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STABILITY MARGIN COMPUTATION FOR NONLINEAR SYSTEMS: A PARAMETRIC APPROACH

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Abstract: This paper studies the existence of limit cycles in a control system which contains nonlinearities and parametric uncertainties. The existence of limit cycles in a control system with a separable nonlinearity can be predicted using the describing function. In this paper, some of the well-known results developed in the area of parametric robust control are used together with the describing function method to analyze the stability problem of uncertain nonlinear systems. Based on the segment lemma, a stability result for a control system with an uncertain nonlinear element and a fixed linear element is first derived. Then, a polynomial method and a graphical method are proposed to determine how much one can perturb the coefficients of the linear element without causing the nonlinear system to have a limit cycle. Examples are given to illustrate the method presented. *Copyright © 2002 IFAC*

Keyword: Nonlinear control systems; Describing functions; Limit cycles; Uncertain dynamic systems; Robust stability; Parametric variation

1. INTRODUCTION

The characteristic features of linear systems such as the proportionality of cause and effect and the principle of superposition no longer hold for nonlinear systems. The proportionality of cause and effect is the basis of harmonic analysis, in which it is known that if the input signal of a linear system is a sinusoidal signal, the output is a sinusoidal signal of different phase and amplitude, but of the same frequency. However, in nonlinear systems, the frequencies present in the output may not be those of the input. There are some phenomena such as periodic motion or the occurrence of limit cycles, chaos, multiple modes of behaviours etc. which can only take place in nonlinear systems.

Analysis and design of nonlinear systems or procedures for finding the solutions of problems involving nonlinear systems, in general, are extremely complicated. Because the analysis tools for nonlinear problems involve more advanced mathematics, one often finds it necessary to use equivalent linearization techniques and to solve a resulting linearized problem. The describing function method is one of the popular equivalent linearization methods (Atherton, 1982). Describing Function based methods are used in nonlinear systems for assessing the system stability where instability is envisaged in the form of limit cycles. However, the classical describing function method was developed for fixed nominal systems and in general is inapplicable when uncertain parameters are present. Within the context of control system with uncertain parameters, the describing function was studied for continuous-time systems in (Ferreres and Fromion, 1998; Fadali and Chachavalvoong, 1995; Impram and Munro, 1998) and for discrete-time systems in (Tan and Atherton, 1999) by using a μ-synthesis framework, the Kharitonov theorem and the mapping theorem. In all of these papers, it is assumed that both the nonlinear and linear elements of the nonlinear system shown in Figure 1 involve parametric uncertainty. However, since the describing function representation of a nonlinear element is an approximate procedure, it is more realistic to represent a nonlinear element by uncertain parameters and take the linear part as a fixed transfer function. In this paper, firstly, the stability analysis of control systems with separable nonlinearity is investigated. A stability result is given using the segment lemma (Bhattacharyya, *et al.*, 1995) assuming that the describing function which represents the nonlinear element is uncertain and the coefficients of the linear element are fixed. Then, two methods are given to determine the maximum value of the perturbation of the coefficients of the linear element while preserving stability.

The paper is organized as follows: In Section 2, some results from parametric robust control are reviewed. The classical describing function method is summarized in Section 3. The stability analysis of control systems with an uncertain describing function and a fixed linear element is investigated in Section 4. Section 5 gives a polynomial method and a graphical method for stability margin computation. Section 6 includes concluding remarks.

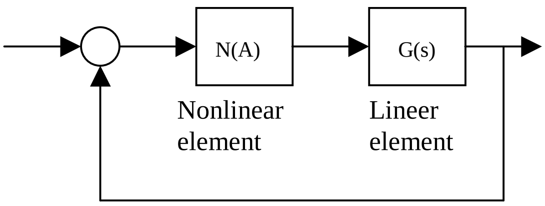

Figure 1: Block diagram of a nonlinear system with separable nonlinearity

2. SOME RESULTS FROM PARAMETRIC ROBUST CONTROL

Most of the research results related to the robustness analysis of systems with parametric perturbations have taken place since the publication of the Kharitonov theorem (Kharitonov, 1979). The Kharitonov theorem is an extension of the Routh stability criterion to interval polynomials. An interval polynomial is a polynomial where each coefficient can vary in a prescribed interval. For example,

$$P(s,q) = q_0 + q_1 s + q_2 s^2 + \cdots + q_n s^n, q_n > 0 \qquad (1)$$

is an interval polynomial where the uncertainty box is $Q = \{q : q_i \in [\underline{q_i}, \overline{q_i}], i = 0,1,2,....., n\}$. The Kharitonov theorem states that an interval polynomial of the form of Eq. (1) is Hurwitz stable if and only if the four Kharitonov polynomials are Hurwitz stable.

