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# EXPLICIT CHARACTERIZATION OF DECENTRALIZED COPRIME FACTORS

**Noboru SEBE** *,[1]

* *Department of Artificial Intelligence,*
*Kyushu Institute of Technology,*
*680-4, Kawazu, Iizuka, Fukuoka, 820-8502 Japan*
*(E-mail: sebe@ai.kyutech.ac.jp)*

Abstract: This paper is concerned with the parametrization of all the decentralized stabilizing controllers. The auxiliary diagonal system, which is defined by the diagonal elements of Bezout factors, plays important roles in the parametrization of decentralized controllers. This paper gives an explicit characterization of the auxiliary system.

Keywords: Decentralized control, Coprime factorization, Parametrization, State-space realization, Transfer functions

## 1. INTRODUCTION

This paper is concerned with the parametrization of all the decentralized stabilizing controllers.

For centralized control systems, the parametrization of all stabilizing controllers is proposed by Youla *et al.* (1976). The parametrization has brought us great advantages in progress of control theory. It clarifies the structure of stabilizing controllers and the restrictions on the performance. It also helps the derivation of $H_\infty$ controllers (Francis, 1987; Doyle *et al.*, 1989; Glover *et al.*, 1991), and the derivation of the conditions for the strong stabilization problem and the simultaneous stabilization problem (Vidyasagar, 1985), etc. The parametrization helps to develop design procedures not only for problems with frequency domain specifications but also for those with time domain specifications. Boyd and Barratt (1991) propose to design a free parameter $Q(s)$, which appears affinely in the parametrization, by a convex optimization. It should be noted that the state space representation of the Bezout factors (Nett *et al.*, 1984) has also achieved the progress in control theory.

For the decentralized control systems, the decentralized Bezout identity and its stable factors, called "d-coprime factors," have been proposed (Gündeş and Desoer, 1990; Date and Chow, 1994). The decentralized Bezout identity has a special structure and the diagonal parts of the coprime factors represent the auxiliary diagonal system. With this auxiliary diagonal system, parametrizations of decentralized controllers have been also proposed. Although the decentralized Bezout identity and the auxiliary diagonal system play important roles in the parametrization of decentralized controllers, the explicit characterization has not been given.

From the viewpoint of controller design, the characterization of of auxiliary diagonal systems is very important. Unlike the centralized case, there are no practical methods to design decentralized controllers based on the parametrization. The first

[1] He was a visiting scholar at LAAS-CNRS, 7 Avenue du Colonel Roche, 31077 Toulouse, France. This work has been done during his stay at LAAS-CNRS. His stay at LAAS-CNRS was sponsored by the Japanese Ministry of Education, Science, Sports and Culture.

reason is that there are constraints on the parameters. The second reason is that the connection between d-coprime factors and their state space representation is not clarified. Thus, there are no computer-oriented computational methods for the parametrization of decentralized controllers.

In this paper, a state space representation of decentralized stable factors is given and the characteristics of the diagonal parts of the decentralized stable factors, i.e. the auxiliary diagonal systems are clarified.

*Notation.*
In this paper, static matrices are used in state space representations of transfer function matrices. And transfer function matrices themselves are also used in this paper. For simplicity, $s$ is dropped for many cases. To avoid misunderstanding, matrices $A$, $B$, $C$, $D$, $E$, $F$, $I$ and $O$ denote static matrices (especially $I$ and $O$ denote an identity and a zero matrices respectively) and the other matrices denote transfer function matrices throughout this paper.

## 2. PRELIMINARIES

This paper considers the decentralized control of a linear time-invariant plant $P(s)$ with $n$ control channels given by

$$\dot{x} = Ax + \sum_{i=1}^{n} B_i u_i, \tag{1}$$

$$y_i = C_i x + \sum_{j=1}^{n} D_{ij} u_j, \quad (i = 1, \dots, n) \tag{2}$$

where $x$, $u_i$ and $y_i$ are the state, the $i$-th local inputs and the $i$-th local outputs of the plant, respectively. In this paper, packed matrix forms are used to represent realizations of systems, and the above realizations are represented by

$$P(s) = \left[\begin{array}{c|ccc} A & B_1 & \cdots & B_n \\ \hline C_1 & & & \\ \vdots & & D_{ij} & \\ C_n & & & \end{array}\right]. \tag{3}$$

