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PARAMETRIC SYNTHESIS OF QUALITATIVE ROBUST CONTROL SYSTEMS USING ROOT LOCUS FIELDS

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Abstract: The paper represents the root locus fields approach to the problem of parametric synthesis of uncertain control systems meeting the given robust quality requirements. The described method consists in location of the system characteristic equations family roots within the given domain *Q* in the complex plane. It can be attained by inscription of the root locus field level lines into the domain *Q*. Considered is the case of linear coefficients correlation. The approach offered is peculiar for making it possible to consider the domains of arbitrary shape both convex and concave described analytically by algebraic curves. *Copyright © 2002 IFAC*

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1. INTRODUCTION

The dynamic system quality characteristics are defined by the position of its characteristic equation roots in the complex plane. Therefore the problem of locating roots in this or that way for attaining required quality characteristics of the system functioning, i.e. solving the *Λ*-stability task, is one of the most important stages in the control systems design process.

The *Λ*-stability tasks within the parametric approach to robustness were preceded by the problems of asymptotic stability, i.e. location of the system roots in the left half-plane region being the particular case of more general *Λ*-stability instance. The algebraic approach to this problem can be represented by the classical works of Barmish (1984) and Soh (1989) who formulated conditions for invariance of strict Hurwitz property for polynomials under coefficient perturbations. The frequency domain approach to the problem can be illustrated by the bright works of Y.Z. Tsypkin and B.T. Polyak with the basic paper (Tsypkin and Polyak, 1991) where the frequency robustness conditions are formulated for various types of uncertainty and the method is developed for computing the robustness margin in one shot.

Further investigations lead to the *Λ*-stability problem. All the tasks that arise in this connection within the parametric approach and relate to robust systems analysis and design may be divided into three main groups: finding the guaranteed roots location domain for the given system, defining conditions for verifying whether roots get into the given region (verifying *Λ*-stability conditions) and locating roots within the given region (ensuring *Λ*-stability).

The first group is represented by the paper of Sirazetdinov (1988) where the domain *Λ* of the guaranteed polynomial roots location is constructed. The second group being most large in number comprises the *Λ*-stability conditions. Kharitonov (1981) formulated conditions of the interval polynomial roots location inside the domain *Λ* based on verifying 2^n polynomials with coefficients taking the limit values of the given intervals. Soh and Foo (1990) gave conditions for interval polynomial roots location within the left sector. Conditions for getting the roots of an interval plant into the convex region in the complex plane were formulated by Shaw and Jayasuriya (1993). The third group of methods for the *Λ*-stability problems solving comprises the paper of Vicino, 1989 who solved the task similar to that considered by Barmish, 1984, but for the *Λ*-stability case. A geometric approach is applied that allows to obtain equations of the hypersurfaces in the coefficient space bounding the regions containing the polynomial roots in the given region *Λ* of the complex plane. However the method implies rather sophisticated algorithms and its realization entails great number of calculations. The task for locating roots of the uncertain dynamic system characteristic equation within the trapeze-shaped domain is solved by Rimsky and Nesenchuk, 1996 using the root locus approach.

This paper belongs to the third group. Its main goal is providing with an instrument for locating the roots of the uncertain control system characteristic polynomial within the given arbitrary domain (including multi-sheet domains) that may be of either convex or concave shape and is bounded by the arbitrary algebraic curve. The method is based on the root locus fields application. Considered is linear correlation of the coefficients which may be of both real and complex types.

2. PROBLEM FORMULATION

Consider a family of dynamic systems characteristic equations like

$$p^n + a_1 p^{n-1} + \ldots + a_{n-1} p + a_n = 0 \qquad (1)$$

where $a_1,\ldots,a_n$ are variable coefficients that may be of either real or complex type and linearly depend on some uncertain system parameter indicated as k being the subject for arbitrary variation; p is a complex variable, $p=\delta+i\omega$. Name the parameter k as the *free parameter*.

After transformation of equation (1) write it relative to k.

$$\phi(p) + k\psi(p) = 0 \qquad (2)$$

where $\phi(p)$ and $\psi(p)$ are polynomials of the complex variable p; k is the system uncertain parameter.

From (2) obtain an expression for k as follows:

$$k = f(p) = -\frac{\phi(p)}{\psi(p)} = u(\delta,\omega) + i\upsilon(\delta,\omega) \qquad (3)$$

where $u(\delta,\omega)$, $\upsilon(\delta,\omega)$ are harmonic functions of two independent real variables δ and ω.

