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REACHABILITY UNDER STATE CONSTRAINTS - THE ELLIPSOIDAL TECHNIQUE

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Abstract:
This report describes the calculation of the reach sets and tubes with hard bounds on the controls and state space variables, with an emphasis on linear nonautonomous systems. The topic is important in the verification of hybrid systems and navigation problems, in set-membership estimation, robust control synthesis and related fields. The suggested scheme for linear systems relies on external approximations by ellipsoidal-valued tubes and induces a smaller computational burden compared with other methods of reach set calculation. In particular such approximations may be expressed through ordinary differential equations with coefficients given in explicit analytical form.This results in exact parametric representation of reach tubes through families of external ellipsoidal tubes. A tight approximation partly requires an online solution of a recurrent optimization problem. Applications to target problems relevant for verification of complex systems are finally indicated.

Keywords: dynamic programming, value function, reachability sets, state constraints, target problems, ellipsoidal calculus, linear systems.

INTRODUCTION

The problem of reachability is one of the key issues in control theory (Lee and Marcus 1967, Krasovski 1970,). A renewed interest in this topic is promoted by recent activities in advanced automation. This is particularly related to the problem of verification of complex systems (Henzinger et al.,1995; Puri et al., 1996, Varaiya 1998). The solutions to these problems must incorporate effective procedures for calculating reach sets and reach tubes for continuous-time systems. A special interest lies in the problem of reachability under state constraints which was relatively less studied and is investigated in this paper. Another demand for the solution to such problems comes from set-membership estimation, problems of navigation, interval analysis in scientific computation and many related issues including control synthesis and robust control.

Among the methods of reachability analysis for linear systems are those based on ellipsoidal techniques. Publications in this area (see, for example Boyd, 1994; Chernousko, 1994) were mostly concentrated on systems without state constraints with a single equation producing a sub-optimal (with respect to volume) ellipsoidal approximation to the exact reach set. However, it turns that ellipsoidal methods allow *exact representations* of the reach sets and tubes for linear systems through *parametrized families of both external and internal ellipsoids* (see Kurzhanski and Valyi,1997; Kurzhanski and Varaiya,2000). But to ensure effective calculation, an important question is how to effectively single out such families of *tight ellipsoidal approximations* to the reach tube or its neighborhood that would touch its surface or the surface of its neighborhood along certain nonintersecting curves which would thus *totally cover* this tube. A crucial point in organizing the calculation is to indicate such a parametrized variety of curves along which the procedure could be realized recurrently in time, without having to calculate the solution "afresh" for every new instant of time. In the absence of state constraints these "good" curves are those that are generated as *trajectories of the original control system that run along the boundary of the reach tube.* They are described by fairly simple ordinary differential equations. Such a move removes an unnecessary computational burden present in other methods and also opens new routes for deriving adequate numerical error estimates and new methods for systems other than those treated here. The suggested approach, indicated in Kurzhanski and Varaiya, 2000, is particularly relevant for hybrid systems since it allows further propagation to systems with resets.

A similar scheme also passes through with state constraints, but in a more complicated manner. Namely, here we need *a special version of the maximum principle under state constraints* which would allow recurrent relations (see Kurzhanski and Filippova, 1993; Kurzhanski and Gusev, 1971). The calculations are again effective when taken along certain "good" curves which will now be the solutions of a linear system, a part of whose coefficients are the optimalizers of a dual optimal control problem presented again in a special recurrent form. The parametrized description of the reach tube is again given by ordinary differential equations, but these will now contain parameters that are the on-line solutions of a recurrent-type dual optimization problem. The optimalizers may also allow generalized delta-functions. However, these difficulties are justified by the fact that the approximation is exact. By dropping the requirement of tightness (exactness) we may rely on simpler equations.

The given external ellipsoidal approximations appear approriate for solving target problems relevant for the verification of new types of complex systems. They allow to check whether the reach set belongs to or stays beyond a certain target set or enabling zone. This is especially true for systems without state constraints where the procedure is reduced to an integration of explicit ODE's with an online optimization of an unconstrained elementary function.

Let us now start from some general considerations.