The objective of this section is to present some basic results in the area of parametric robust control which are based on the Kharitonov theorem. A complete and very up-to date investigations of this area of research can be found in the books (Bhattacharyya, *et al.*, 1995; Barmish, 1994; Ackermann, 1993; Djaferis, 1995). Consider a unity feedback system with

$$C(s) = \frac{N_c(s)}{D_c(s)} \qquad (2)$$

and

$$G(s,q,r) = \frac{N(s,r)}{D(s,q)} = \frac{r_m s^m + r_{m-1} s^{m-1} + \cdots + r_0}{q_n s^n + q_{n-1} s^{n-1} + \cdots + q_0} \qquad (3)$$

where $r_i \in [\underline{r_i}, \overline{r_i}]$, i=0,1,2,....,m and $q_i \in [\underline{q_i}, \overline{q_i}]$, i=0,1,2,....n. The numerator, $N_c(s)$, and the denominator, $D_c(s)$, are fixed polynomials in *s*. Let the Kharitonov polynomials associated with $N(s,r)$ and $D(s,q)$ be $N_1(s)$, $N_2(s)$, $N_3(s)$, $N_4(s)$ and $D_1(s)$, $D_2(s)$, $D_3(s)$ and $D_4(s)$, respectively. By taking all combinations of the $N_i(s)$ and $D_j(s)$ for $i, j = 1,2,3,4$, one obtains the sixteen Kharitonov plants family as

$$G_K(s) = \{G_{ij}(s) \mid G_{ij}(s) = \frac{N_i(s)}{D_j(s)}\} \qquad (4)$$

The Kharitonov segments for the numerator and denominator of $G(s,q,r)$ can be written as

$$\lambda N_i(s) + (1-\lambda) N_j(s), \ \lambda D_i(s) + (1-\lambda) D_j(s) \qquad (5)$$

where $\lambda \in [0,1]$ and $(i, j) \in \{(1,2), (1,3), (2,4), (3,4)\}$. And the following 32 subsets of the family of interval plants $G(s,q,r)$ can be obtained by using the Kharitonov segments. These subsets are

$$G_E(s) = \frac{N_i(s)}{\lambda D_j(s) + (1-\lambda) D_k(s)} \cup \frac{\lambda N_j(s) + (1-\lambda) N_k(s)}{D_i(s)} \qquad (6)$$

where $\lambda \in [0,1]$, $i = 1,2,3,4$ and $(j,k) \in \{(1,2), (1,3), (2,4), (3,4)\}$. The closed loop characteristic equation of the system is $\delta(s) = D_c(s) D(s,q) + N_c(s) N(s,r)$. Then the closed loop system is stable for all $C(s)G(s,q,r)$ if and only if it is stable for all $C(s)G_E(s)$ (Bhattacharyya, *et al.*, 1995). If the controller is a proper first order controller such as $C(s) = K(s+a)/(s+b)$ then $C(s)G(s,q,r)$ is stable if $C(s)G_K(s)$ is stable (Barmish, 1994). It was shown that the outer boundary of the Nyquist envelope of a stable strictly proper interval plant is covered by the Nyquist plots of the sixteen Kharitonov plants (Hollot and Tempo, 1994). The whole boundary of the Nyquist and Nichols envelopes of $G(s,q,r)$ and $C(s)G(s,q,r)$ are generated from the boundary of $G_E(s)$ and $C(s)G_E(s)$ and the Bode envelope can be obtained from the rectangular value sets of the numerator and the denominator of the interval plant.

