LOCAL STATIC OUTPUT FEEDBACK STABILIZATION OF A CLASS OF MINIMUM-PHASE NONLINEAR SYSTEMS

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Abstract: The problem of (local) static output feedback stabilization for a class of minimum phase relative degree two systems is studied. It is shown that, under natural assumptions, it is possible to design a stabilizing static output feedback control law and to give an estimate of the resulting region of attraction. The general results are illustrated via examples.

Keywords: Nonlinear control, output feedback stabilization, minimum phase systems

1. INTRODUCTION

In this work, the problem of local static output feedback stabilization of single-input single-output systems described by equations of the form

$$\dot{z} = f(z, \xi_1, \dots, \xi_r)
\dot{\xi}_1 = \xi_2
\vdots
\dot{\xi}_r = \tilde{F}(z, \xi_1, \dots, \xi_r) + \tilde{G}(z, \xi_1, \dots, \xi_r) v
y = \xi_1$$
(1)

in which $z \in \mathbb{R}^{n-r}$, $\xi_i \in \mathbb{R}$ for all $i = 1, \ldots, r$, $v \in \mathbb{R}$ is the control input, $f(z, \xi_1, \ldots, \xi_r)$ is a smooth vector valued function such that $f(0, \ldots, 0) = 0$, $\tilde{F}(z, \xi_1, \ldots, \xi_r)$ and $\tilde{G}(z, \xi_1, \ldots, \xi_r)$ are smooth scalar functions such that $\tilde{F}(0, \ldots, 0) = 0$ and $\tilde{G}(0, \ldots, 0) = 0$, is studied. System (1) has been widely studied in the nonlinear control community. Necessary and sufficient conditions for global equivalence of a general single-input nonlinear system to the form (1) has been given in (Byrnes and Isidori,

1991), whereas semiglobal stabilization, using partial state feedback, has been studied in (Byrnes and Isidori, 1991; Teel, 1996). In (Sussmann, 1990) it has been shown that, if r > 1 global stabilization either with full state feedback or with partial state feedback cannot be achieved in general. This limitation is associated with the presence of the peaking phenomenon, discussed in detail in (Sussmann and Kokotovic, 1991), and with the fact that the z subsystem with the ξ_i regarded as input may not be Inputto-State-Stable. In the case r = 1 semiglobal stabilization can be achieved by high gain static output feedback (Byrnes and Isidori, 1991), provided that the system $\dot{z} = f(z,0)$ is globally asymptotically stable; whereas global stabilizers can (in principle) be constructed using a simple back-stepping argument (Krstic *et al.*, 1995). In the case r = 2 global asymptotic stabilization using full state feedback has been studied in (Astolfi, 1998).

The case in which the z sub-system is described by an equation of the form $\dot{z} = f(z, C\xi)$ for some C, has been widely studied in (Kokotovic and Sussmann, 1989; Saberi *et al.*, 1990), whereas the case in which the z sub-system is driven only by one of the ξ_i

 $^{^{1}\,}$ This work has been partly supported by the CEC TMR-Network NACO2.

variables, *i.e.* $\dot{z} = f(z, \xi_i)$ for some *i*, has been studied in (Isidori, 1995, Chapter 9). Finally, some analysis tools for the considered class of systems have been given in (Teel, 1996) and the problem of semiglobal (robust) stabilization using dynamic output feedback has been recently solved in (Isidori, 2000).

The present work has been motivated by some of the results in (Byrnes and Isidori, 1991; Byrnes et al., 1991). Therein, among others, the problem of semiglobal stabilization by means of high gain partial state feedback for system (1) has been studied, and it has been shown that, if r = 1 semiglobal stabilization by static output feedback is achievable. Here the problem of static output feedback stabilization for system (1) in the case of r = 2 is addressed. In the case of linear systems it is well known that this problem is solvable by high gain feedback if (and only if) the socalled center of the asymptotes of the root-locus lies in the left half of the complex plane. This interpretation does not have an immediate nonlinear counterpart. Nevertheless, it will be shown that, under suitable assumptions, a nonlinear notion of center of the asymptotes can be given. Finally, contrary to the results in (Byrnes and Isidori, 1991; Byrnes et al., 1991), but in agreement with the conclusions in (Sussmann, 1990), in the present context only local asymptotic stability can be achieved. However together with the stabilizing control law we will provide an estimate for the domain of attraction of the zero equilibrium. More precisely, the following problem is studied.