1. REACHABILITY PROBLEM: GENERAL ISSUES

Consider a controlled system described by the ordinary differential equation,

$$\dot{x} = f(t, x, u), \quad t_0 \leq t \leq \tau, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state and $u \in \mathbb{R}^m$ the control restricted by inclusions (hard bounds)

$$\text{(a)} \quad u(t) \in \mathcal{P}(t); \quad \text{(b)} \quad Hx(t) \in \mathcal{Y}(t), \tag{2}$$

for all $t \geq t_0$. Here $\mathcal{P}(t), \mathcal{Y}(t)$ are set-valued functions, with values in $\text{comp}\mathbb{R}^m, \text{comp}\mathbb{R}^p$—the varieties of compact sets in $\mathbb{R}^m, \mathbb{R}^p$ respectively, continuous in the Hausdorff metric; $x(t_0) \in \mathcal{X}^0 \subseteq \text{comp}\mathbb{R}^n$. Function $f(t, x, u)$ is supposed to ensure uniqueness and uniform prolongability of solutions to any finite interval of time for any $x(t_0) = x^0,\ u(t) \in \mathcal{P}(t),\ t \geq t_0$.

Definiton 1.1. Given set-valued position $\{t_0, \mathcal{X}^0\}$, $G\mathcal{X}^0 \cap \mathcal{Y}(t_0) \neq \emptyset$, the **reach set** (or "attainability domain") $\mathcal{X}(\tau, t_0, \mathcal{X}^0)$ **at time** $\tau > t_0$, under constraints (2), from position $\{t_0, \mathcal{X}^0\}$, is the set

$$\mathcal{X}[\tau] = \mathcal{X}(\tau, t_0, \mathcal{X}^0) = \{x[\tau]\}$$

of all states $x[\tau] = x(\tau, t_0, x^0),\ x^0 = x(t_0) \in \mathcal{X}(t_0) = \mathcal{X}^0$ reachable at time τ by system (1), from some $x^0 \in \mathcal{X}^0$, through all possible controls u that ensure constraints (2). The set-valued function $\tau \mapsto \mathcal{X}[\tau] = \mathcal{X}(\tau, t_0, \mathcal{X}^0)$ is known as the **reach tube under state constraint** from position $\{t_0, \mathcal{X}^0\}\square$

If $\mathcal{X}[\tau] \neq \emptyset$ for $\tau \geq t_0$, then $\mathcal{X}[t] \neq \emptyset$ for any $t \in [t_0, \tau)$.

Definiton 1.2. The **reach set for interval** $[t_0, \tau]$ from position $\{t_0, \mathcal{X}^0\}$ is the union

$$\mathcal{X}_\tau(t_0, \mathcal{X}^0) = \cup_t \{\mathcal{X}(t, t_0, \mathcal{X}^0) | t \in [t_0, \tau] \geq \} \square$$

Thus $\mathcal{X}_\tau(t_0, \mathcal{X}^0)$ is the set of points attainable from some point $x^0 \in \mathcal{X}^0$ at some instant $t \in [t_0, \tau]$, with some control $u(t)$ restricted by the inclusions of (2). For time-invariant systems $\mathcal{X}_\tau(t_0, \mathcal{X}^0) = \mathcal{X}_{\tau - t_0}(\mathcal{X}^0)$. It is important to introduce schemes for calculating reach sets.

2. A RELATED OPTIMIZATION PROBLEM: THE VALUE FUNCTION

Problem I. Calculate reach sets $\mathcal{X}(\tau, t_0, \mathcal{X}^0)$ and $\mathcal{X}(t_0, \mathcal{X}^0)\square$

A fairly general approach is to calculate reach sets through an*optimization scheme* . Namely, one may look for the *value function*

$$V(\tau, x) = \min_u\{\max\{\phi_0(t_0, x(t_0)),$$

$$\max_t\{\phi(t, x(t))|t \in [t_0, \tau]\}\}\}, \qquad (3)$$

under restriction $x(\tau) = x$, with minimum taken over all measurable functions $u(t)$ that ensure (2(a)) . Here

$$\phi_0(t_0, x) = d^2(x, \mathcal{X}^0),\ \phi(t, x) = d^2(Hx(t), \mathcal{Y}(t)),$$

$$d^2(x, \mathcal{X}) = \min_z\{(x - z, x - z)|z \in \mathcal{X}\},$$

is the square of the *distance function* $d(x, \mathcal{X})$.