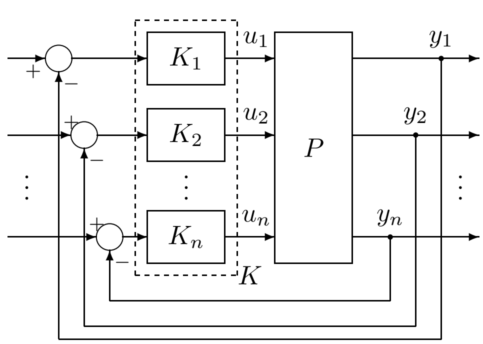


Fig. 1. Decentralized control system

The decentralized control problem is to find $n$ local controllers

$$K_i(s) = \left[\begin{array}{c|c} A_{ki} & B_{ki} \\ \hline C_{ki} & D_{ki} \end{array}\right], \qquad (i = 1, \dots, n) \tag{4}$$

to stabilize the given plant.

If a decentralized controller $K(s)$, defined as

$$K = \mathrm{diag}\{K_1, K_2, \dots, K_n\}, \tag{5}$$

stabilizes the given plant $P(s)$, then, obviously $P(s)$ and $K(s)$ have the doubly coprime factorization (Vidyasagar, 1985). Furthermore, according to the structure of the decentralized stabilizing controllers, some of the doubly coprime factorizations can be given as

$$P = NM^{-1} = \tilde{M}^{-1}\tilde{N}, \tag{6}$$

$$K = XY^{-1} = \tilde{Y}^{-1}\tilde{X}, \tag{7}$$

$$\begin{bmatrix} \tilde{Y} & -\tilde{X} \\ \tilde{N} & \tilde{M} \end{bmatrix} \begin{bmatrix} M & X \\ -N & Y \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} \tag{8}$$

where

$$X = \mathrm{diag}\ \{X_1, X_2, \dots, X_n\}, \tag{9}$$

$$Y = \mathrm{diag}\ \{Y_1, Y_2, \dots, Y_n\}, \tag{10}$$

$$\tilde{X} = \mathrm{diag}\ \{\tilde{X}_1, \tilde{X}_2, \dots, \tilde{X}_n\}, \tag{11}$$

$$\tilde{Y} = \mathrm{diag}\ \{\tilde{Y}_1, \tilde{Y}_2, \dots, \tilde{Y}_n\}. \tag{12}$$

As the coprime factors of $K(s)$ have the diagonal structures (9)-(12), the next lemma holds.

*Lemma 1.* (Date and Chow, 1994) The coprime factors in (8) also satisfy

$$\begin{bmatrix} \tilde{Y} & -\tilde{X} \\ \tilde{N}_d & \tilde{M}_d \end{bmatrix} \begin{bmatrix} M_d & X \\ -N_d & Y \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}, \tag{13}$$

where $N$, $M$, $\tilde{N}$ and $\tilde{M}$ are partitioned according to the sizes of inputs and outputs, and

$$N_d = \mathrm{diag}\ \{N_{11}, N_{22}, \dots, N_{nn}\}, \tag{14}$$

$$M_d = \mathrm{diag}\ \{M_{11}, M_{22}, \dots, M_{nn}\}, \tag{15}$$

$$\tilde{N}_d = \mathrm{diag}\ \{\tilde{N}_{11}, \tilde{N}_{22}, \dots, \tilde{N}_{nn}\}, \tag{16}$$

$$\tilde{M}_d = \mathrm{diag}\ \{\tilde{M}_{11}, \tilde{M}_{22}, \dots, \tilde{M}_{nn}\}. \tag{17}$$

Date and Chow (1994) call this doubly coprime factorization as “d-coprime factorization.” With this doubly coprime factorization, they give a parametrization of decentralized stabilizing controllers, and show the connection to the other attempts on parametrization (Manousiouthakis, 1989; Gündeş and Desoer, 1990; Özgüler, 1990).

To show the importance of $N_d$, $M_d$, $\tilde{N}_d$, $\tilde{M}_d$, another parametrization of decentralized controllers is reviewed here.