Problem 1. Given system (1) with r = 2. Find (if possible) a static output feedback control law v = -K(y) and a set $\Omega \ni (0, 0, \dots, 0)$, such that

- the origin is a locally asymptotically (exponentially) stable equilibrium of the closed loop system;
- for any initial conditions (z⁰, ξ⁰) ∈ Ω, the corresponding trajectory converges to the origin.

Throughout the paper, the following standing assumptions will be made.

Assumption 1. The system $\dot{z} = f(z, 0, 0, ..., 0)$ has z = 0 as a globally asymptotically, locally exponentially, stable equilibrium and we know a smooth positive definite and proper function V(z) and positive constants $\alpha_1, \alpha_2, \delta$ and γ , such that

$$\begin{aligned} \alpha_1 \, \|z\|^2 &\leq V(z) \leq \alpha_2 \, \|z\|^2, \\ V_z f(z, 0, 0, \dots, 0) &\leq -\delta \|z\|^2, \\ \|V_z\| &\leq \gamma \|z\|. \end{aligned}$$

If this condition is fulfilled, the system (1) is said to be globally (exponentially) minimum phase.

Assumption 2. There are smooth functions $\psi_1(\xi_1)$, $\psi_2(\xi_1)$, and $\beta(z, \xi_1, \xi_2)$ such that

$$\dot{F}(z,\xi_1,\xi_2) = -\beta(z,\xi_1,\xi_2)\xi_2 + \psi_1(\xi_1),$$
$$\tilde{G}(z,\xi_1,\xi_2) = \psi_2(\xi_1)$$

with $\psi_2(\xi_1) \neq 0$ for all ξ_1 . Moreover, there exists $\underline{\beta} > 0$ such that $\beta(z, \xi_1, \xi_2) \geq \underline{\beta} > 0$ for all z, ξ_1 and ξ_2 .

If this assumption is satisfied, there exists a static output feedback such that the resulting controlled system is described by the equations

$$\dot{z} = f(z, \xi_1, \xi_2)
\dot{\xi}_1 = \xi_2
\dot{\xi}_2 = -\beta(z, \xi_1, \xi_2)\xi_2 + u
y = \xi_1.$$
(2)

Moreover, note that, under mild regularity conditions, it is always possible to factor (in a non-unique way) the term $f(z, \xi_1, \xi_2)$ as $f_0(z) + F(z, \xi_1, \xi_2)\xi$.

Assumption 3. The vector $F(z, \xi_1, \xi_2)$ is such that, for some known k_0 and $k_1(||z||)$,

$$||F(z,\xi_1,\xi_2)|| \le k_0 + k_1(||z||) ||\xi||.$$
(3)

Remark 1. Under the stated Assumptions, system (2) with u = 0 is locally stable, whereas, system (2) with u = -Ky and K > 0 is locally exponentially stable. However, goal of the paper, as said in the statement of Problem 1, is to provide an estimate of the region of attraction of the origin, as a function of K.

Remark 2. Note that condition (3) is imposed only for convenience of exposition and to obtain simple and workable formulae. All results of the paper can be extended to the more general case in which, for some integer $p \ge 0$,

$$||F(z,\xi_1,\xi_2)|| \le k_0 + k_1(||z||) ||\xi||^p$$

The present paper is organized as follows. In Section 2 the meaning of Assumption 2 is considered more precisely. In Section 3 a way to construct a feedback law

$$u = -Ky \tag{4}$$

achieving local (exponential) stability of the zero equilibrium of the closed loop system (2)-(4) and to derive an estimate Ω of its domain of attraction is given. Finally, in Section 4 some examples illustrate the main results and in Section 5 concluding remarks are provided.

2. ON ASSUMPTION 2

In this section the assumption on the function $\beta(z, \xi_1, \xi_2)$ to be positive is briefly motivated.

Lemma 1. Consider system (2) and assume the system is linear. Then the system is output feedback stabilizable by high gain output feedback if and only if $\beta(z, \xi_1, \xi_2) = \beta > 0$.

Proof. As the system (2) has relative degree two and positive high frequency gain, the positive root locus will have one asymptote parallel to the imaginary axis. A simple computation shows that this asymptote intersects the real axis at the point $-\beta/2$, hence the claim.

In the case of nonlinear systems the above root locus argument does not carry over. However, using the general results in (Astolfi and Colaneri, 2000), a necessary condition for (local) static output feedback stabilizability for system (2) is as follows.