Lemma 2.1. The following relation is true:

$$\mathcal{X}(\tau, t_0, \mathcal{X}^0) = \{x : V(\tau, x) \le 0\}\square$$

This follows from the definition of the reach set $\mathcal{X}(\tau, t_0, \mathcal{X}^0)$ which is thus a *level set* for $V(\tau, x)$. We also use the notation $V(\tau, x) = V(\tau, x|t_0, V(t_0, \cdot))$, emphasizing the dependence of $V(\tau, x)$ on the boundary condition $V(t_0, x) = d^2(x, \mathcal{X}^0)$. Denote

$$\min_t\{V(t, x|t_0, V(t_0, (\cdot))|t \in [t_0, \tau]\}$$

$$= \mathcal{V}(\tau, x|t_0, V(t_0, (\cdot)).$$

Here and further we assume that function $V(t, x)$ is continuous in $\{t, x\}$.

Lemma 2.2. The reach set $\mathcal{X}(t_0, \mathcal{X}^0)$ is the level set

$$\mathcal{X}(t_0, \mathcal{X}^0) = \{x : \mathcal{V}(\tau, x|t_0, V(t_0, (\cdot)) \le 0\}\square$$

.

Next is an important property of value functions.

Theorem 2.1. $V(\tau, x)$ satisfies **the principle of optimality**, which has the semigroup form

$$V(\tau, x|t_0, V(t_0, \cdot)) =$$

$$= V(\tau, x|t, V(t, \cdot|t_0, V(t_0, \cdot))), \qquad (4)$$

$t_0 \le t \le \tau\square$

This property is established through a conventional argument (Fleming and Soner, 1993) and its consequence is a similar property for reach sets. The solution of the reachability problem now depends on the properties of the "classical" or "viscosity" solutions of the "forward" HJB equation which follows from (4).

Denote

$$\mathcal{H}(t, x, V, u) =$$

$$V_t(t, x) + (V_x(t, x), f(t, x, u)).$$

Then the HJB equation is

$$V_t(t, x) + \max_u(V_x(t, x), f(t, x, u)) = 0 \quad (5)$$

when $V(t, x) \ne \phi(t, x)$ and

$$\max_u\{\min\{\mathcal{H}(t, x, V, u),$$

$$\mathcal{H}(t, x, \phi, u)|u \in \mathcal{P}(t)\}\} = 0, \qquad (6)$$

when $V(t, x) = \phi(t, x)$. The boundary condition is

$$V(t_0, x) = \max\{\phi_0(t_0, x), \phi(t_0, x)\}.$$

Here V_t, V_x stand for the partial derivatives of $V(t, x)$, if these exist. Otherwise (5), (6) is a symbolic relation for the generalized HJB equation which has to be described in terms of subdifferentials, Dini derivatives or their equivalents. However, the typical situation is that V *is not differentiable.* The treatment of equation (5),(6) then involves the notion of generalized "viscosity" solution for this equation or their equivalents (Fleming and Soner, 1993).

Among the reachability-related issues important, for example, for system verification are some target problems.

Problem II. Given time τ, target set $\mathcal{D} \in \text{comp}\mathbb{R}^n$ and set $\mathcal{X}^0 = \mathcal{X}[t_0]$, verify which of the following relations is true:

$$\text{(i)}\mathcal{X}(\tau, t_0, \mathcal{X}^0) \subseteq \mathcal{D},$$

(all the reachable points are in $\mathcal{D}$ at time τ);

$$\text{(ii)}\mathcal{X}(\tau, t_0, \mathcal{X}^0) \cap \mathcal{D} \ne \emptyset,$$

(some of the reachable points are in $\mathcal{D}$ at time τ);

$$\text{(iii)}\mathcal{X}(\tau, t_0, X^0) \cap \mathcal{D} = \emptyset$$

(set $\mathcal{D}$ is not reachable at time τ) $\square$

The above may be checked through next facts.

Theorem 2.2. Conditions (i)-(iii) of Problem II, for a fixed time τ, are respectively equivalent to the relations:

$$\max_x\{d^2(x, \mathcal{D})|V(\tau, x) \le 0\} = 0,$$

$$\min_x\{d^2(x, \mathcal{D})|V(\tau, x) \le 0\} = 0,$$

$$\min_x\{d^2(x, \mathcal{D})|V(\tau, x) \le 0\} > 0.$$

Conditions (i),(ii) are true *for some* $\tau \in [t_0, t_1]$ if respectively

$$\min_\tau \max_x\{d^2(x, \mathcal{D})|V(\tau, x) \le 0, \tau \in [t_0, t_1]\} = 0,$$

$$\min_\tau \min_x \{d^2(x, \mathcal{D}) | V(\tau, x) \leq 0, \tau \in [t_0, t_1]\} = 0,$$