3. DESCRIBING FUNCTION ANALYSIS

The describing function of a nonlinear element can be defined as the ratio of the fundamental component of the output to the magnitude of an applied sinusoidal input. Consider an input $x(t) = A\sin(\omega t)$ to the nonlinear element, the output can be expressed in a Fourier series as follows:

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}[a_n \cos(n\omega t) + b_n \sin(n\omega t)] \qquad (7)$$

where

$$a_n = \frac{1}{\pi}\int_0^{2\pi} y(t)\cos(n\omega t)d(\omega t) \qquad (8)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} y(t)\sin(n\omega t)d(\omega t) \qquad (9)$$

For an odd nonlinearity a_0 is zero. Using the fundamental component, $y_1(t) = a_1\cos(\omega t) + b_1\sin(\omega t)$, of $y(t)$, the nonlinearity is represented with the describing function as

$$N(A,\omega) = \frac{b_1 + ja_1}{A} \qquad (10)$$

Now, the characteristic equation of the system shown in Figure 1 can be written as

$$1 + N(A,\omega)G(s) = 0 \qquad (11)$$

Generally, in practice, $N(A,\omega)$ does not depend on ω. Therefore, a graphical method which is based on the intersections of $G(j\omega)$ and $-1/N(A)$ in the complex plane can be used to solve Eq. (11). From Eq. (11), the following equation can be obtained

$$G(j\omega) = -\frac{1}{N(A)} \qquad (12)$$

Thus, the possible (A,ω) solutions can be investigated by plotting $G(j\omega)$ and $-1/N(A)$ together. If intersections exist, the system may have limit cycles at (A,ω) corresponding to the intersection points. The stability of the limit cycle can be assessed by applying the Nyquist criterion. In this case, the single $(-1,0)$ critical point is replaced by a locus of critical points, which are given by $-1/N(A)$.

Since the describing function technique is an approximate method, it may give inaccurate results. The accuracy of the describing function method depends on two factors which are the distortion produced by the nonlinearity assuming a sinusoidal input and the frequency characteristic of the linear element. Therefore, the results which are produced by the describing function method can be relied upon if the linear subsystem is sufficiently low pass.

4. NONLINEAR UNCERTAIN SYSTEMS

In this section, it is assumed that the nonlinear system of Figure 1 has a fixed linear transfer function and an uncertain describing function for the nonlinear element. Using the segment lemma, an exact stability result is presented. As was mentioned before, within the context of uncertain systems, the describing function analysis of uncertain systems has been investigated in (Ferreres and Fromion, 1998; Fadali and Chachavalvoong, 1995; Impram and Munro, 1998; Tan and Atherton, 1999). However, it is interesting to point out that in all of these papers, it has been assumed that both linear and nonlinear elements include parametric uncertainty. The stability analysis of such a system is generally difficult due to multilinear uncertainty structure of the resultant characteristic equation. Converting a multilinear uncertainty structure to an interval polynomial structure gives conservative results.

Since Eq. (10) is a scalar quantity, an uncertain describing function can be written as

$$N(A,k) = [\underline{k_r}, \overline{k_r}] + j[\underline{k_i}, \overline{k_i}] \qquad (13)$$

For a memoryless nonlinearity, the interval describing function is

$$N(A,k) = [\underline{k_r}, \overline{k_r}] \qquad (14)$$

Assume that the linear part of the nonlinear system of Figure 1 is a fixed transfer function of the form

$$G(s) = \frac{N(s)}{D(s)} \qquad (15)$$

Then, from Eq. (11), the characteristic polynomial of the system

$$1 + N(A,k)G(s) = D(s) + N(A,k)N(s) = 0 \qquad (16)$$

should be Hurwitz stable for the stability of the nonlinear system of the form given Figure 1. Eq. (16) can be represented by a segment of polynomials which has the following end point polynomials:

$$v_1(s) = D(s) + \underline{k}_r N(s)$$
$$v_2(s) = D(s) + \overline{k}_r N(s) \qquad (17)$$

