (p-p_{s+1})}$ is non-singular and $\hat{c}_2(x) \in \mathbb{R}^{(p-p_{s+1}) \times (m-(p-p_{s+1}))}$.

We will consider a driftless dynamic compensator of the general form:

$$\begin{aligned} \dot{w} &= v_1 \\ u &= \begin{bmatrix} F(y)y \\ 0 \end{bmatrix} + \begin{bmatrix} G(y)w \\ 0 \end{bmatrix} + \beta(y)v \end{aligned} \tag{4}$$

where $w \in \mathbb{R}^{n_c}$ is the vector of the compensator states, $v \in \mathbb{R}^m$ is the vector of new inputs, $F(y), G(y)$ and $\beta(y)$ are matrices of dimensions $(p - p_{s+1}) \times m$, $(p - p_{s+1}) \times n_c$, $m \times m$ respectively. The compensator in Eq. 4 will be designed to modify the constraints $0 = \hat{k}(x) + \hat{c}(x)u$ such that

- (i) the resulting constraints are independent of the inputs v, and
- (ii) differentiating these constraints once, the resulting algebraic equations are solvable in z.

The requirement in (i) is necessary to obtain a regular system. The requirement in (ii) implies that the feedback modified DAE has an index $\bar{\nu}_d = 2$, and thus it is solvable and no additional underlying constraints are present.

3.1 *Requirement (i)*

Let's focus on the requirement that the new constraints are to be independent of the new inputs v. Clearly, for this to happen, we must have:

$$\hat{c}(x)\beta(y) = 0 \qquad (5)$$

Note that this can be easily satisfied if any of the following holds:

* we allow for measurements of all states,
* $\hat{c}(x)$ is a function of only the output variables,
* $\hat{c}(x)$ is a constant matrix.

Then, we can directly construct β to be in the null space of $\hat{c}$.

In the case where none of the above holds, the following proposition provides conditions under which requirement (i) is possible with output feedback.

Proposition: *Consider a DAE system of the form in Eq. 1 for which the algorithm in (Kumar and Daoutidis, 1999a) yields the equivalent DAE system in Eq.2, subject to the dynamic compensator of Eq. 4. Consider also a partition of* β *as* $[\beta_1(y)^T \ \beta_2(y)^T]^T$ *where* $\beta_1(y) \in \mathbb{R}^{(p-p_{s+1})\times m}$ *and* $\beta_2(y) \in \mathbb{R}^{(m-(p-p_{s+1}))\times m}$*, and a decomposition of* $-[\hat{c}_1(x)]^{-1}\hat{c}_2(x)$ *as* $M(x)+P(y)$*, where* P *is a matrix involving only the output variables* y*. The resulting constraints are independent of the new inputs* v *if and only if there exists* $\beta_2(y)$ *such that:*

$$M(x)\beta_2(y) = 0 \qquad (6)$$

Proof: Let's assume that there exists β_2 such that Eq.6 holds. Then the matrix $\beta = [\beta_1(y)^T \ \beta_2(y)^T]^T$, where $\beta_1(y) = P(y)\beta_2(y)$, satisfies Eq. 5, which proves the sufficiency.

For Eq. 5 to hold, we need:

$$\beta_1(y) = [M(x)+P(y)]\beta_2(y)$$

Let's consider two specific representations of the state vector denoted by x^1 and x^2 such that $x_i^2 = x_i^1, \ i = 1,\ldots,m$, but $x_i^2 \neq x_i^1, \ i = m+1,\ldots,n$. Then:

$$\beta_1(Hx^1) = \beta_1(Hx^2) = \beta_1(x_1^1,\ldots,x_m^1)$$
$$\beta_2(Hx^1) = \beta_2(Hx^2) = \beta_2(x_1^1,\ldots,x_m^1)$$

which implies:

$$M(x^1)\beta_2(Hx^1) = M(x^2)\beta_2(Hx^2) = 0$$

i.e., $\beta_2(Hx)$ is such that $M(x)\beta_2(Hx) = 0$, which completes the proof.

3.2 *Requirement* (ii)

Once the requirement (i) is satisfied, the new constraints which do not involve z are:

$$\hat{k}(x) + \hat{c}_1(x)[F(y)y + G(y)w] = 0 \qquad (7)$$

or equivalently,

$$[\hat{c}_1(x)]^{-1}\hat{k}(x) + F(y)y + G(y)w = 0 \qquad (8)$$

For simplicity of notation, let's denote $[\hat{c}_1(x)]^{-1}\hat{k}(x)$ by $\underline{\hat{k}}(x)$.