Condition (iii) is true *for all* $\tau \in [t_0, t_1]$ if

$$\min_\tau \min_x \{d^2(x, \mathcal{D}) | V(\tau, x) \leq 0, \tau \in [t_0, t_1]\} > 0 \ \square$$

The last theorem shows the role of value functions for solving the target problems of the above. The calculation of $V(t, x)$ in the general case may be done through the HJB equation (5). It is a fairly complicated procedure in general. However, in case of linear systems, an effective ellipsoidal technique may be applied. Our main goal is to use this technique and thus bypass in this case the main stumbleblock which is the calculation of viscosity solutions.

3. REACHABILITY IN LINEAR SYSTEMS

Consider the linear system

$$\dot{x} = A(t)x + B(t)u, \quad t_0 \leq t \leq \tau, \tag{7}$$

where matrices $A(t), B(t)$ are continuous and the system is *completely controllable* (see Lee and Marcus, 1967). The control $u = u(t)$ and state constraint are ellipsoidal

$$\text{(a)} \ \ u(t) \in \mathcal{P}(t) = \mathcal{E}(q(t), Q(t)),$$

$$\text{(b)} \ \ H(t)x(t) \in \mathcal{Y}(t) = \mathcal{E}(y(t), Y(t)),$$

where an ellipsoid with center q and shape matrix $Q \geq 0$ is denoted as $\mathcal{E}(q, Q) = \{u : (u-q, Q^{-1}(u-q) \leq 1\}$. Functions $Q(t) > 0, Y(t) > 0$ are continuous and absolutely continuous respectively. It is also assumed that $\mathcal{X}^0 = \mathcal{E}(x^0, X^0)$.

For the class of linear systems considered here the following basic property will be used.

Lemma 3.1. The set $\mathcal{X}[\tau]$ may be treated as the intersection

$$\mathcal{X}[\tau] = \cap\{\mathcal{X}_M(\tau, t_0, \mathcal{E}(x^0, X^0)) | M(\cdot)\} \tag{8}$$

of the cuts $\mathcal{X}_M(\tau, t_0, \mathcal{E}(x^0, X^0))$ of the solution tubes $\mathcal{X}_M(\cdot) = \{\mathcal{X}_M[t] : t \geq t_0\}$ to the differential inclusion

$$\dot{x} \in (A(t) - M(t)H')x + M(t)\mathcal{E}(y(t), Y(t)))$$
$$+\mathcal{E}(B(t)q(t), B(t)Q(t)B'(t)), \tag{9}$$

$t \geq t_0, \quad x^0 \in \mathcal{E}(x^0, X^0)$, over all $M(\cdot) \in \mathcal{M}$ - the class of piecewise-continuous matrix functions $M(t)$ defined for $t \in [t_0, \tau]$. The sets $\mathcal{X}_M(\tau, t_0, \mathcal{E}(x^0, X^0))$ are convex and compact.

The support function

$$\rho(l | \mathcal{X}(\tau, t_0, \mathcal{E}(x^0, X^0)) = \tag{10}$$

$$= \inf\{\rho(l | \mathcal{X}_M(\tau, t_0, \mathcal{E}(x^0, X^0)) | M(\cdot)\} \square$$

(Recall that $\rho(l|\mathcal{X}) = \max\{(l, x) | x \in \mathcal{X}\}$).

These relations are *different from traditional forms of the maximum principle under state constraints*, and are specially adapted to the investigation of the dynamics of reach sets. Paper of Kurzhanski and Filippova, 1993, where this scheme was introduced, indicates that here $M(\cdot)$ may as well be taken in the class $\mathcal{M}_\mathcal{C}$ of continuous functions or even in the class $\mathcal{M}_\mathcal{D}$ of continuously differentiable functions.

The essence of this result is at that the reach set $\mathcal{X}[\tau]$ under state constraints may be presented as an intersection of reach sets $\mathcal{X}_M(\tau, t_0, \mathcal{E}(x^0, X^0))$ for a parametrized system (9) *without state constraints.* The intersection is over the parametrizing functions $M(\cdot)$. We then just have to approximate the reach sets $\mathcal{X}_M(\tau, t_0, \mathcal{E}(x^0, X^0))$ by external ellipsoids. It is also important that the suggested procedure is *recurrent*, as we shall see in the sequel.