*Lemma 2.* (Sebe, 1998) All the decentralized controllers which stabilize the given plant $P(s)$ are parametrized as

$$(\tilde{T}\tilde{Y} - \tilde{Q}\tilde{N}_d)^{-1}(\tilde{T}\tilde{X} + \tilde{Q}\tilde{M}_d), \tag{18}$$

where

$$\begin{pmatrix} \det(\tilde{T}\tilde{Y} - \tilde{Q}\tilde{N}_d) \not\equiv 0, \\ \tilde{T} + \tilde{Q}R \in \mathcal{U}, \quad \tilde{T}, \tilde{Q} \in \mathcal{D} \end{pmatrix}, \tag{19}$$

and

$$R = \tilde{N}M_d - \tilde{M}N_d. \tag{20}$$

($\mathcal{U}$ denotes the set of unimodular matrices and $\mathcal{D}$ denotes the set of stable block diagonal transfer function matrices.)

*Definition 3.* For given $P(s)$ and $K(s)$, define the auxiliary diagonal system $P_d(s)$ as

$$P_d = N_d M_d^{-1} \quad (= \tilde{M}_d^{-1}\tilde{N}_d), \tag{21}$$

where $N_d$, $M_d$, $\tilde{N}_d$, $\tilde{M}_d$ are defined in (14)-(17).

The auxiliary system $P_d(s)$ plays an important role in the parametrization of decentralized stabilizing controllers given by Lemma 2. If the parameter $\tilde{T}$ is fixed as $\tilde{T} = I$, then (18) becomes a parametrization of decentralized controllers which stabilize the auxiliary system $P_d(s)$. And the constraint (19) becomes $I + \tilde{Q}R \in \mathcal{U}$, which implies decentralized controllers stabilize both the actual system $P(s)$ and the auxiliary system $P_d(s)$ simultaneously. The constraint (19) is a condition for simultaneous stabilization.

It is easy to see that

$$\|\tilde{Q}\|_\infty < (\|R\|_\infty)^{-1} \tag{22}$$

is a sufficient condition for $I + \tilde{Q}R$ to be an unimodular matrix, where $\|\cdot\|_\infty$ denote the $H_\infty$-norm of $(\cdot)$. Thus, under the above condition on $\tilde{Q}$, it would be able to tune $\tilde{Q}$ similarly to the convex optimization methods for centralized control systems. Note that this procedure designs controllers for the auxiliary system $P_d(s)$. The auxiliary system $P_d(s)$ would be important for decentralized controller design.

Date and Chow (1994) shows that $P_d(s)$ is uniquely detemined by given $P(s)$ and $K(s)$. But what is the system $P_d(s)$? Does $P_d(s)$ have a real meaning? The characteristics of the auxiliary system $P_d(s)$ has not been clarified yet. The reasons are

- The auxiliary system $P_d(s)$ is defined not only by the given $P(s)$, but also an initially given decentralized controller $K(s)$.
- The definition of $P_d(s)$ is complicated, extracting the diagonal elements from the coprime factors and re-constructing the system as a fraction of the elements.

The purpose of this paper is to reveal the property of the auxiliary system $P_d(s)$.

## 3. MAIN RESULTS

*Definition 4.* For given $P(s)$ and $K(s)$, define the auxiliary systems $H_i(s)$ as transfer functions from $u_i$ to $y_i$, where all the local loops are closed by $K_i(s)$ except the $i$-th loop. The block diagram of $H_i(s)$ is shown in Fig. 2. With the systems $H_i(s)$ $(i = 1, \ldots, n)$, let us define the system $H(s)$ as

$$H = \text{diag}\,\{H_1, H_2, \ldots, H_n\}. \tag{23}$$

In most designs of decentralized control systems, especially in design procedures called "independent designs," controllers are designed for "noninteractive models" of plants, which consists of decoupled subsystems. For these design procedures, it is very important to evaluate the effect from the other loops to ensure the stability and/or the robust stability of closed-loop systems. Thus, many classical and modern design procedures (Rosenbrock, 1969; Araki and Nwokah, 1975; Grosdidier and Morari, 1986; Skogestad and Morari, 1989) use $H_i(s)$ to analyze the (robust) stability of closed-loop systems. Furthermore, these design procedures also use the (robust) stability conditions on the difference between $H_i(s)$ and $P_{ii}(s)$ as design specifications for each local loop. With the additional design specifications, designed controllers (robustly) stabilize the given plant. The auxiliary system $H(s)$ is also important for designs of decentralized controllers.