Lemma 2. Consider system (2). The system is static output feedback stabilizable only if there exists a positive definite function $W(z, \xi_1, \xi_2)$ such that, for any (z, ξ_2) , $\dot{W} \mid_{y=0} = \dot{W} \mid_{\xi_1=0} \le 0$.

Despite its simplicity, the above condition cannot be easily tested. However, if the study is restricted to a special class of positive definite functions, it yields a simpler result.

Lemma 3. Consider system (2). Suppose

$$W(z,\xi_1,\xi_2) = V(z) + \xi' P\xi,$$

where $\xi = col(\xi_1, \xi_2)$, V(z) is the function given in Assumption 1,

$$P = \begin{bmatrix} \frac{1}{\alpha} + dK + \frac{\alpha}{K} & \frac{1}{K} \\ \frac{1}{K} & \frac{1}{K\alpha} + d \end{bmatrix}, \quad (5)$$

with K > 0, $\alpha > 0$ and d > 0. Then

$$\dot{W}|_{y=0} = \dot{W}|_{\xi_1=0} \le 0$$

implies $\beta(0, 0, \xi_2) > 0$.

Remark 3. It is worth noting that the assumption $\beta(z, \xi_1, \xi_2) > 0$ is stronger then the condition given in the above Lemma.

3. MAIN RESULTS

This section explains how to construct a locally stabilizing control law and how to estimate its domain of attraction. The construction is broken up in four steps.

- It is shown that W, with W(·) as in Lemma 3, can be upper bounded by a function independent from the parameter K.
- An estimate of the level sets of $W(\cdot)$ as a function of K is given.

- An estimate of the set Γ in which $\dot{W} < 0$ is given.
- The set Ω (whenever it exists) is constructed.

3.1 Upper bound on \dot{W}

Consider the positive definite (and proper) function

$$W(z,\xi_1,\xi_2) = V(z) + \xi' P\xi$$

where V(z) is as in Assumption 1 and P is as in Lemma 3, *i.e.*

$$P = \begin{bmatrix} \frac{1}{\alpha} + dK + \frac{\alpha}{K} & \frac{1}{K} \\ \frac{1}{K} & \frac{1}{K\alpha} + d \end{bmatrix}$$

with $\alpha > 0$, d > 0 and K > 0. A simple computation shows that

$$\dot{W} = V_z \dot{z} - 2 \|\xi_1\|^2 + \left(\frac{2}{K} - 2\beta(z,\xi_1,\xi_2)(\frac{1}{K\alpha} + d)\right) \xi_2^2.$$
(6)

The next result shows that, with a proper selection of α and d it is possible to upper bound \dot{W} with a function which does not depend on K, provided K is sufficiently large.

Proposition 1. Consider \dot{W} in equation (6). Then, for any $0 < \eta < 2$ there exists $K_0 > 0$ such that for any $K > K_0, \dot{W} - V_z \dot{z} \le -(2 - \eta) ||\xi||^2$.

Remark 4. Note that the bound on $\dot{W} - V_z \dot{z}$ does not depend upon K, that K_0 depends on d and that d is a free parameter, provided that $d > 1/\beta$.

If $\beta(z, \xi_1, \xi_2)$ is constant, then it is possible to obtain a sharper bound, as detailed in the following statement.

Corollary 1. Suppose $\beta(z, \xi_1, \xi_2) = \underline{\beta} > 0$. Then if $d = \frac{1}{\beta}, \dot{W} = V_z \dot{z} - 2||\xi||^2$.

3.2 Estimate of the level sets of W

Consider the candidate Lyapunov function

$$W(z,\xi_1,\xi_2) = V(z) + \xi' P\xi$$

. The level sets of W are the sets

$$\{(z,\xi) \in (I\!\!R^{n-2}, I\!\!R^2) / W(z,\xi_1,\xi_2) = c^2/2\}$$

where c > 0 is a constant. For any (z, ξ_1, ξ_2) , one has

$$V(z) + \lambda_{min}(P) \|\xi\|^2 \le W(z, \xi_1, \xi_2)$$
$$W(z, \xi_1, \xi_2) \le V(z) + \lambda_{max}(P) \|\xi\|^2.$$

Thus, the boundary of the level sets of W is enclosed in the region described by the two inequalities

$$V(z) + \lambda_{min}(P) \|\xi\|^2 \le c^2/2 V(z) + \lambda_{max}(P) \|\xi\|^2 \ge c^2/2$$
(7)

Note now that, following the conclusions in Proposition 1, K has to be larger than a certain K_0 and that $\lambda_{min}(P)$ and $\lambda_{max}(P)$ are functions of K and d. Therefore, for any fixed d, there exists K^* such that the inequalities (7) with $K = K^*$ yield the largest possible level set for W.