4. EXTERNAL ELLIPSOIDS UNDER STATE CONSTRAINTS

We will now approximate the set-valued solution $\mathcal{X}_M(t, t_0, \mathcal{E}(x^0, X^0))$ of system (9) (for a fixed function $M(\cdot)$)) by ellipsoids. This may be done following the techniques of Kurzhanski and Valyi, 1997, and Kurzhanski and Varaiya, 2000.

Theorem 4.1 Given $M(t), t \geq t_0$, the reach set

$$\mathcal{X}_M(t, t_0, \mathcal{E}(x^0, X^0)) \in \mathcal{E}(x^*_M(t), X_M(t)), \tag{11}$$

where

$$\dot{x}^* = (A(t) - M(t)H')x^* +$$
$$+B(t)q(t) + M(t)y(t), \ x^*(t_0) = x^0, \tag{12}$$

$$\dot{X}_M = (A(t) - M(t)H')X_M + X_M(A(t) - M(t)H')' +$$
$$+(\pi_Q(t) + \pi_Y(t))X_M + (\pi_Q(t))^{-1}B(t)Q(t)B'(t) +$$
$$(\pi_Y(t))^{-1}M(t)Y(t)M(t), \ \ X(t_0) = X^0. \tag{13}$$

The inclusion (11) is true for any $\pi_Q(t) > 0, \pi_Y(t) > 0$.$\square$

Definiton 4.1. An ellipsoid $\mathcal{E}(x^*_M(t), X_M(t))$ is said to be **tight** relative to $\mathcal{X}_M(t, t_0, \mathcal{E}(x^0, X^0))$ if

$$\rho(\pm l | \mathcal{E}(x^*_M(t), X_M(t))) = \rho(\pm l | \mathcal{X}_M(t, t_0, \mathcal{E}(x^0, X^0))$$

for some $l \in \mathbb{R}^n$.$\square$

As indicated in Kurzhanski and Varaiya, 2000, each of the reach tubes

$$\mathcal{X}_M[t] = \mathcal{X}_M(t, t_0, \mathcal{E}(x^0, X^0)), \ t \geq t_0,$$

may be included into ellipsoidal-valued tubes that are tight for each $t \geq t_0$, so that

$$\rho(l(t)|\mathcal{E}(x^*_M(t), X_M(t)))$$
$$= \rho(l(t)|\mathcal{X}_M(t, t_0, \mathcal{E}(x^0, X^0))$$

for some continuous curve $l(t)$. If the infimum in (10) could be substituted for a minimum, we would also have

$$\rho(l(t)|\mathcal{X}(t, t_0, \mathcal{E}(x^0, X^0)) = \quad (14)$$

$$= \min\{\rho(l(t)|\mathcal{X}_M(\tau, t_0, \mathcal{E}(x^0, X^0))|M(\cdot)\}.$$

Thus, firstly we have to ensure that the minimum is attained. This will always be true if we broaden the class $\mathcal{M}$ to the class of products $\mathcal{M}_\Lambda = \{M(\cdot)d\Lambda/dt\}$ where $M(\cdot) \in \mathcal{M}_\mathcal{C}$ and $\Lambda(t)$ is a nondecreasing piecewise - constant scalar function of bounded variation on $[t_0, \tau]$, $\forall\tau$: $\mathrm{Var}^{\tau}_{t_0}\Lambda(t) \leq \infty$, while $d\Lambda/dt$ is a generalized derivative which produces delta - functions. Necessary conditions for the presence of delta-functions in the minimizer for problem (14) may be found in Kurzhanski, 1977. Then, for calculating the value $\rho(l(t)|\mathcal{X}(t, t_0, \mathcal{E}(x^0, X^0))$, we will need online to solve an optimization problem of type (14), in the class $M(\cdot) \in \mathcal{M}_\Lambda$ if the mentioned necessary conditions are fulfilled or in the class $M(\cdot) \in \mathcal{M}$ in the opposite case. Secondly, we will have to ensure that all our procedures will be recurrent and will not have to be resolved "afresh" for every new instant of time. Further on, due to limit in space, we present relations when the minimum in (14) is attained in the class $M(\cdot) \in \mathcal{M}$.