*Theorem 5.* For a given plant $P(s)$ and a decentralized stabilizing controller $K(s)$, let us define the auxiliary systems $P_d(s)$ and $H(s)$ as (21) and (23) respectively. Then,

$$P_d(s) = H(s). \tag{24}$$

**PROOF.** See Appendix.

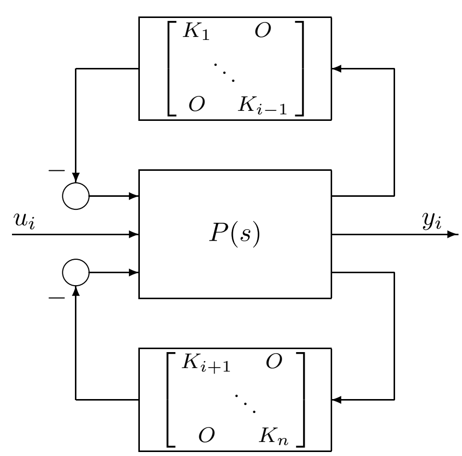


Fig. 2. Block diagram of $H_i(s)$

Now, let us discuss about designs of decentralized controllers based on the parametrization, and the meaning of Theorem 5.

Date and Chow (1993) suggested the direction of decentralized controller designs based on the parametrization shown in Lemma 2. Theorem 5 shows that such a procedure designs local controllers for $H_i(s)$, which are defined by current (already designed) decentralized controllers. In other words, the design procedure based on the parametrization would be an iterative design procedure, i.e., updating decentralized controllers iteratively. This iterative design procedure requires the evaluation of the difference between the $H_i(s)$ defined by current controllers and $H_i(s)$ defined by updated controllers before the update. Unlike the ordinary "independent design" procedures, this evaluation is very difficult. Until now, independent and iterative design procedures are proposed by Miyamoto and Vinnicombe (1997) and Sebe (1998). But the relations between these design procedures and the decentralized parametrization are not clarified. The relations will provide a deep insights into independent and iterative design procedures.

## 4. NUMERICAL EXAMPLE

In this section, a numerical example is given to verify the result. Let a given plant $P(s)$ and a decentralized (static) stabilizing controller $K(s)$ be

$$P(s) = \begin{bmatrix} \frac{s-3}{(s-1)(s-2)} & \frac{-1}{(s-1)(s-2)} \\ \frac{2}{(s-1)(s-2)} & \frac{s}{(s-1)(s-2)} \end{bmatrix} = \left[\begin{array}{cc|cc} 0 & -1 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right], \quad (25)$$

$$K(s) = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}. \quad (26)$$

Assume the d-coprime factors of $K(s)$ be

$$\tilde{X} = X = \mathrm{diag}\{1, 4\}, \quad (27)$$

$$\tilde{Y} = Y = \mathrm{diag}\{1, 1\}. \quad (28)$$

Then, the d-coprime factors of $P(s)$, which satisfy Bezout identity (8) are given by

$$\begin{bmatrix} M \\ \hline N \end{bmatrix} = \left[\begin{array}{cc} \frac{s^2+s+2}{s^2+2s+3} & \frac{1}{s^2+2s+3} \\ \frac{-8}{s^2+2s+3} & \frac{s^2-2s-1}{s^2+2s+3} \\ \hline \frac{s+1}{s^2+2s+3} & \frac{-1}{s^2+2s+3} \\ \frac{2}{s^2+2s+3} & \frac{s+1}{s^2+2s+3} \end{array}\right]. \quad (29)$$

Extracting the diagonal elements from the above coprime factors, the elements of the auxiliary diagonal system $P_d(s)$ are given by

$$N_{11}M_{11}^{-1} = \frac{s+1}{s^2+s+2}, \quad (30)$$

$$N_{22}M_{22}^{-1} = \frac{s+1}{s^2-2s-1}. \quad (31)$$

The systems (30) and (31) are identical to $H_1(s)$ and $H_2(s)$ respectively.

## 5. CONCLUSION

An explicit characterization of auxiliary diagonal systems which appears in the decentralized coprime factors is given in this paper. This characterization will provide additional insights into the parametrization of decentralized controllers.

The result also brings advantages in computational aspects, such as the reduction of computational complexity and the improvement of accuracy.


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## REFERENCES


Araki, M. and O. I. Nwokah (1975). Bounds for closed-loop transfer functions of multivariable systems. *IEEE Trans. Automatic Control* **AC-20**(10), 666–670.