For simplicity, assume that $V(z) = ||z||^2/2$. As a result, the sets described by inequalities (7) are ellipses. $\lambda_{max}(P)$, as functions of K, has a global minimum, which depends on d. It is therefore possible to set K^* and d^* such that

$$\lambda_{max}(P(K^{\star}, d^{\star})) = \min_{d > 1/\underline{\beta}, \ K > K_0(d)} \lambda_{max}(P(K, d)).$$

Note that this selection has the goal to render the region given by the second of the inequalities (7) as large as possible. However, if the above (global) minimization problem is not solvable, any pair (K, d) allows to compute a subset of the region of attraction of the origin.

3.3 *Estimate of the set* Γ *where* $\dot{W} < 0$

In this section an estimate of the set where \dot{W} is negative is given. By Proposition 1, \dot{W} can be upper bounded by $\dot{W}(z,\xi_1,\xi_2) \leq V_z \dot{z} - (2-\eta) ||\xi||^2$, with $0 < \eta < 2$ and $\dot{z} = f_0(z) + F(z,\xi_1,\xi_2)\xi$. Moreover, by Assumptions 1 and 3, one has

$$\dot{W}(z,\xi_1,\xi_2) \le -\delta ||z||^2 + \gamma ||z|| k_0 ||\xi|| + \gamma ||z|| k_1(||z||) ||\xi||^2 - (2 - \eta) ||\xi||^2 = \Lambda(z,\xi).$$

 $\Lambda(z,\xi)$ can be seen as a quadratic function in $||\xi||$. Thus, if its discriminant θ is non-positive for $||z|| = ||z^*||$ then $\Lambda(z,\xi)$ is non-positive for $||z|| = ||z^*||$ and all ξ , whereas if θ is positive for some ||z||, $\Lambda(z,\xi) = 0$ is equivalent to

$$\|\xi\| = \frac{-\gamma \|z\| k_0 \pm \sqrt{\theta}}{2(-(2-\eta) + \gamma \|z\| k_1(\|z\|))}$$

where the constants γ , δ , η , k_0 and the function $k_1(||z||)$ are known. As a result, it is trivial to determine the regions of the plane $(||z||, ||\xi||)$ where $\Lambda(z, \xi)$ is negative, positive or equal to zero. Note finally, that

$$\Phi = \{(z,\xi) \in \mathbb{R}^{n-2} \times \mathbb{R}^2 \mid \Lambda(z,\xi) < 0\} \subseteq \{(z,\xi) \in \mathbb{R}^{n-2} \times \mathbb{R}^2 \mid \dot{W} < 0\} = \Gamma.$$

3.4 Construction of the set Ω

In this section an estimate Ω of the region of attraction of the zero equilibrium of the system (2) with u = -Ky is provided.

Such a region of attraction can be characterized as the largest level set of W contained in the set Γ , defined in Subsection 3.3. However, as only estimates of the level set of W and of Γ are available, the set Ω has to be defined as

$$\{(z,\xi_1,\xi_2) \mid V(z) + \lambda_{max}(P(K)) \|\xi\|^2 \le (c^*)^2/2\},\$$

where $c^* > 0$ is the maximum value of c such that

$$\{(z,\xi_1,\xi_2) \mid V(z) + \lambda_{min}(P(K)) \|\xi\|^2 \le c^2/2\} \subset \Phi.$$

Note that c^* may be zero, see Examples 3 and 4, hence it is not possible to give an estimate of the region of attraction of the origin. Instead, if $c^* > 0$, then Ω is such that for all initial conditions $(z^0, \xi^0) \in \Omega$ the trajectories of the closed loop system remain in Ω and converge to zero.

Remark 5. Note that Ω is a simple to compute estimate of the region of stability of the closed loop system. This implies that the real region of attraction is larger. However, the exact characterization of the latter is in general impossible. Finally, in some cases (see Example 1) it is possible to compute precisely the set Γ , *i.e.* the set where \dot{W} is negative, hence a more accurate estimate of the region of attraction of the zero equilibrium can be obtained.

4. SOME EXAMPLES

Some examples now illustrate the results developed. The first two examples are adapted from (Sussmann, 1990).