We now make the following observations:

- Let
$$rank\begin{bmatrix}\bar{l}(x)\\ \underline{\underline{L}}_b\underline{\hat{k}}(x)\end{bmatrix} = p^*$$
where $p_{s+1} \leq p^* \leq p$ and
$$\underline{\underline{L}}_b\underline{\hat{k}}(x) = \begin{bmatrix} L_{b_1}\underline{\hat{k}}_1(x) & \ldots & L_{b_p}\underline{\hat{k}}_1(x)\\ \vdots & & \vdots\\ L_{b_1}\underline{\hat{k}}_q(x) & \ldots & L_{b_p}\underline{\hat{k}}_q(x)\end{bmatrix}$$
where $L_{b_j}\underline{\hat{k}}_i(x)$ denotes the standard Lie derivative, b_j is the jth column of $b(x)$ and $q = p - p_{s+1}$. Then, without loss of generality, the vector $\underline{\hat{k}}(x)$ can be rearranged as $\underline{\hat{k}}(x) = \left[\underline{\hat{k}}^1(x)^T\underline{\hat{k}}^2(x)^T\right]^T$ where $\underline{\hat{k}}^1(x)$ and $\underline{\hat{k}}^2(x)$ are vectors of dimensions p^*-p_{s+1} and $p-p^*$ respectively, such that
$$rank\underline{\underline{L}}_b\underline{\hat{k}}^1(x) = p^* - p_{s+1}$$
- If moreover
$$rank\begin{bmatrix}\bar{l}(x)\\ \underline{\underline{L}}_b\underline{\hat{k}}(x)\\ Hb(x)\end{bmatrix} = p$$
then $rankHb(x) = p-p^*$. The matrix $Hb(x)$ corresponds to the m first rows of the matrix $b(x)$, so that the previous rank condition implies that, among the m first rows of $b(x)$, $p-p^*$ rows are linearly independent. Thus, there exists a matrix $\Gamma \in \mathbb{R}^{(p-p^*)\times m}$ with full row rank that selects the $p-p^*$ linearly independent rows of $Hb(x)$ such that:
$$rank\begin{bmatrix}\bar{l}(x)\\ \underline{\underline{L}}_b\underline{\hat{k}}^1(x)\\ \Gamma Hb(x)\end{bmatrix} = p \qquad (9)$$

The following theorem states the result on the design of the dynamic output feedback compensator:

Theorem: *Consider a DAE system of the form in Eq. 1 with the equivalent DAE system of Eq. 2 where the condition of the proposition holds. Then, the system in Eq. 2 can be modified through a dynamic output feedback compensator of the form in Eq.4 to obtain an index-two regular system, if and only if:*

$$rank \begin{bmatrix} \bar{l}(x) \\ \underline{\underline{L}}_b[[\hat{c}_1(x)]^{-1}\hat{k}(x)] \\ Hb(x) \end{bmatrix} = p \qquad (10)$$

If this condition is satisfied, then the dynamic output feedback compensator:

$$\begin{aligned} \dot{w} &= v_1 \\ u &= \begin{bmatrix} F(y)y \\ 0 \end{bmatrix} + \begin{bmatrix} w \\ 0 \end{bmatrix} + \beta(y)v \end{aligned} \qquad (11)$$

where

$$F = \begin{bmatrix} 0 \\ \Gamma \end{bmatrix}, \qquad (12)$$

$\beta(y)$ is such that $\hat{c}(x)\beta(y) = 0$, and Γ is as chosen in Eq. 9, yields an index-two modified DAE system.

Proof: The differentiation with respect to time of the constraints in Eq. 8 yields new constraints for which the resulting matrix coefficient for z has the form:

$$\begin{bmatrix} I_{p_s+1} & 0 & 0 \\ 0 & I_{p-p_s+1} & \frac{\partial F}{\partial y}.y + \frac{\partial G}{\partial y}.w + F \end{bmatrix} \begin{bmatrix} \bar{l}(x) \\ \underline{\underline{L}}_b\hat{k}(x) \\ Hb(x) \end{bmatrix}$$

where

$$\frac{\partial F}{\partial y}.y = \begin{bmatrix} \sum_{k=1}^{m} \frac{\partial F_{1,k}}{\partial y} y_k \\ \vdots \\ \sum_{k=1}^{m} \frac{\partial F_{p-p_s+1,k}}{\partial y} y_k \end{bmatrix}$$

with

$$\frac{\partial F_{i,k}}{\partial y} = \begin{bmatrix} \frac{\partial F_{i,k}}{\partial y_1} & \cdots & \frac{\partial F_{i,k}}{\partial y_m} \end{bmatrix}$$

and, $F_{i,j}$ is the $(i,j)th$ element of F. To have an index-2 DAE system, the matrix coefficient for z has to be invertible. Given that the first matrix in the above product has full row rank, it follows that the condition of Eq. 10 should hold.