Boyd, S. P. and C. H. Barratt (1991). *Linear Controller Design; –Limits of Performance–*. Prentice Hall.

Date, R. A. and J. H. Chow (1993). A parametrization approach to optimal $H_2$ and $H_\infty$ decentralized control problems. *Automatica* **29**(2), 457–463.

Date, R. A. and J. H. Chow (1994). Decentralized stable factors and a parameterization of decentralized controllers. *IEEE Trans. Automatic Control* **39**(2), 347–351.

Doyle, J. C., K. Glover, P. P. Khargonekar and B. A. Francis (1989). State-space solutions to standard $H_2$ and $H_\infty$ control problems. *IEEE Trans. Automatic Control* **34**, 831–847.

Francis, B. A. (1987). *A Course in $H_\infty$ Control Theory*. Vol. 88 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag.

Glover, K., D. J. N. Limebeer, J. C. Doyle, E. M. Kasenally and M. G. Safonov (1991). A characterization of all solutions to the four block general distance problem. *SIAM J. Control and Optimization* **29**(2), 283–324.

Grosdidier, P. and M. Morari (1986). Interaction measures for systems under decentralized control. *Automatica* **22**(3), 309–319.

Gündeş, A. N. and C. A. Desoer (1990). *Algebraic Theory of Linear Feedback Systems with Full and Decentralized Compensators*. Vol. 142 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag.

Manousiouthakis, V. (1989). On parametrization of all decentralized stabilizing controllers. In: *Proceedings of American Control Conference*. pp. 2108–2111.

Miyamoto, S. and G. Vinnicombe (1997). Fixed structure $H_\infty$ control design for multiple plants, —a coprime factorization/LMI based approach. In: *Proceedings of 4th Europican Control Conference*. Brussels, Bergium.

Nett, C. N., C. A. Jacobson and M. J. Balas (1984). A connection between state-space and doubly coprime fractional representations. *IEEE Trans. Automatic Control* **29**(9), 831–832.

Özgüler, A. B. (1990). Decentralized control: A stalbe proper fractional approach. *IEEE Trans. Automatic Control* **35**(10), 1109–1117.

Rosenbrock, H. H. (1969). Design of multivariable control systems using the inverse Nyquist array. *Proceedings of IEE* **116**(11), 1929–1936.

Sebe, N. (1998). Decentralized $H_\infty$ controller design. In: *Proceedings of 37th IEEE Conference on Decision and Control*. pp. 2810–2815.

Skogestad, S. and M. Morari (1989). Robust performance of decentralized control systems by independent designs. *Automatica* **25**(1), 119–125.

Vidyasagar, M. (1985). *Control System Synthesis: A Factorization Approach*. MIT Press.

Youla, D. C., H. A. Jabr and J. J. Bongiorno Jr. (1976). Modern Wiener-Hopf design of optimal controllers, Part II; multivariable case. *IEEE Trans. Automatic control* **AC-21**(3), 319–338.

## APPENDIX

*Lemma 6.* Assume state space representations of the given systems are

$$K(s) = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right], \quad P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]. \qquad (32)$$

And let a realization of the left coprime factors of $K(s)$ $(= \tilde{Y}^{-1}\tilde{X})$ be

$$[\tilde{Y} \quad \tilde{X}] = \left[\begin{array}{c|cc} A_k - F_kC_k & F_k & -(B_k - F_kD_k) \\ \hline -C_k & I & D_k \end{array}\right], \qquad (33)$$

where $A_k - F_kC_k$ is a stable matrix. Then, a realization of the right coprime factors of $P(s)$ which satisfy Bezout identity

$$\tilde{Y}M + \tilde{X}N = I, \qquad (34)$$

is given by

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{cc|c} A - BE^{-1}D_kC & -BE^{-1}C_k & BE^{-1} \\ B_k\tilde{E}^{-1}C & A_k - B_kDE^{-1}C_k & -F_k + B_kDE^{-1} \\ \hline -E^{-1}D_kC & -E^{-1}C_k & E^{-1} \\ \tilde{E}^{-1}C & -DE^{-1}C_k & DE^{-1} \end{array}\right], \qquad (35)$$

where

$$E = I + D_kD, \quad \tilde{E} = I + DD_k \qquad (36)$$

Note that this lemma is an extension of the result in Nett *et al.* (1984).