Thus, the segment which represents Eq. (16) can be written as

$$(1-\lambda)v_1(s) + \lambda v_2(s) \qquad (18)$$

where $\lambda \in [0,1]$. Then, from the segment lemma, the stability of nonlinear system can be checked as follows: Let $v_1(s)$ and $v_2(s)$ be stable polynomials. Then the line segment $(1-\lambda)v_1(s) + \lambda v_2(s)$ is Hurwitz stable for all $\lambda \in [0,1]$ if and only if there exists no real $\omega > 0$ such that all of the following three conditions are met

$$v_1^e(\omega)v_2^o(\omega) - v_2^e(\omega)v_1^o(\omega) = 0$$
$$v_1^e(\omega)v_2^e(\omega) \le 0$$
$$v_1^o(\omega)v_2^o(\omega) \le 0 \qquad (19)$$

where ($v_1^e(\omega)$, $v_1^o(\omega)$) and ($v_2^e(\omega)$, $v_2^o(\omega)$) are the even and odd parts of $v_1(s)$ and $v_2(s)$, respectively.

Example 1

Consider the nonlinear system of Figure 1 and assume that the nonlinearity is as shown in Figure 2. Its describing function is

$$N(A) = (k_1 - k_2)\, f(\frac{\delta}{A}) + k_2 \qquad (20)$$

where $f(\frac{\delta}{A}) \in [0,1]$, $A > \delta$. If $k_1 > k_2$ then $N(A) \in [k_2, k_1]$ and if $k_2 > k_1$ then $N(A) \in [k_1, k_2]$.

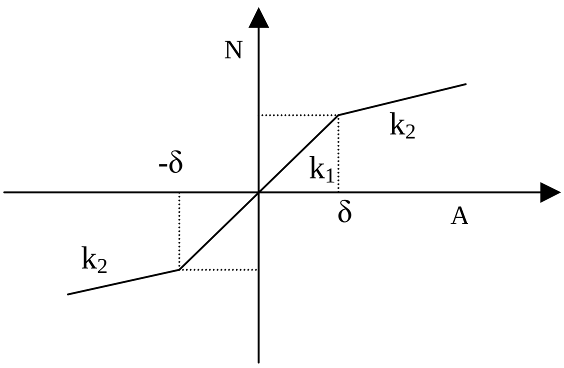

Figure 2: Characteristic of saturation nonlinearity

Now, assume that the nonlinearity of Figure 1 is a saturation with $k_1 = 2$, $k_2 = 0$ and $\delta = 1$ and the transfer function of the linear part is given by

$$G(s) = \frac{as^2 + bs + c}{ds^4 + es^3 + fs^2 + gs + h} \qquad (21)$$

where the nominal values of the parameters are: $a_0 = 1$, $b_0 = 3$, $c_0 = 60$, $d_0 = 1$, $e_0 = 3$, $f_0 = 35$, $g_0 = 40$ and $h_0 = 50$. Hence the nominal transfer function of the linear element is

$$G(s) = \frac{s^2 + 3s + 60}{s^4 + 3s^3 + 35s^2 + 40s + 50} \qquad (22)$$

From Eq. (20), the uncertain describing function is $N(A,k) \in [0,2]$. Then the characteristic equation of the system can be written as

$$1 + N(A,k)G(s) = D(s) + N(A,k)N(s) = s^4 +$$
$$3s^3 + 35s^2 + 40s + 50 + [0,2](s^2 + 3s + 60) = 0 \qquad (23)$$

From Eq. (17),

$$v_1(s) = s^4 + 3s^3 + 35s^2 + 40s + 50$$
$$v_2(s) = s^4 + 3s^3 + 37s^2 + 46s + 170 \qquad (24)$$

Thus, the stability of the nonlinear system is equivalent to the stability of the line segment $(1-\lambda)v_1(s) + \lambda v_2(s)$ where $\lambda \in [0,1]$. To apply the segment lemma it is necessary to find the positive real roots of the polynomial

$$v_1^e(\omega)v_2^o(\omega) - v_2^e(\omega)v_1^o(\omega) = 190\omega^2 - 4500 = 0$$

There is one positive real root which is $\omega = 4.8666$. However, for $\omega = 4.8666$ it can be seen that
$v_1^e(\omega)v_2^e(\omega) = 3.17x10^4 > 0$
and
$v_1^o(\omega)v_2^o(\omega) = 777.88 > 0$.