If we assume that the condition of Eq. 10 is fulfilled, the direct substitution of the feedback law of Eq. 11 into the DAE system of Eq. 2 results in the following algebraic equations:

$$\begin{aligned} 0 = &\begin{bmatrix} \bar{k}(x) \\ \hat{k}(x) \end{bmatrix} + \begin{bmatrix} \bar{l}(x) \\ 0 \end{bmatrix} z + \begin{bmatrix} \bar{c}_1(y)F(y)y \\ \hat{c}_1(y)F(y)y \end{bmatrix} \\ &+ \begin{bmatrix} \bar{c}_1(x)w \\ \hat{c}_1(y)w \end{bmatrix} + \begin{bmatrix} \bar{c}(x)\beta(y) \\ \hat{c}(x)\beta(y) \end{bmatrix} v \end{aligned}$$

Given that the matrix $\beta(y)$ is constructed to satisfy requirement (i), the constraints that do not involve the new inputs v take the form:

$$0 = \hat{k}(x) + \hat{c}_1(x)[F(y)y + w]$$

or, equivalently,

$$0 = [\hat{c}_1(x)]^{-1}\hat{k}(x) + F(y)y + w$$

Given the definition of $F(y)$, those constraints are:

$$0 = [\hat{c}_1(x)]^{-1}\hat{k}(x) + \begin{bmatrix} 0 \\ \Gamma y \end{bmatrix} + w$$

Differentiating these constraints with respect to time once yields a new set of algebraic equations in which the matrix coefficient for the algebraic variables z is:

$$\begin{bmatrix} \bar{l}(x) \\ \underline{\underline{L}}_b\hat{k}^1(x) \\ \underline{\underline{L}}_b\hat{k}^2(x) + \Gamma Hb(x) \end{bmatrix}$$

Clearly, the algebraic equation is solvable for z given that the previous matrix is non-singular by construction (see Eq. 10), which completes the proof.

Corollary: *Consider a DAE system of the form in Eq. 1 with the equivalent DAE system of Eq. 2 where the condition of the proposition holds. Assuming that:*

$$rank \begin{bmatrix} \bar{l}(x) \\ \underline{\underline{L}}_b[[\hat{c}_1(x)]^{-1}\hat{k}(x)] \end{bmatrix} = p \qquad (13)$$

the following output feedback compensator:

$$\begin{aligned} \dot{w} &= v_1 \\ u &= \begin{bmatrix} w \\ 0 \end{bmatrix} + \beta(y)v \end{aligned} \qquad (14)$$

where $\beta(y)$ is such that $\hat{c}(x)\beta(y) = 0$ yields an index-two modified DAE system.

Proof: When the condition of Eq. 13 is satisfied, Γ can be chosen as $\Gamma = 0$, so that $F = 0$ in Eq. 11, yielding the compensator of Eq. 14.

Remark: Note that, if we allow for measurements of all the states (i.e. H is the identity matrix), then, the condition of the proposition is obviously satisfied and the condition of the theorem is also satisfied since we always have:

$$rank \begin{bmatrix} \bar{l}(x) \\ b(x) \end{bmatrix} = p$$

(see (Kumar and Daoutidis, 1999a), pp:59 – 62).

Once the non-regular DAE system is modified with the output feedback compensator of Eq. 11, the resulting regular index-two DAE system can be used directly for the derivation of a state-space realization and the formulation of an output feedback controller synthesis problem on the basis of this realization (see e.g. (Kumar and Daoutidis, 1999*a*)).

4. EXAMPLE

In a network of processes (reactors, separation systems) or a single staged process, where the individual units are connected with recycle streams, large recycle flowrates typically induce a two time scale behavior: the dynamics of the individual units evolve in a fast time scale while the dynamics of the overall network or process evolve in a slow time scale. This slow dynamics is typically described by a high-index DAE which can be non-regular depending on the choice of manipulated inputs (Kumar and Daoutidis, 2001).