It is required to obtain the values of the uncertain (variable) parameter k which ensure location of equation (1) roots within the given domain Q bounded by the arbitrary closed algebraic curve $q(\delta,\omega)$, containing both convex and concave regions. Name the domain Q as the *quality domain*.

3. LOCATION OF THE CHARACTERISTIC EQUATION ROOTS WITHIN THE GIVEN ARBITRARY DOMAIN

3.1. Circular image root locus field for dynamic system.

Let the parameter k (see (2) and (3)) to vary continuously along a circle located in the complex plane k that is also named as the *free parameter plane*. It means that the image of equation (3) root locus is represented by a circle. The corresponding equation of the circular image root locus (CRL) obtained on the basis of the mapping function (3) is represented in general as

$$f_k(\delta,\omega,a,b,\rho)=0 \qquad (4)$$

where a and b are coordinates of the image center by the axes u and υ correspondingly; ρ is the radius of the image circle. Complex potential (Rimsky and Nesenchuk, 1996) of the scalar root locus field is set at any point of the extended plane of the variable parameter by means of setting the root locus image existence over the whole plane. Therefore supposing that the root locus (4) image is set in the whole plane k by defining the infinite variation interval $-\infty \le \rho \le +\infty$, write the function of the scalar stationary circular root locus field in the general form as follows:

$$f^*=f^*(\delta,\omega)$$

and the field level lines equation as follows:

$$f^*(\delta,\omega)=\rho^2. \qquad (5)$$

The latter equation is similar to (4) at $a=const$ and $b=const$.

Due to the conformity of the mapping realized by function (3), the circular field level lines are formed by the closed curves located in the complex variable plane p. These curves are located concentrically around the points named as *field localization centers* (Rimsky and Nesenchuk, 1996) which map the field image center (the circular image center) onto the plane p.

Definition 1. The points mapping the center of the circular image defined in the free parameter plane onto the plane of the system fundamental frequencies using the rational function $p=g(k)$, reverse to function (3), are named as *localization centers* of the circular root locus field.

Definition 2. *Local level lines* of the circular root locus field are the lines bounding the closed simply-connected regions mapped by function (3) and not containing the mapping function branching points as the inner ones.

Definition 3. *Global level lines* of the circular root locus field are the lines bounding the closed simply-connected regions mapped by function (3) and containing the mapping function branching points as the inner ones.

Fig. 1 represents the fragment of the circular root locus field for the dynamic system of [4;0] class. The localization centers of the field represented by the level lines $L_1(L_1', L_1'', L_1''')$, L_2, L_3 are located at points C_1, C_2, C_3 and C_4. The level lines bind correspondingly the multi–sheet domains (W_1, W_2 and W_3 in fig. 1) in the plain *p*. Every domain is the mapping of the disc image with a particular radius.

The line $L_1(L_1', L_1'', L_1''')$ is local one and lines L_2 and L_3 are of the global type.