Example 1. Consider the system

$$\dot{z} = -z + z^3 ||\xi||^2
\dot{\xi}_1 = \xi_2
\dot{\xi}_2 = -\xi_2 + u
y = \xi_1$$
(8)

Let $W(z, \xi_1, \xi_2) = V(z) + \xi' P \xi$, with $V(z) = z^2/2$ and P as in Lemma 3, with $\alpha = 1$ and d = 1. Note moreover that $\beta = 1$. Thus, it is easily shown that

$$\dot{W} = -z^2 + z^4 ||\xi||^2 - 2||\xi||^2.$$

Note now that $\dot{W} = 0$ is equivalent to

$$\|\xi\| = \frac{\|z\|}{\sqrt{z^4 - 2}},$$

when $||z|| > 2^{1/4}$. Hence, the set Γ can be easily computed.

To compute the level sets of W, note that $K_0 = 0$ and that the eigenvalues of P(K, 1) are given by

$$\lambda_{\min,\max}(K) = \frac{2K + K^2 + 2 \pm \sqrt{K^4 + 4}}{2K}$$

As a result, λ_{max} reaches its minimum for $K = \sqrt{2}$. Thus, any level surface of W with $K = \sqrt{2}$ is between the two ellipses defined by the equations

$$z^{2}/2 + \lambda_{min}(\sqrt{2}) \|\xi\|^{2} \le c^{2}/2$$
$$z^{2}/2 + \lambda_{max}(\sqrt{2}) \|\xi\|^{2} \ge c^{2}/2,$$

for some c > 0. To find Ω we need to compute the largest value of c, *i.e.* c^* , such that the ellipse described by the equation

$$z^2/2 + \lambda_{min}(\sqrt{2}) \|\xi\|^2 \le c^2/2$$

is enclosed in the set Γ . A simple computation shows that $c^* \approx 2.035$, and this allows to compute explicitly Ω , *i.e.*

$$\Omega = \{ (z,\xi) \in \mathbb{R} \times \mathbb{R}^2 | \frac{z^2}{2} + \lambda_{max}(\sqrt{2}) ||\xi||^2 \le c^{\star^2}/2 \}.$$

Example 2. Consider the system

$$\dot{z} = (z\xi_1 - 1) z^3 + (z\xi_1 + \xi_2^2 - 1) z
\dot{\xi}_1 = \xi_2
\dot{\xi}_2 = -\xi_2 + u
y = \xi_1$$
(9)

Let $W(z, \xi_1, \xi_2) = V(z) + \xi' P \xi$, where $V(z) = z^2/2$, P is as in Lemma 3, and $\alpha = 1$ and d = 1. Then, \dot{W} can be upper bounded, using Young's inequality, by the function $\Lambda(z,\xi) = -z^2 + z^8 + 2z^2 ||\xi||^2 - 2||\xi||^2$, which is non positive on the set $\Phi = \{(z,\xi) \mid ||z|| \leq 1\}$. The level sets of W have the same structure as in Example 1. Hence, to compute the set Ω we only need to find c^* . For, note that the set

$$\{(z,\xi_1,\xi_2) \mid z^2/2 + \lambda_{min}(\sqrt{2}) \|\xi\|^2 \le c^2/2\}$$

has to be enclosed in Φ , consequently $c^* = 1$. As a result

$$\Omega = \{ (z,\xi) \in I\!\!R \times I\!\!R^2 | \frac{z^2}{2} + \lambda_{max}(\sqrt{2}) ||\xi||^2 \le 1/2 \}.$$

Remark 6. For the system considered in this example, it has been proved in (Sussmann, 1990) that, no matter what control is selected, all points (z, ξ_1, ξ_2) such that $z\xi_1 \ge 2$ are *explosive* points. Obviously, these points cannot be part of the region of attraction of the origin. Consistently with this observation, it is easy to see that

$$\{(z,\xi_1,\xi_2) \mid z\xi_1 \ge 2\} \cap \Omega = \emptyset$$

For both Examples 1 and 2, simulations confirm that for any initial conditions $(z^0, \xi^0) \in \Omega$, the trajectories of the closed loop system remain in Ω and converge to zero. Moreover, there are points close to Ω but not in Ω yielding trajectories that converge to the origin, thus showing that the real region of attraction is larger than Ω . However, all points sufficiently away from Ω result in trajectories that do not converge to the origin.