Let's consider a distillation column with N trays (numbered from top to bottom), to which a saturated liquid containing a mixture of three components with mole fractions x_{1f}, x_{2f} of components 1 and 2 respectively, is fed at (molar) flowrate F_0 on tray N_f. The heavy component 3 is the desired product and is removed at the bottom from the reboiler at a flowrate B, while the lighter components 1 and 2 are removed at the top from the condenser at a flowrate D. In this column, a large vapor boilup V_B and liquid recycle R are used compared to the feed, distillate and bottom product flowrates, to attain a high purity of the desired component 3 in the bottom product.

The presence of large vapor boilup V_B and liquid recycle R, and hence, large internal liquid and vapor flowrates in the column, compared to the inlet and outlet flowrates from the column, induces a time-scale separation in the column dynamics with the dynamics of individual stages evolving in a fast time scale, and the dynamics of the overall column in a slow time scale (Kumar and Daoutidis, 1999*b*). Note that a large liquid recycle R implies an equally large vapor boilup V_B at the nominal steady state. On the other hand, the feed flowrate F_0, the distillate flowrate D and the bottom product flowrate B are of the same order magnitude. Thus, defining the singular perturbation parameter $\epsilon = D_{nom}/R_{nom}$, and $\kappa_1 = V_{Bnom}/R_{nom} = O(1)$, where the subscript *nom* refers to nominal steady state values and $O(.)$ is the standard order of magnitude notation, the terms involving the large parameter $(1/\epsilon)$ can be isolated in the model. Then, the process model, under standard modeling assumptions, takes the following general form (Kumar and Daoutidis, 1999*b*):

$$\dot{x} = f(x) + g^s(x)u^s + \frac{1}{\epsilon} g^l(x)u^l \qquad (15)$$

where x is the vector of state variables (compositions and holdups in each stage), $u^s = [D\ B]^T \in \mathbb{R}^2$ is the vector of manipulated inputs corresponding to small flowrates and $u^l = [\bar{R}\ \bar{V}_B]^T \in \mathbb{R}^2$ is the vector of manipulated inputs corresponding to large flowrates where $\bar{R} = R/R_{nom}$ and $\bar{V}_B = V_B/V_{Bnom}$.

The control of a two-time-scale system such as the one in Eq. 15 is naturally addressed through the derivation of separate controllers in the fast and the slow time scales. In the **fast time scale** $(\tau = t/\epsilon)$, in the limit $\epsilon \to 0$, the inputs u^s have no effect on the fast dynamics; only the outputs u^l can be used for control. In particular, the liquid holdups in the condenser and the reboiler (M_C and M_R) behave like integrators and need to be stabilized, which is easily achieved by using simple proportional controllers:

$$\begin{aligned} \bar{R} &= 1 - \bar{K}_{c1}(M_{Cnom} - M_C) \\ \bar{V}_B &= 1 - \bar{K}_{c2}(M_{Rnom} - M_R) \end{aligned}$$

In the **slow time scale t**, the dynamics are obtained from Eq. 15 in the limit $\epsilon \to 0$ and take the form:

$$\begin{aligned} \dot{x} &= f(x) + b(x)z + g^s(x)u^s \\ 0 &= \bar{g}^l(x)u^l \end{aligned} \qquad (16)$$

where $0 = \bar{g}^l(x)u^l$ are $2N+3$ linearly independent constraints, and $z = lim_{\epsilon \to 0}\bar{g}^l(x)u^l \in \mathbb{R}^{2N+3}$ is the vector of the linearly independent algebraic variables.

In this slow-time scale, there are two manipulated inputs D and B that affect the slow dynamics. At this time scale, the total holdup (or, equivalently, one of M_R or M_C) also needs to be stabilized as it is not affected by the large flowrates (Kumar and Daoutidis, 1999*b*). The specification of one more output, e.g. the top or the bottom composition, leads to a well-defined control configuration and the resulting DAE system is regular. However, in order to achieve control of both the top and bottom compositions, we need an additional manipulated input. A natural approach to this end is to treat the setpoints for the condenser/reboiler holdups used in the fast proportional control as additional manipulated input variables, leading to a cascaded control configuration. In this case, the DAE system that describes the slow dynamics of the column is non-regular, since the algebraic constraints explicitly involve the setpoints for the reboiler/condenser holdups, which are manipulated inputs. Considering M_{Cnom} as an additional manipulated input, and $M_C, x_{3,B}, x_{3,D}$ as the controlled outputs where $x_{3,B}, x_{3,D}$ are the mole fraction of component 3 in the reboiler and the condenser respectively, the DAE system of Eq. 16 can be rewritten as:

$$\begin{aligned} \dot{x} &= f(x) + b(x)z + g(x)u \\ 0 &= \begin{bmatrix} \bar{k}^1(x) \\ \underline{k}^1(x) \end{bmatrix} + \begin{bmatrix} \bar{c}^1 \\ 0 \end{bmatrix} u \\ y_i &= x_i, \quad 1 \le i \le 3 \end{aligned}$$

where $u = \begin{bmatrix} M_{Cnom} & D & B \end{bmatrix}^T$ is the vector of manipulated inputs,

$$\begin{aligned} \bar{c}^1 &= \begin{bmatrix} \bar{K}_{c1} & 0 & 0 \end{bmatrix} \\ \bar{k}^1(x) &= [-1 - \bar{K}_{c1} M_C - \kappa_1 \bar{K}_{c2} M_{Rnom} \\ &\quad + \kappa_1 (1 + \bar{K}_{c2} M_R)] \end{aligned}$$

and the $2N + 2$ linearly independent constraints that do not involve the inputs are:

$$\underline{k}^1(x) = \begin{bmatrix} \kappa_1 \bar{V}_B (y_{3,1} - x_{3,D}) \\ \kappa_1 \bar{V}_B (y_{1,1} - x_{1,D}) \\ \kappa_1 \bar{V}_B (y_{1,i+1} - y_{1,i} + x_{1,i-1} - x_{1,i}) \\ \kappa_1 \bar{V}_B (y_{3,i+1} - y_{3,i} + x_{3,i-1} - x_{3,i}) \\ \vdots \end{bmatrix}$$

where $1 \le i \le N$, $x_{1,i}, x_{3,i}$ are the liquid mole fractions of 1 and 3 in tray i, and $y_{1,i}, y_{3,i}$ are the vapor mole fractions in tray i. The differentiation of the constraints $\underline{k}^1(x) = 0$ with respect to time yields $K\dot{x} = 0$ where K is a $(2N + 2) \times (2N + 6)$ full row rank matrix and leads to the following DAE system:

$$\begin{aligned} \dot{x} &= f(x) + b(x)z + g(x)u \\ 0 &= \begin{bmatrix} Kf(x) \\ \bar{k}^1(x) \end{bmatrix} + \begin{bmatrix} Kb(x) \\ 0 \end{bmatrix} z + \begin{bmatrix} Kg(x) \\ \bar{c}^1 \end{bmatrix} u \\ 0 &= \underline{k}_1(x) \\ y_i &= x_i, \quad 1 \le i \le 3 \end{aligned}$$

which is in the form of Eq. 2.

Notice that since $\hat{c}(x) = \bar{c}^1$ is a constant, β can be easily constructed to be in the null space of $\hat{c}(x)$. Let's consider the following $\beta = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ that does not depend on y. It can also be verified (details are omitted due to lack of space) that:

$$\begin{aligned} rank \begin{bmatrix} \bar{l}(x) \\ \underline{\underline{L}}_b \hat{\underline{k}}(x) \\ Hb(x) \end{bmatrix} &= rank \begin{bmatrix} Kb(x) \\ \underline{\underline{L}}_b [\bar{k}^1(x)/\bar{K}_{c1}] \\ Hb(x) \end{bmatrix} \\ &= rank \begin{bmatrix} Kb(x) \\ \underline{\underline{L}}_b [\bar{k}^1(x)/\bar{K}_{c1}] \end{bmatrix} \\ &= 2N + 3 \end{aligned}$$

so that the condition of the corollary is satisfied too. Thus, we obtain the following dynamic compensator:

$$\begin{aligned} \dot{w} &= v_1 \\ u &= \begin{bmatrix} w \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} v \end{aligned}$$

which achieves the desired regularization. In this special case, the compensator corresponds to adding an integrator to the channel of the manipulated input M_{Cnom}. The resulting regular DAE system can be used as the basis for designing a non-linear output feedback controller, which, coupled with the proportional controllers in the condenser and reboiler, will comprise the overall control scheme for the column.

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