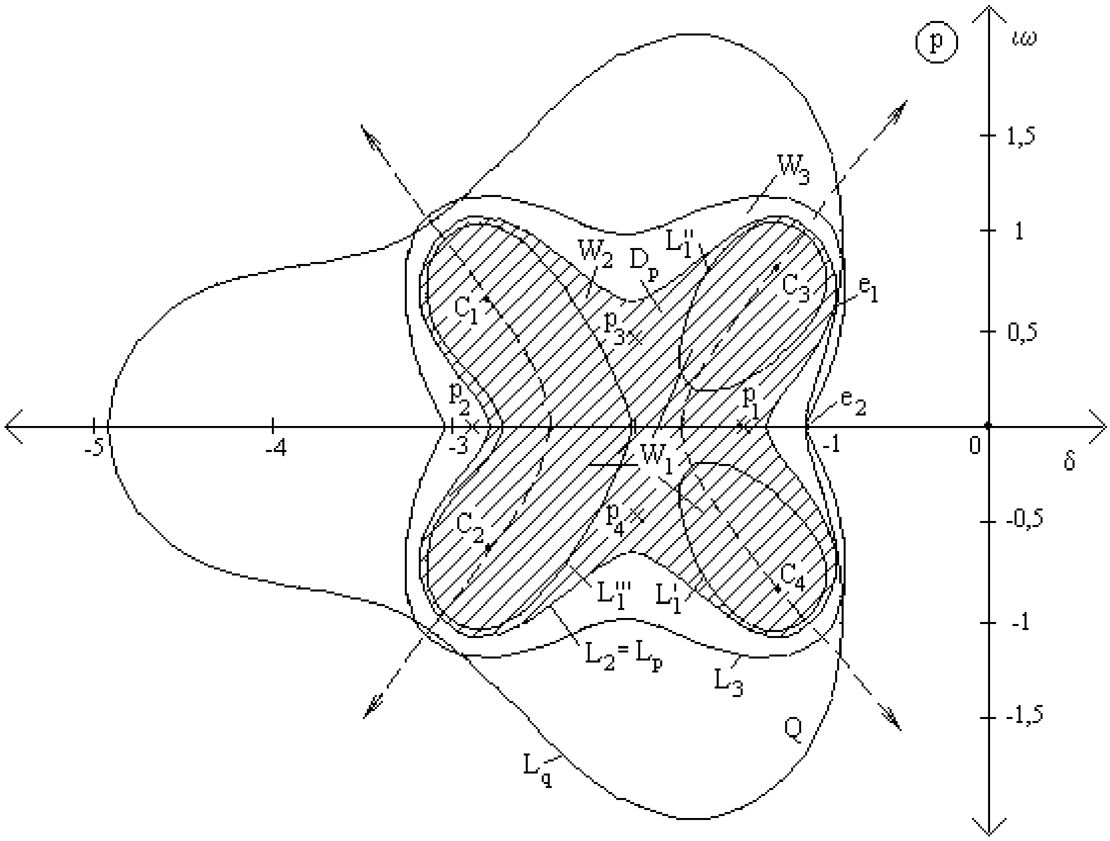

Fig. 1. The domain D_p of the dynamic system characteristic equation roots belonging to the given domain *Q*.

3.2. Field orientation.

Find the domain D_p of roots values belonging to the given quality domain *Q*, as the domain bounded by the level line L_p of the given system circular field. The image of this line is a circle located in the complex plane *k* and bounding the circular (disc) domain D_k of the corresponding parameter *k* values. Thus the following condition should be satisfied:

$$k \in D_k \rightarrow p_i \in Q, i = \overline{1, n}.$$

The task is solved by inscription of level lines of the preliminary oriented circular field into the preset domain *Q*. The inscribed line L_p will be the bound of the required domain D_p.

For the task solving algorithm implementation it is first necessary to set orientation (location) of the scalar field CRL relative to the domain *Q* in such a manner to ensure the inscription of the field level lines into *Q*. Proceeding from the geometric considerations one can conclude that the desired circular field location is attained when all localization centers of the field are located inside the domain *Q*. Let the circular image center is located on the real axis *u* of the complex plane *k*. It means that the corresponding field localization centers are located either at zeroes of function (3), i.e. at the poles of the open loop transfer function

$$G(p)=\psi(p)/\phi(p) \qquad (6)$$

or on the portions of the system Teodorchik-Evans Root Locus (TERL) branches located in the plane of system fundamental frequencies.

Therefore it is evident that the enough condition for desired field localization centers orientation is location (may be arbitrary) of transfer function (6) poles p_j, $j = \overline{1, n}$ within the given domain *Q*.

For the example in fig. 1 the field localization centers C_1, C_2, C_3 and C_4 are located on the TERL branches which are depicted in the figure by the dashed lines.

Figure above demonstrates an example of the dynamic system stationary circular root locus field orientation in relation to the given domain Q.

In the event that if the poles of (6) are located within the bounds of Q, the field orientation is not required.

3.3. Field level line inscription.

After setting, if necessary, the poles of (6) and the field localization centers (image center) it is possible to start the procedure of the level line inscription into the given region.

The circular root locus field level line inscribed into the given domain Q is the line being completely located within this domain and having the corresponding circular image of the maximal radius. This line, that is designated as L_p, bounds the required domain D_p of the characteristic equation roots values that is completely located inside the domain Q bounds and mapping the disk image D_k, bounding the domain of the required variable parameter k values.

It is evident that the line L_q bounding the domain Q and the inscribed line $\boldsymbol{L_p}$ have the common point e (the touch point) at which they have common tangent. For calculating this point coordinates δ_e *and* ω_e write the equation of a tangent to the curve $f^*(\delta,\omega)$ in general representation:

$$\frac{\partial f^*(\delta,\omega)}{\partial \delta}(d-\delta)+\frac{\partial f^*(\delta,\omega)}{\partial \omega}(c-\omega)=0, \quad (7)$$

where c and d are current coordinates of a point located on a tangent; δ and ω are the touch point coordinates.