The next two examples illustrate the fact that Ω may be empty. Note that this does not mean that the closed loop system is not locally exponentially stable (around zero), but simply that the Lyapunov argument discussed in the paper is not applicable.

Example 3. Consider the system

$$\dot{z} = -z + (4 + ||z||) ||\xi||
\dot{\xi}_1 = \xi_2
\dot{\xi}_2 = -2\xi_2 + u
y = \xi_1$$
(10)

Let $W(z, \xi_1, \xi_2) = V(z) + \xi' P \xi$, with $V(z) = z^2/2$ and P as before, and $\alpha = 2$ and d = 1/2. Then

$$\dot{W} = -z^2 + (4 + ||z||) |z||\xi|| - 2||\xi||^2,$$

and $\dot{W} = 0$ is equivalent to

$$\|\xi\| = \frac{(4+\|z\|) \ z \pm \ \|z\| \ \sqrt{(4+\|z\|)^2 - 8}}{4}.$$

Note now that $(4 + ||z||)^2 - 8$ is always positive, hence in any neighborhood of the origin there are points in which $\dot{W} > 0$. As a result $\Omega = \emptyset$.

Remark 7. A simple analysis shows that all trajectories of the system in closed loop with u = -Ky, and K > 0, converge to the origin. This is a consequence of the fact that the \dot{z} equation is globally Lipschitz in z, hence trajectories exist for all time. To have a better understanding of the behavior of the closed loop system note that for any initial conditions, ξ_1 and ξ_2 converge exponentially to zero. Hence, the \dot{z} equation can be rewritten as $\dot{z} = -\rho(z,t)z + 4r(t)$, where $r(t) \geq 0$ for all t, and r(t) converges exponentially to zero. This system is a time varying system, driven by an exponentially decaying signal, with the property that $\lim_{t\to\infty} \rho(z,t) = 1$, where $\rho(z,t) = 1 - 1$ sign(z)r(t). Hence z converges exponentially to the origin. However, the system is not uniformly stable, *i.e.* the trajectories may have large overshots.

Example 4. Consider the system

$$\dot{z} = -z + (4 + ||z||^2) ||\xi||
\dot{\xi}_1 = \xi_2
\dot{\xi}_2 = -2\xi_2 + u
y = \xi_1$$
(11)

Let $W(z, \xi_1, \xi_2) = V(z) + \xi' P \xi$, with $V(z) = z^2/2$ and P as before, and $\alpha = 2$ and d = 1/2. Then \dot{W} is as in Example 3. As a result $\Omega = \emptyset$.

Remark 8. Unlike Example (3), the \dot{z} equation is not globally Lipschitz in z. Thus, not all trajectories of the closed loop system converge to the origin, or exist for all time. This conclusion can be also confirmed with simulations.

To conclude the section, an example that does not satisfy Assumption 1 is given.

Example 5. Consider the system

$$\dot{z} = -z^{3} - qz^{5}\xi_{1}\xi_{2}
\dot{\xi}_{1} = \xi_{2}
\dot{\xi}_{2} = -\xi_{2} + u
y = \xi_{1}$$
(12)

with q > 0. Note that the system is not exponentially minimum phase. To apply the methodology described above, let $W(z, \xi_1, \xi_2) = V(z) + \xi' P \xi$, where $V(z) = z^2/2$ and P is as in Lemma 3, and $\alpha = 1$ and d = 1. Then, \dot{W} can be upper bounded, using Young's inequality, by the function

$$\Lambda(z,\xi) = -z^4 + \frac{q}{2}z^6 ||\xi||^2 - 2||\xi||^2$$

Using similar considerations as in Example 1, and setting, for simplicity, q = 1/10, we conclude that

$$\begin{split} \Omega &= \{(z,\xi) \in I\!\!R \times I\!\!R^2 \,|\, \frac{z^2}{2} + (2 + \sqrt{2}) \; \|\xi\|^2 \leq \frac{(c^\star)^2}{2} \}, \\ \text{with } c^\star &\approx 4. \end{split}$$

5. CONCLUSION

The problem of local static output feedback stabilization for a certain class of nonlinear minimum phase relative degree two systems has been addressed. It has been shown that, under suitable assumptions, it is possible to find a feedback control law u = -Kyand a set Ω of initial conditions, such that the closed loop system is locally exponentially stable and Ω is an estimate of the region of attraction of the zero equilibrium. The general theory has been illustrated via examples.

Note finally that all results have been stated assuming that the system is in normal form. However, coordinates free description of the results can be also given. This issue is left for further investigation.

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