Therefore, the nonlinear system is stable. The frequency response of $G(j\omega)$ and describing function $-1/N(A,k)$ are shown in Figure 3. From Figure 3, it is seen that the describing function plot does not intersect with $G(j\omega)$. Hence, the system is asymptotically stable as concluded using the segment lemma.

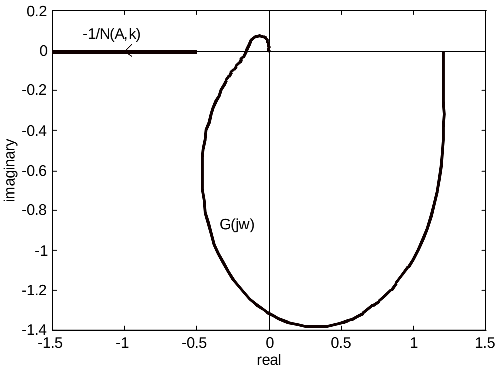

Figure 3: Graphical prediction of limit cycle

5. STABILITY MARGIN COMPUTATION

A main problems in robustness of control systems is to find the maximum allowable perturbation bounds of parameters of a system while preserving stability. The interest in this area has greatly increased since the publication of the Kharitonov theorem. Although the stability of an interval system can be checked by Kharitonov's test, there is no direct indication as to what extent the bounds of parameters can be increased before the system becomes unstable. In this section, the answer of "how much can we perturb the coefficients of the linear subsystem of Figure 1 while simultaneously preserving the stability of the nonlinear system?" or "how much can we perturb the coefficients of the linear subsystem of Figure 1 without forcing the nonlinear system to have a limit cycle?" is given. Two approaches are presented. The first one is based on the robust stability of uncertain polynomials and the next one is based on the Nyquist envelope of interval transfer functions.

5.1 Polynomial Approach

Consider the nominal transfer function of the linear part of Figure 1 as

$$G(s) = \frac{x_m s^m + x_{m-1} s^{m-1} + ... + x_0}{y_n s^n + y_{n-1} s^{n-1} + ... + y_0} \tag{25}$$

and assume that $N(A,k) = [\underline{k_r}, \overline{k_r}]$. Then, the open loop transfer function of the system is

$$G(s)N(A,k) = \frac{[\underline{k_r} x_m, \overline{k_r} x_m] s^m + \cdots + [\underline{k_r} x_0, \overline{k_r} x_0]}{y_n s^n + y_{n-1} s^{n-1} + \cdots + y_0} \tag{26}$$

Assume that the nonlinear system with $N(A,k) = [\underline{k_r}, \overline{k_r}]$ and $G(s)$ of Eq. (25) is Hurwitz stable. Given any allowable variations in the coefficients $\varepsilon > 0$, Eq. (26) can be written with an interval transfer function as

$$G(s,a,b) = \frac{b_m s^m + b_{m-1} s^{m-1} + ... + b_0}{a_n s^n + a_{n-1} s^{n-1} + ... + a_0} \tag{27}$$

where $b_i \in [\underline{b_i}, \overline{b_i}]$, $\underline{b_i} = \underline{k_r} x_i - \varepsilon$, $\overline{b_i} = \overline{k_r} x_i + \varepsilon$, $i = 0,1,2,....,m$ and $a_i \in [\underline{a_i}, \overline{a_i}]$, $\underline{a_i} = y_i - \varepsilon$, $\overline{a_i} = y_i + \varepsilon$, $i = 0,1,2,....,n$. The closed loop characteristic equation of the system is

$$\delta(s,a,b) = \sum_{i=m+1}^{n} a_i s^i + \sum_{i=0}^{m} (a_i + b_i) s^i \tag{28}$$

which is an interval polynomial. From the Kharitonov theorem, assume that $\delta_1(s)$, $\delta_2(s)$, $\delta_3(s)$ and $\delta_4(s)$ are Kharitonov polynomials of Eq. (28). Thus, for any $\varepsilon > 0$, if all these four Kharitonov polynomials are Hurwitz stable then the stability of the perturbed system is guaranteed. If the maximum perturbations of $\delta_1(s)$, $\delta_2(s)$, $\delta_3(s)$ and $\delta_4(s)$ to retain stability are respectively ε_1, ε_2, ε_3 and ε_4, then from

$$\varepsilon_{\max} = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} \tag{29}$$

the maximum allowable perturbation can be determined.