On the basis of equations (5) and (7) compose the system of equations

$$\left.\begin{aligned} &q(\delta,\lambda)=0;\\ &q_k(\delta,\lambda,d)=0;\\ &g_k(\delta,\lambda,d)=0 \end{aligned}\right\} \quad (8)$$

where the expression

$$\lambda = c/d = \omega/\delta$$

is the tangent of the angle of inclination of a tangent to the axis δ (equation of a tangent straight line $\omega=\lambda\delta$). The first equation of the system is the equation of the line L_q, bounding the given region Q, the second and the third equations describe the tangents to the bounding lines L_q and L_p correspondingly.

As a result of system (8) solving the coordinates δ_e and $\omega_e=\lambda\delta$ for the lines L_q and L_p tangent point are determined. Radius ρ of a circle bounding the domain D_k of the parameter k values and having the mapping D_p (the domain of equation (1) roots location) in the plane p is calculated by formula (5). The radius ρ that is found completely defines the required domain D_p of roots values.

When solving system (8) several values of δ_e, ω_e and ρ may be found, as several lines may be tangent to the bound, and also one line may have several tangent points with a bound. It is descriptively demonstrated in fig. 1 where the lines L_2 and L_3 have the tangent points with a bound (points e_2 and e_3 correspondingly). In such cases it is evidently necessary to select the minimal value of ρ that corresponds to the inscribed domain, i.e. to the domain completely located within the limits of Q. In fig. 1 the inscribed line is represented by the line $L_2=L_p$. It touches the domain Q bound in the touch point e_2.

Consider the numerical *example*. Suppose the control system dynamics is described by the characteristic equation like

$$p^4+8{,}1p^3+24{,}3p^2+32{,}1p+15{,}7+k=0 \quad (9)$$

where k is the uncertain parameter being the subject for arbitrary variation.

The bound Q of roots location is described by the algebraic equation

$$\begin{aligned}&\omega^6+3\omega^4\delta^2+15\omega^4\delta+20{,}6\omega^4+3\omega^2\delta^4+30\omega^2\delta^3+\\ &+113\omega^2\delta^2+158\omega^2\delta++45{,}3\omega^2+\delta^6+15\delta^5+\\ &+91{,}9\delta^4+303\delta^3+588\delta^2+644\delta+279. \quad (10)\end{aligned}$$

Bound (10) is represented in fig. 1 by the line L_q. It is required to determine the domain of the parameter k values ensuring location of the roots of (9) within the quality domain Q, bounded by the curve L_q (10).

The free component of (9) is selected as the uncertain parameter for making the example to be more descriptive. It is evident that the free parameter k may enter into any coefficient of (9) or into all of them together.

To solve the formulated task the coefficients of the polynomials $\phi(p)= p^4+8{,}1p^3+24{,}3p^2+32{,}1p+15{,}7$ and $\psi=1$ (according to formulas (2) and (9)) are entered to the input of the corresponding software package. The curve L_q and function (6) poles (points p_1, p_2, p_3, p_4 in the figure) for the given system are displayed. The poles are located within the domain Q, i.e. new poles location is not required. The image center for the circular field is selected at the point with coordinates u=2, υ=0. Corresponding localization centers are located at points C_1, C_2, C_3 и C_4. As a result of the program execution the value ρ=3 is calculated. It means that if the parameter k values are located within the domain bounded by the circle having the center located at the point with coordinates u=2, υ=0 and radius ρ=3, all the values of polynomial (9) are located within the domain Q bounded by curve (10). The corresponding domain D_p of roots location bounded by the level line L_q is shown in figure 1.

4. CONCLUSION

The paper deals with the method for parametric synthesis of uncertain control systems meeting the requirements of robust quality. The method is based on locating roots of the dynamic control system characteristic equation within the given domain bounded by the algebraic curve of arbitrary configuration. Coefficients depend linearly on some uncertain parameter being the subject for arbitrary variation. The essence of the proposed analytical method consists in application of root locus fields that makes it possible to determine the domain of the uncertain parameter values (including both types, real and complex values) which ensure location of all the roots within the given domain. The method is peculiar for its simplicity and descriptiveness. Unlike the existing parametric methods dealing with the convex regions only, the proposed method considers the roots domains of arbitrary shape bounded by the analytically represented algebraic curves. Solutions are also provided for the cases when the given dynamic system is unstable.

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