Following the schemes of Kurzhanski and Varaiya, we will approximate set $\mathcal{X}(t, t_0, \mathcal{E}(x^0, X^0)$ and also its upper bound $\mathcal{X}_M(t, t_0, \mathcal{E}(x^0, X^0))$ only along "good" curves of type $l(t) = G'_{M_l}(t_0, t)l,\ l \in \mathbb{R}^n$. Here $M_l(\cdot)$ is the minimizer for the following dual "Problem DP":

$$\min\{\rho(l(t)|\mathcal{X}_M(t, t_0, \mathcal{E}(x^0, X^0))|M(\cdot) \in \mathcal{M}\}$$

with $l(t) = G'_{M_l}(t_0, t)l$ and l given.

Under our assumptions, with output $y \in YY$ of dimension $p \geq 2$, the solution $M_l(\cdot)$ to the minimization problem is unique. Moreover, $M_l(t) \equiv 0$ whenever $Hx(t) \in \mathrm{int}\mathcal{E}(y(t), Y(t))$.

Under our scheme we will have the following relations.

Theorem 4.2. The inclusions

$$\mathcal{X}(t, t_0, \mathcal{E}(x^0, X^0)) \subseteq \mathcal{X}_{M_l}(t, t_0, \mathcal{E}(x^0, X^0)),$$

$$\mathcal{X}_{M_l}(t, t_0, \mathcal{E}(x^0, X^0) \subseteq \mathcal{E}(x^*_{M_l}(t), X_{M_l}(t))),$$

are true. Along the function $l(t) = G'_{M_l}(t_0, t)l,\ l \in \mathbb{R}^n$ one has

$$\rho(l(t))|\mathcal{X}(t, t_0, \mathcal{E}(x^0, X^0)) = \quad (15)$$

$$= \rho(l(t)|\mathcal{X}_{M_l}(t, t_0, \mathcal{E}(x^0, X^0)) =$$
$$= \rho(l(t)|\mathcal{E}(x^*_{M_l}(t), X_{M_l}(t))) = (l(t), x_l(t))\square$$

With l given, $x_l(t)$ belongs to each of the tubes

$$\mathcal{X}(t, t_0, \mathcal{E}(x^0, X^0));\ \ \mathcal{X}_{M_l}(t, t_0, \mathcal{E}(x^0, X^0)),$$
$$\mathcal{E}(x^*_{M_l}(t), X_{M_l}(t)),\ t \geq t_0.$$

The curves $x_l(t),\ t \geq t_0$, do not intersect. They cover the surface of the reach tube

$$\mathcal{X}(t, t_0, \mathcal{E}(x^0, X^0)),\ \ t \geq t_0.$$

Direct calculation also shows the following.

Theorem 4.3 The parameters $x^*_{M_l}(t), X_{M_l}(t)$ of each ellipsoid $\mathcal{E}(x^*_{M_l}(t), X_{M_l}(t))$ that ensure (15) may be found from equations (12), (13), where $M(\cdot) = M_l(\cdot)$, and $\pi_Q(t), \pi_Y(t)$ are taken as

$$\pi_Q(t) = \pi_{QM_l}(t) = \quad (16)$$

$$= \frac{(l, G_{M_l}(t_0, t)B(t)Q(t)B'(t)G'_{M_l}(t_0, t)l)^{1/2}}{(l, X_{M_l}(t)l)^{1/2}},$$

$$\pi_Y(t) = \pi_{YM_l}(t) = \quad (17)$$

$$= \frac{(l, G_{M_l}(t_0, t)M(t)Y(t)M'(t)G'_{M_l}(t_0, t)l)^{1/2}}{(l, X_{M_l}(t)l)^{1/2}},$$

and the dimension of y is $p \geq 2$. Here $G_{M_l}(t, s)$, is the transition matrix for the homogeneous system $dx = (A(t) - M_l(t))xdt\square$