**Proof of Theorem 5** The uniqueness of the auxiliary system $P_d(s)$ is already shown by Date and Chow (1994). The proof of Theorem 5 is only to perform the calculation shown in Date and Chow (1994) by state space representations. For simplicity of presentation, we will develop the results for a 2-channel system. Results for the $n$-channel systems can be derive analogously, and hence, will be omitted here.

Let us assume

$$K(s) = \left[\begin{array}{c|c} A_k & B_k \\ \hline C_k & D_k \end{array}\right] = \left[\begin{array}{cc|cc} A_{k1} & O & B_{k1} & O \\ O & A_{k2} & O & B_{k2} \\ \hline C_{k1} & O & D_{k1} & O \\ O & C_{k2} & O & D_{k2} \end{array}\right], \qquad (37)$$

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right]. \qquad (38)$$

Let us also assume

$$F_k = \mathrm{diag}\{F_{k1}, F_{k2}\}, \qquad (39)$$

where $A_{ki} - F_{ki}C_{ki}$ are stable matrices.

From Lemma 6, the right coprime factors of the given $P(s)$, which satisfy (8), can be given by (35). Extracting the $(1,1)$ blocks from $M$ and $N$, the state space representation of $M_{11}$ and $N_{11}$ is

$$\begin{bmatrix} M_{11} \\ N_{11} \end{bmatrix} = \left[\begin{array}{cc|c} A - BE^{-1}D_kC & -BE^{-1}C_k & BE^{-1}\begin{bmatrix} I \\ O \end{bmatrix} \\ B_k\tilde{E}^{-1}C & A_k - B_kDE^{-1}C_k & -\begin{bmatrix} F_{k1} \\ O \end{bmatrix} + B_kDE^{-1}\begin{bmatrix} I \\ O \end{bmatrix} \\ \hline -[I \quad O]E^{-1}D_kC & -[I \quad O]E^{-1}C_k & [I \quad O]E^{-1}\begin{bmatrix} I \\ O \end{bmatrix} \\ [I \quad O]\tilde{E}^{-1}C & -[I \quad O]DE^{-1}C_k & [I \quad O]DE^{-1}\begin{bmatrix} I \\ O \end{bmatrix} \end{array}\right]. \qquad (40)$$

Then, the state space representation of $N_{11}M_{11}^{-1}$ is given by

$$N_{11}M_{11}^{-1} = \left[\begin{array}{ccc|c} A - B_2E_{22}^{-1}D_{k2}C_2 & O & -B_2E_{22}^{-1}C_{k2} & B\begin{bmatrix} I \\ -E_{22}^{-1}D_{k2}D_{21}\end{bmatrix} \\ * & A_{k1} - F_{k1}C_{k1} & * & * \\ B_{k2}\tilde{E}_{22}^{-1}C_2 & O & A_{k2} - B_{k2}D_{22}E_{22}^{-1}C_{k2} & B_{k2}\tilde{E}_{22}^{-1}D_{21} \\ \hline C_1 - D_{12}E_{22}^{-1}D_{k2}C_2 & O & -D_{12}E_{22}^{-1}C_{k2} & D_{11} - D_{12}E_{22}^{-1}D_{k2}D_{21} \end{array}\right], \tag{41}$$

where

$$E_{22} = I + D_{k2}D_{22}, \quad \tilde{E}_{22} = I + D_{22}D_{k2}. \tag{42}$$

Neglecting the unobservable modes, (41) can be reduced to

$$N_{11}M_{11}^{-1} = \left[\begin{array}{cc|c} A - B_2E_{22}^{-1}D_{k2}C_2 & -B_2E_{22}^{-1}C_{k2} & B_1 - B_2E_{22}^{-1}D_{k2}D_{21} \\ B_{k2}\tilde{E}_{22}^{-1}C_2 & A_{k2} - B_{k2}D_{22}E_{22}^{-1}C_{k2} & B_{k2}\tilde{E}_{22}^{-1}D_{21} \\ \hline C_1 - D_{12}E_{22}^{-1}D_{k2}C_2 & -D_{12}E_{22}^{-1}C_{k2} & D_{11} - D_{12}E_{22}^{-1}D_{k2}D_{21} \end{array}\right]. \tag{43}$$

It is easy to verify that the realization (43) is identical to that of $H_1(s)$. Similarly, $N_{22}M_{22}^{-1} = H_2(s)$ holds. □