5.2 Graphical Approach

Since the outer boundary of the Nyquist envelope of a proper interval plant is covered by the Nyquist plots of the sixteen Kharitonov plants, it is possible to find $\varepsilon_{\max}$ graphically. For any variations in the coefficients of Eq. (25), say $\varepsilon > 0$, the interval plant representations of Eq. (27) can be written. Then, from the Nyquist plots of the Kharitonov plants, it can be checked whether if there is any intersection between $-1/N(A,k)$ and the Nyquist plots of the Kharitonov plants. If there is no any intersection then increase the value of ε and repeat the process until there is intersection. The value of ε for which the Nyquist plots of the Kharitonov plants and $-1/N(A,k)$ start to intersect is equal to $\varepsilon_{\max}$.

Example 2

Consider the nonlinear system given in Example 1 which has been shown not to possess a limit cycle. In this example, it is aimed to find how much one can perturb the parameters (a, b, c, d, e, f, g, h) of the transfer function of the linear element of Eq. (22) around their nominal values while preserving stability. Using the polynomial approach given above, it was computed that the value of $\varepsilon_{\max}$ is equal to 0.385. For this value of perturbation, the Nyquist plots of the Kharitonov plants and

$-1/N(A,k)$ are shown in Figure 4. From Figure 4, it can be observed that the polynomial approach gives a conservative result since $-1/N(A,k)$ does not touch the Nyquist plots of the Kharitonov plants. The conservative nature of the polynomial approach comes from the fact that since the system has an uncertain describing function, any parameter variation in a linear element gives an open loop transfer function with a multilinear uncertainty structure. However, the multilinear uncertainty structure is converted to the interval uncertainty structure via Eq. (27). On the other hand, the graphical approach gives an exact result. Using the graphical method, it was computed that $\varepsilon_{\max}$ is equal to 0.42. Using this value, the Nyquist plots of the Kharitonov plants and $-1/N(A,k)$ are shown in Figure 5 where it can be seen that $-1/N(A,k)$ touches the Nyquist plots of the Kharitonov plants.

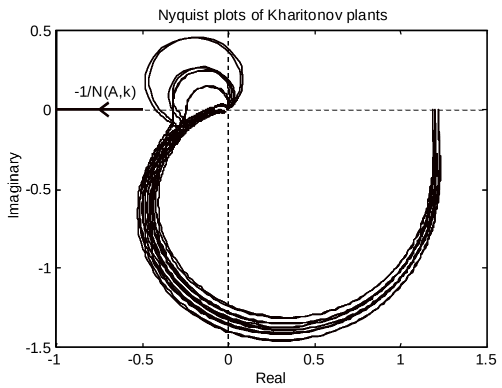

Figure 4: The Nyquist plots of the Kharitonov plants and –1/N(A,k) (polynomial approach)

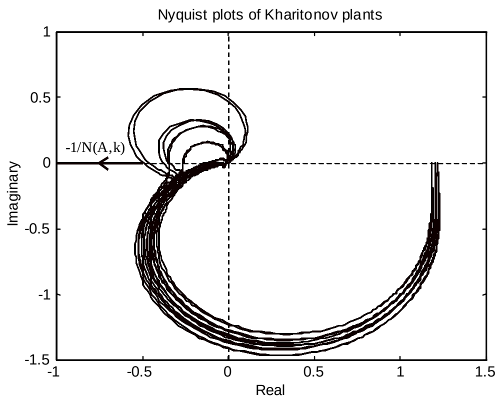

Figure 5: The Nyquist plots of the Kharitonov plants and –1/N(A,k) (graphical approach)

6. CONCLUSION

In this paper, the describing function analysis of nonlinear systems with parametric uncertainty has been studied. It has been first shown that the characteristic equation of a nonlinear system with a fixed linear element and an uncertain describing function is a line segment of a polynomial whose stability can be checked by the segment lemma. Then, two methods have been presented for stability margin computation of nonlinear uncertain systems. The first one is based on the robust stability of uncertain polynomials. The second method is a graphical method which is based on the Nyquist envelope of interval transfer functions.

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