Thus, we have solved Problem I in two steps: first, due to Theorem 4.1, we have externally approximated $\mathcal{X}[\tau]$ with $\mathcal{X}_M(\tau, t_0, \mathcal{E}(x^0, X^0))$, then we have externally approximated $\mathcal{X}_M(\tau, t_0, \mathcal{E}(x^0, X^0)$ with $\mathcal{E}(x^*_M(t), X_M(t))$. Parameters $\pi_Q(t), \pi_Y(t)$ being selected due to Theorem 4.3, for a fixed $M_l(t)$, the ellipsoid $\mathcal{E}(x^*_{M_l}(t), X_{M_l}(t))$ will be tight relative to $\mathcal{X}_{M_l}(\tau, t_0, \mathcal{E}(x^0, X^0))$. In order that $\mathcal{X}_{M_l}(\tau, t_0, \mathcal{E}(x^0, X^0)), \mathcal{E}(x^*_{M_l}(t), X_{M_l}(t))$ would ensure (15), function $M(\cdot)$ has to be a minimizer of Problem DP. Namely, $M(\cdot) = M_l(\cdot)$ for a certain $l \in \mathbb{R}^n$. Finding such solutions for all $l \in \mathbb{R}^n$, we will find the desired reach tube $\mathcal{X}[t],\ t \geq t_0$.

*Remark 4.1.*In the *absence of state constraints* one should set $M(t) \equiv 0$ or $M_l(t) \equiv 0$. Under these conditions relations (12), (13), (16), (17) will give a *complete description of the reach tubes* in terms of solutions to a system of differential equations with *explicitly given* coefficients as shown in Kurzhanski and Varaiya, 2000 □

5. THE TARGET PROBLEMS

To indicate some solution routes to Problem II we will use the following relations:

$$\max\{d^2(x, \mathcal{D})|x \in \mathcal{E}(x^*(t), X_M(t))\} =$$
$$= \max\{(l, x^*(t)) + (l, X_M(t)l)^{1/2} + (l, Hl)^{1/2} -$$

$$-\frac{1}{4}(l,l)|l \in \mathbb{R}^n\} = \phi_+(t, M(\cdot)),$$

$$\min\{d^2(x,\mathcal{D})|x \in \mathcal{E}(x^*(t), X_M(t))\} =$$

$$= \max\{(l, x^*(t)) - (l, X_M(t)l)^{1/2} - (l, Hl)^{1/2} -$$

$$-\frac{1}{4}(l,l)|l \in \mathbb{R}^n\} = \phi_-(t, M(\cdot)).$$

This allows to formulate the next assertion which follows from Theorem 2.2.

Theorem 5.1 Conditions (i) and (iii) of Problem II for a fixed time τ are respectively ensured by the relations (with $M(\cdot) \in \mathcal{M}$):

$$\exists M(\cdot) \;\Rightarrow\; \phi_+(\tau, M(\cdot)) \leq 0, \qquad (18)$$

$$\exists M(\cdot) \;\Rightarrow\; \phi_-(\tau, M(\cdot)) > 0 \qquad (19)$$

□

Thus, you just have to check the fulfillment of (18) or (19) for only one function $M(\cdot)$, that the respective property (i) or (iii) would be verified. The practical calculations are then within the techniques of ellipsoidal calculus. They are especially simple in the absence of state constraints, when $M(\cdot) = 0$ and $\phi_+(t), \phi_-(t)$ do not depend upon $M(\cdot)$: you just check the inequalities $\phi_+(\tau) \leq 0$ or $\phi_-(\tau) > 0$ that the respective properties would be verified!

Theorem 5.2

Condition (i) is ensured *for some* τ if

$$\exists M(\cdot) \;\Rightarrow \min_{\tau}\{\phi_+(\tau, M(\cdot))|\tau \in [t_0, t_1]\} \leq 0.$$

Condition (iii) is ensured *for all* τ if

$$\exists M(\cdot) \;\Rightarrow \min_{\tau} \phi_-(\tau, M(\cdot))|\tau \in [t_0, t_1]\} > 0 \quad \square$$

The calculation algorithm again follows from ellipsoidal calculus.

6. CONCLUSION

This paper presents techniques for calculating reach sets under hard bounds on the controls and state space variables. For linear systems ellipsoidal techniques are further specified. The suggested scheme introduces parametrized varieties of tight ellipsoidal-valued tubes that approximate the exact reach tube or its neighbourhood from above, touching it along specially selected "good" curves that cover the entire exact tube. This leads to recurrent relations that simplify calculations compared to other approaches. The proofs rely on relations equivalent to a special "recurrent" version of the maximum principle under state constraints. The results are then applied to target problems relevant for verification of complex systems. The given descriptions are relevant for parallel calculations and computer animation. Presented results may also be applied to "guaranteed" (set-membership) state estimation where the measurement equation acts as an on-line state constraint.

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